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C^∞-REGULARITY OF INTERFACE OF SOME ONE-DIMENSIONAL NONLINEAR DEGENERATE PARABOLIC EQUATIONS

YOUNGSANG KO

ABSTRACT. In this article, we survey the regularity of the interfaces of some nonlinear degenerate parabolic equations.

1. Introduction

We consider the Cauchy problem of the form

$$u_t = \frac{\partial}{\partial x}\left(\frac{\partial u^m}{\partial x}\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\right) \quad \text{in} \quad S = \{(x,t) \in \mathcal{R} \times \mathcal{R}^+\} \tag{1.1}$$

where $m > 0$, $p > 1 + \frac{1}{m}$.

Equations like (1.1) were studied by many authors and arise in different physical situations, for the detail see [6]. An important quantity of the study of equation (1.1) is the local velocity of propagation $V = -v_x|v_x|^{p-2}$, whose expression in terms of u can be obtained by writing the equation as a conservation law in the form

$$u_t + (uV)_x = 0.$$

In this way we get

$$V = -v_x|v_x|^{p-2},$$

where the nonlinear potential $v(x,t)$ is

$$v = \frac{m(p-1)}{m(p-1)-1} u^{m-\frac{1}{p-1}} \tag{1.2}$$

and by direct computation v satisfies

$$v_t = (m(p-1)-1)v|v_x|^{p-2}v_{xx} + |v_x|^p. \tag{1.3}$$

In [6], it was shown that V satisfies

$$V_x \le \frac{1}{(p-1)(m+1)t},$$

which can also be written as

$$(v_x|v_x|^{p-2})_x \ge -\frac{1}{(p-1)(m+1)t} \tag{1.4}$$

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Without loss of generality we may consider the case where u_0 vanishes on $\mathcal{R}^-$ and is a continuous positive function, at least, on an interval $(0, a)$ with $a > 0$. Let

$$P[u] = \{(x,t) \in S : u(x,t) > 0\}$$

be the positivity set of a solution u. Then $P[u]$ is bounded to the left in (x,t)-plane by the left interface curve $x = \zeta(t)$[6], where

$$\zeta(t) = \inf\{x \in \mathcal{R} : u(x,t) > 0\}.$$

Moreover there is a time $t^* \in [0, \infty)$, called the waiting time, such that $\zeta(t) = 0$ for $0 \le t \le t^*$ and $\zeta(t) < 0$ for $t > t^*$. It is shown [6] that t^* is finite(possibly zero) and $\zeta(t)$ is a nonincreasing C^1 function on (t^*, ∞). Actually it is shown that $\zeta'(t) < 0$ for every $t > t^*$, i.e., a moving interface never stop.

For the interface of the porous medium equation

$$\begin{cases} u_t = \triangle(u^m) & \text{in} \quad \mathcal{R}^n \times [0,\infty), \\ u(x,0) = u_0 & \text{on} \quad \mathcal{R}^n \end{cases}$$

much more is known. D. G. Aronson and J. L. Vazquez [2] and independently K. Höllig and H. O. Kreiss [7] showed the interfaces are smooth after the waiting time. S. Angenent [1] showed that the interfaces are real analytic after the waiting time. In dimensions $n \ge 2$, L. A. Caffarelli and N. J. Wolanski [4] showed under some nondegeneracy conditions on the initial data, the interface can be described by a $C^{1,\alpha}$ function when $t > T$, for some $T > 0$. Very recently, P. Daskalopoulos and R. Hamilton [5] showed the interface is smooth when $0 < t < T$, for some $T > 0$.

On the other hand much less is known for the equation (1.1). For dimensions $n \ge 2$, Zhao Junning [11] showed, under some nondegeneracy conditions on the initial data, the interface is Lipschitz continuous and we [8] improved this result, showing that, under the same hypotheses, the interface is a $C^{1,\alpha}$ surface after some time.

In this paper we show the interfaces of the solutions of (1.1) are smooth after the waiting time. In establishing C^∞ regularity of the interfaces, we follow the ideas of Aronson and Vazquez. They showed the C^∞ regulatity by establishing the bounds for $v^{(k)}$ for $k \ge 2$, where $v = \frac{m}{m-1}u^{m-1}$ represents the pressure of the gas flow through a porous medium, while u represents the density.

2. The Upper and Lower Bound for v_{xx} and v_{xxx}

Let $q = (x_0, t_0)$ be a point on the left interface, so that $x_0 = \zeta(t_0)$, $v(x, t_0) = 0$ for all $x \le \zeta(t_0)$, and $v(x, t_0) > 0$ for all sufficiently small $x > \zeta(t_0)$. We assume the left interface is moving at q. Thus $t_0 > t^*$. We shall use the notation

$$R_{\delta,\eta} = R_{\delta,\eta}(t_0) = \{(x,t) \in \mathcal{R}^2 : \zeta(t) < x \le \zeta(t) + \delta, t_0 - \eta \le t \le t_0 + \eta\}.$$

Proposition 2.1. *Let q be the point as above. Then there exist positive constants C, δ and η depending only on p, q, m and u such that*

$$v_{xx} \ge C \quad \textit{in} \quad R_{\delta,\eta/2}.$$

Proof. From (1.4) we have, $v_{xx} \ge -\dfrac{1}{(m+1)(p-1)^2|v_x|^{p-2}t}$. But from Lemma 4.4 in [6] v_x is bounded away and above from zero near the interface where $u(x,t) > 0$. □

By constructing a a barrier for v_{xx} in $R_{\delta,\eta}$ of the form

$$\phi(x,t) \equiv \frac{\alpha}{x-\zeta(t)} + \frac{\beta}{x-\zeta^*(t)},$$

for some nonnegative constants α and β and using standard parabolic theories we have

Proposition 2.2. [9] *Let $q = (x_0, t_0)$ be as before. Then there exist positive constants C_2,δ and η depending only on p, q and u such that*

$$v_{xx} \le C_2 \quad in \quad R_{\delta,\eta/2}.$$

Next, we find the estimates of $v^{(3)} \equiv \left(\frac{\partial}{\partial x}\right)^3 v$. By a direct computation we have,

$$\begin{aligned} L_3(v^{(3)}) \ &= v_t^{(3)} - Mvv_x^{p-2}v_{xx}^{(3)} - (A+B)v_x^{(3)} - Cv^{(3)} - D(v^{(3)})^2 \\ &-Ev_x^{p-3}v_{xx}^3 - M(p-2)(p-3)(p-4)vv_x^{p-5}v_{xx}^4 = 0 \end{aligned} \tag{2.1}$$

where A, B, C, D and E are functions depending on v, v_x, and v_{xx}.

Suppose that $q = (x_0, t_0)$ is a point on the left interface for which equation (2.1) in [9] holds. Fix $\epsilon \in (0, a)$ and take $\delta_0 = \delta_0(\epsilon) > 0$ and $\eta_0 = \eta(\epsilon) \in (0, t_0 - t^*)$ such that $R_0 \equiv R_{\delta_0,\eta_0}(t_0) \subset P[u]$. Then by rescaling and interior estimate we have

Proposition 2.3. [9] *There are constants $K \in \mathcal{R}^+$, $\delta \in (0, \delta_0)$, and $\eta \in (0, \eta_0)$ depending only on m, p, q and C_2 such that*

$$|v^{(3)}(x,t)| \le \frac{K}{x-\zeta(t)} \quad in \quad R_{\delta,\eta}.$$

We now turn to the barrier construction. If $\gamma \in (0, \delta)$ we will use the notation

$$R^\gamma_{\delta,\eta} = R^\gamma_{\delta,\eta}(t_0) \equiv \{(x,t) \in \mathcal{R}^2 : \zeta(t) + \gamma \le x \le \zeta(t) + \delta, t_0 - \eta \le t \le t_0 + \eta\}.$$

Proposition 2.4. *Let R_{δ_1,η_1} be the region constructed in the proof of Proposition 2.2 with*

$$0 < \delta_1 < \frac{(p-1)a^{\frac{1}{p-1}}}{12M(p-2)K}. \tag{2.2}$$

For $(x,t) \in R^\gamma_{\delta_1,\eta_1}$, let

$$\phi_\gamma(x,t) \equiv \frac{\alpha}{x-\zeta(t)-\gamma/3} + \frac{\beta}{x-\zeta^*(t)} \tag{2.3}$$

where ζ^* is given by the equation (2.9) in [9], *and α and β are positive constant less than $K/2$. Then there exist $\delta \in (0, \delta_1)$ and $\eta \in (0, \eta_1)$ depending only on a, m, p and C_2 such that*

$$L_3(\phi_\gamma) \ge 0 \quad in \quad R^\gamma_{\delta,\eta}$$

for all $\gamma \in (0, \delta)$.

Proposition 2.5. *(Barrier Transformation). Let δ and η be as in Proposition 2.4 with the additional restriction that*

$$\eta < \frac{\delta}{6\epsilon}, \tag{2.4}$$

where ϵ is as in Proposition 2.4. Suppose that for some nonnegative constants α and β

$$v^{(3)}(x,t) \le \frac{\alpha}{x-\zeta(t)} + \frac{\beta}{x-\zeta^*(t)} \quad in \quad R_{\delta,\eta}. \tag{2.5}$$

Then $v^{(3)}$ also satisfies

$$v^{(3)}(x,t) \le \frac{2\alpha/3}{x-\zeta(t)} + \frac{\beta+2\alpha/3}{x-\zeta^*(t)} \quad in \quad R_{\delta,\eta}. \tag{2.6}$$

Proof. By Remark 3.1 in [9], for any $\gamma \in (0,\delta)$ since $\beta + 2\alpha/3 \le K$ the function

$$\phi_3(x,t) = \frac{2\alpha/3}{x-\zeta-\gamma/3} + \frac{\beta+2\alpha/3}{x-\zeta^*}$$

satisfies $L_3(\phi_3) \ge 0$ in $R^{\gamma}_{\delta,\eta}$. On the other hand, on the parabolic boundary of $R^{\gamma}_{\delta,\eta}$ we have $\phi_3 \ge v^{(3)}$. In fact, for $t = t_1$ and $\zeta_1 + \gamma \le x \le \zeta_1 + \delta$, with $\zeta_1 = \zeta(t_1)$, we have

$$\phi_3(x,t_1) = \frac{2\alpha}{x-\zeta_1-\gamma/3} + \frac{\beta+2\alpha/3}{x-\zeta_1} > \frac{4\alpha/3}{x-\zeta_1} + \frac{\beta}{x-\zeta_1} > v^{(3)}(x,t_1)$$

while for $x = \zeta + \delta$ and $t_1 \le t \le t_2$ we get, in view of (2.4),

$$\begin{aligned} \phi_3(\zeta+\delta,t) \quad &\ge \quad \frac{2\alpha/3}{\delta-\gamma/3} + \frac{\beta}{\zeta+\delta-\zeta^*} + \frac{2\alpha/3}{\delta+6\epsilon\eta} \\ &\ge \quad \frac{2\alpha/3}{\delta} + \frac{\delta}{\zeta+\delta-\zeta^*} + \frac{\alpha/3}{\delta} \ge v^{(3)}(\zeta+\delta,t). \end{aligned}$$

Finally, for $x = \zeta + \gamma$, $t_1 \le t \le t_2$ we have

$$\phi_3(\zeta+\delta,t) = \frac{2\alpha/3}{\gamma-\gamma/3} + \frac{\beta+2\alpha/3}{\zeta+\gamma-\zeta^*} \ge \frac{\alpha}{\gamma} + \frac{\beta}{\zeta+\gamma-\zeta^*} \ge v^{(3)}(\zeta+\gamma,t).$$

By the comparison principle we get

$$\phi_3 \ge v^{(3)} \quad \text{in} \quad R^{\gamma}_{\delta.\eta}$$

for any $\gamma \in (0,\delta)$, and (2.6) follows by letting $\gamma \downarrow 0$. □

Then we have the following

Proposition 2.6. *Let $q = (x_0, t_0)$ be a point on the interface as before. Then there exist constants C_3, δ and η depending only on p, q and u such that*

$$\left| \left(\frac{\partial}{\partial x}\right)^3 v \right| \le C_3 \quad in \quad R_{\delta,\eta/2}.$$

Proof. By Proposition 2.3, we have an estimate for $v^{(3)}$ of the form (2.5) with $\alpha = K$ and $\beta = 0$. Iterating this estimate by Proposition 2.5 we obtain the sequence of estimates

$$v^{(3)}(x,t) \le \frac{\alpha_n}{x-\zeta} + \frac{\beta_n}{x-\zeta^*(t)}$$

with $\alpha_n = (2/3)^n K$ and $\beta_n = \sum_{i=1}^{n} (2/3)^i K$. Then if we let $nrightarrow\infty$ we obtain an upper bound for $v^{(3)}$ of the form

$$v^{(3)}(x,t) \le \frac{2K}{x-\zeta^*(t)} \qquad \text{in} R_{\delta,\eta}.$$

As in the proof of Proposition 2.2, this implies $v^{(3)}$ is bounded above in $R_{\delta,\eta/2}$. Similary, we obtain a lower bound for $v^{(3)}$ and Proposition 2.6 is peoved.

3. Main Result

In this section we prove the interface is a C^∞ function in (t^*, ∞). First we find the estimates of the derivatives of the form

$$v^{(j)} \equiv \left(\frac{\partial}{\partial x}\right)^j v$$

for $j \geq 4$. For the porous medium equation, we have [2] the following equation:

$$\begin{aligned} L_j v^{(j)} \quad &\equiv \quad v_t^{(j)} - (m-1)vv_{xx}^{(j)} - (2 + j(m-1))v_x v_x^{(j)} - c_{mj} v_{xx} v^{(j)} \\ &\quad - \sum_{l=3}^{j^*} d_{mj}^l v^{(l)} v^{(j+2-l)} = 0 \end{aligned}$$

for $j \geq 3$ in $P[u]$, where $j^* = [j/2] + 1$, and the c_{mj} and d_{mj}^l are constants which depend only on their indices, but whose precise values are irrelevant. Note that L_j is linear in $v^{(j)}$. On the other hand for our equation by a direct computation, we have the following equation for $j \geq 4$,

$$\begin{aligned} L_j v^{(j)} \quad &= \quad v_t^{(j)} - Mvv_x^{p-2} v_{xx}^{(j)} - ((j-2)A + B)v_x^{(j)} - C_{pj} v^{(j)} \\ &\quad - F(v, v_x, \ldots, v^{(j-1)}) = 0 \end{aligned} \tag{3.1}$$

where A, B and M are as before, and C_{pj} involves only v and derivatives of order $< j$. Note that equation (3.1) is linear in $v^{(j)}$. Hence our result is

Theorem 3.1. *Let $q = (x_0, t_0)$ be a point on the interface as in Proposition 2.1. For each integer $j \geq 2$ there exist constants C_j, δ and η depending only on j, m, p, q and u such that*

$$\left| \left(\frac{\partial}{\partial x}\right)^j v \right| \leq C_j \quad in \quad R_{\delta, \eta/2}.$$

The proof also proceeds by induction on j. Suppose that $q = (x_0, t_0)$ is a point on the left interface for which equation (2.1) in [9] holds. Fix $\epsilon \in (0, a)$ and take $\delta_0 = \delta_0(\epsilon) > 0$ and $\eta_0 = \eta(\epsilon) \in (0, t_0 - t^*)$ such that $R_0 \equiv R_{\delta_0, \eta_0}(t_0) \subset P[u]$. Assume that there are constants $C_k \in \mathcal{R}^+$ for $k = 3, \ldots, j-1$ such that

$$|v^{(k)}| \leq C_k \quad \text{on} \quad R_0 \quad for \quad k = 2, \ldots, j-1. \tag{3.2}$$

Observe that by Propositions 2.1, 2.2 and 2.6, (3.2) holds for $k = 2$ and $k = 3$. Then by using the ideas in proving propsitions 2.3, 2.4, 2.5 and 2.6, we can prove Theorem 3.1.

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Department of Mathematics, Kyonggi University, Suwon, Kyonggi-do, Korea
E-mail address: ysgo@kuic.kyonggi.ac.kr