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THE COHOMOLOGY RING OF THE GKM GRAPH OF A FLAG MANIFOLD

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1. Introduction

Let T be a torus of dimension n and M a closed smooth T-manifold. The equivariant cohomology of M, denoted $H_T^*(M)$, contains a lot of geometrial information on M. Moreover it is often easier to compute $H_T^*(M)$ than $H^*(M)$ by virtue of the Localization Theorem which implies that the restriction map

$$\iota^*\colon H_T^*(M) \to H_T^*(M^T) \tag{1.1}$$

to the T-fixed point set M^T is often injective, in fact, this is the case when $H^{odd}(M) = 0$. When M^T is isolated, $H_T^*(M^T) = \oplus_{p\in M^T} H_T^*(p)$ and hence $H_T^*(M^T)$ is a direct sum of copies of a polynomial ring in n variables because $H_T^*(p) = H^*(BT)$.

Therefore we are in a nice situation when $H^{odd}(M) = 0$ and M^T is isolated. Goresky-Kottwitz-MacPherson [2] (see also [3, Chapter 11]) found that under the further condition that the weights at a tangential T-module are pairwise linearly independent at each $p \in M^T$, the image of ι^* in (1.1) above is determined by the fixed point sets of codimension one subtori of T when $\mathbb{Q}$ is tensored in cohomology. Their result motivated Guillemin-Zara [4] to associate a labeled graph $\mathcal{G}_M$ with M and define the "cohomology" ring $\mathcal{H}^*(\mathcal{G}_M)$ of $\mathcal{G}_M$, which is a subring of $\oplus_{p\in M^T} H^*(BT)$. Then the result of Goresky-Kottwitz-MacPherson can be stated that $H_T^*(M)\otimes\mathbb{Q}$ is isomorphic to $\mathcal{H}^*(\mathcal{G}_M)\otimes\mathbb{Q}$ as graded rings when M satisfies the conditions mentioned above.

The result of Goresky-Kottwitz-MacPherson can be applied to many important T-manifolds M such as flag manifolds and compact smooth toric varieties etc. When M is such a nice manifold, $H_T^*(M)$ is often known to be isomorphic to $\mathcal{H}^*(\mathcal{G}_M)$ without tensoring with $\mathbb{Q}$ (see [1], [5], [6] for example). We determine the ring structure of $\mathcal{H}^*(\mathcal{G}_M)$ or $\mathcal{H}^*(\mathcal{G}_M)\otimes\mathbb{Z}[\frac{1}{2}]$ when M is a flag manifold of classical type directly without using the fact that $H_T^*(M)$ is isomorphic to $\mathcal{H}^*(\mathcal{G}_M)$ ([7]). In my talk, I introduced the result when M is a flag manifold of type A. This is a joint work with Mikiya Masuda and the details can be found in [7].

2. Labeled graph and its cohomology for type A_{n-1}

Let $\{t_i\}_{i=1}^n$ be a basis of $H^2(BT)$, so that $H^*(BT)$ can be identified with a polynomial ring $\mathbb{Z}[t_1, t_2, \dots, t_n]$. We take an inner product on $H^2(BT)$ such that the basis $\{t_i\}$ is orthonormal.

Then

$$\Phi(A_{n-1}) := \{\pm(t_i - t_j) \mid 1 \le i < j \le n\} \tag{2.1}$$

is a root system of type A_{n-1}.

Definition. The labeled graph associated with $\Phi(A_{n-1})$, denoted $\mathcal{A}_n$, is a graph with labeling ℓ defined as follows.

- The vertex set of $\mathcal{A}_n$ is the permutation group S_n on {1,2,. . . ,n}.
- Two vertices w, w' in $\mathcal{A}_n$ are connected by an edge $e_{w,w'}$ if and only if there is a transposition $(i, j) \in S_n$ such that $w' = w(i, j)$, in other words,

  $$w'(i) = w(j), \;\; w'(j) = w(i) \;\; and \;\; w'(r) = w(r) \;\; for \;\; r \neq i, \; j.$$

- The edge $e_{w,w'}$ is labeled by $\ell(e_{w,w'}) := t_{w(i)} - t_{w'(i)}$.

Definition. The cohomology ring of $\mathcal{A}_n$, denoted $\mathcal{H}^*(\mathcal{A}_n)$, is defined to be the subring of $\mathrm{Map}(V(\mathcal{A}_n), H^*(BT)) = \bigoplus_{v \in V(\mathcal{A}_n)} H^*(BT)$, where $V(\mathcal{A}_n)$ denotes the set of vertices of $\mathcal{A}_n$, *i.e.* $V(\mathcal{A}_n) = S_n$, satisfying the following condition:

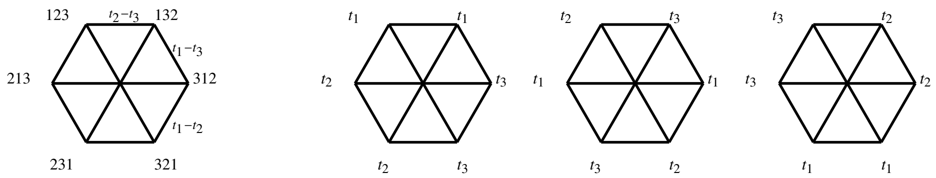
> $f \in \mathrm{Map}(V(\mathcal{A}_n), H^*(BT))$ is an element of $\mathcal{H}^*(\mathcal{A}_n)$ if and only if $f(v) - f(v')$ is divisible by $\ell(e)$ in $H^*(BT)$ whenever the vertices v and v' are connected by an edge e in $\mathcal{A}_n$.

For each $i = 1, \ldots, n$, we define elements τ_i, t_i of $\mathrm{Map}(V(\mathcal{A}_n), H^*(BT))$ by

$$\tau_i(w) := t_{w(i)}, \quad t_i(w) := t_i \quad \text{for } w \in S_n. \tag{2.2}$$

In fact, both τ_i and t_i are elements of $\mathcal{H}^2(\mathcal{A}_n)$.

Example. The case $n = 3$. The root system $\Phi(A_2)$ is $\{\pm(t_i - t_j) | 1 \le i < j \le 3\}$. The labeled graph $\mathcal{A}_3$ and τ_i for $i = 1, 2, 3$ are as follows.



The labeled graph $\mathcal{A}_3$ $\quad$ τ_1 $\quad$ τ_2 $\quad$ τ_3

Theorem 2.1. *Let $\mathcal{A}_n$ be the labeled graph associated with the root system $\Phi(A_{n-1})$ of type A_{n-1} in* (2.1). *Then*

$$\mathcal{H}^*(\mathcal{A}_n) = \mathbb{Z}[\tau_1, \cdots, \tau_n, t_1, \cdots, t_n]/(e_i(\tau) - e_i(t) \mid i = 1, \cdots, n),$$

where $e_i(\tau)$ (resp. $e_i(t)$) is the i^{th} elementary symmetric polynomial in $\tau_1, \cdots, \tau_n$ (resp. $t_1, \cdots, t_n$).

To prove this theorem, we need the following two lemmas.

Lemma 2.2. *$\mathcal{H}^*(\mathcal{A}_n)$ is generated by $\tau_1, \cdots, \tau_n, t_1, \cdots, t_n$ as a ring.*

We abbreviate the polynomial ring $\mathbb{Z}[\tau_1, \cdots, \tau_n, t_1, \cdots, t_n]$ as $\mathbb{Z}[\tau, t]$. The canonical map $\mathbb{Z}[\tau, t] \to \mathcal{H}^*(\mathcal{A}_n)$ is a grade preserving homomorphism which is surjective by Lemma 2.2. Let $e_i(\tau)$ (resp. $e_i(t)$) denote the i^{th} elementary symmetric polynomial in $\tau_1, \cdots, \tau_n$ (resp. $t_1, \cdots, t_n$). It easily follows from (2.2) that $e_i(\tau) = e_i(t)$ for $i = 1, \cdots, n$. Therefore the canonical map above induces a grade preserving epimorphism

$$\mathbb{Z}[\tau, t]/\big(e_1(\tau) - e_1(t), \; \cdots, e_n(\tau) - e_n(t)\big) \to \mathcal{H}^*(\mathcal{A}_n). \tag{2.3}$$

Remember that the Hilbert series of a graded ring $A^* = \oplus_{j=0}^{\infty} A^j$, where A^j is the degree j part of A^* and of finite rank over $\mathbb{Z}$, is a formal power series defined by

$$F(A^*, s) := \sum_{j=0}^{\infty} (\operatorname{rank}_{\mathbb{Z}} A^j) s^j.$$

In order to prove that the epimorphism in (2.3) is an isomorphism, it suffices to verify the following lemma because the modules in (2.3) are both torsion free.

Lemma 2.3. *The Hilbert series of the both sides at* (2.3) *coincide, in fact, they are given by* $\frac{1}{(1-s^2)^{2n}} \prod_{i=1}^{n} (1 - s^{2i})$.

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