Trends in Mathematics - New Series
Information Center for Mathematical Sciences
Volume 10, Number 2, 2008, pages 71–78

2008 International Workshop
on Dynamical Systems and
Related Topics

C^1-STABLY WEAK SHADOWING CHAIN COMPONENTS ARE PARTIALLY HYPERBOLIC

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Abstract. Let f be a diffeomorphism of a closed C^∞ three-dimensional manifold. In this paper, we introduce the notion of C^1-stably weak shadowing for a closed f-invariant set, and prove that C^1-generically, for an aperiodic chain component C_f of f isolated in the chain recurrent set, if $f_{|C_f}$ is C^1-stably weak shadowing, then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and an open and dense subset $\mathcal{V}$ of $\mathcal{U}(f)$ such that for any $g \in \mathcal{V}$, there is a chain component (of g nearby C_f) which is partially hyperbolic.

1. Introduction.

The weak shadowing property of dynamical systems was introduced in [4], and it was shown that this property is C^0-generic in the space of diffeomorphisms of a closed C^∞ manifold. Of course, every homeomorphism having the shadowing property has the weak shadowing property, but its converse is not true. Indeed, an irrational rotation map on the unit circle has the weak shadowing property but it does not have the shadowing property. The study of the weak shadowing property led to the discovery of its non-trivial relations with some classical properties introduced and studied in the geometric theory of differentiable dynamical systems (for instance, [13] and [15]).

In the shadowing theory, a lot of attention has been paid for the characterization of the systems under the C^1-open condition on various type of the shadowing properties. The C^1-open condition means that the property under consideration is preserved by C^1-small perturbations.

Let M be a closed C^∞ manifold, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. If $\mathcal{P}$ is some property of diffeomorphisms, let us denote by $\mathcal{P}(M)$ the set of diffeomorphisms of M having the property $\mathcal{P}$, and by int$\mathcal{P}(M)$ the C^1-interior of $\mathcal{P}(M)$.

Let $\mathcal{S}$ and $\mathcal{WS}$ be the shadowing property and the weak shadowing property, respectively. The sets int$\mathcal{S}(M)$ and int$\mathcal{WS}(M)$ have been studied intensively. For example, it was shown by the author [16] that any diffeomorphism in int$\mathcal{S}(M)$ satisfies both Axiom A and the strong transversality condition; that is, it is structurally stable. Since structural stability implies shadowing [13] and the set of structurally stable systems is C^1-open, the inverse statement is also true. Thus, the set of structurally stable diffeomorphisms coincides with int$\mathcal{S}(M)$.

The situation becomes more complicated if we pass to the set int$\mathcal{WS}(M)$. Let M^2 be a closed C^∞ surface. It was shown by the author [17, Theorem 1] that if a diffeomorphism is in int$\mathcal{WS}(M^2)$, then it satisfies both Axiom A and the no-cycle condition; that is, it is Ω-stable. But the inverse statement does not hold: the Plamenevskaya example [15] shows that there exist Ω-stable diffeomorphisms of the two-dimensional torus $\mathbb{T}^2$ without the weak shadowing property.

We say that $f \in \mathrm{Diff}(M)$ is *topologically transitive* if there is a dense orbit, and the property is denoted by $\mathcal{T}$. Clearly, $\mathcal{T} \subset \mathcal{WS}$. Observe that the above formulated statement does not hold in higher dimensions. In fact, Mañé [11] constructed a nonempty open subset Θ of $\mathrm{Diff}(\mathbb{T}^3)$ such that $\Theta \subset \mathcal{T}(\mathbb{T}^3)$ (and hence $\Theta \subset \mathrm{int}\mathcal{WS}(\mathbb{T}^3)$) but not Anosov so that diffeomorphisms in Θ are not Ω-stable (every element of Θ has periodic points with different indices). Here $\mathbb{T}^3$ is the 3-dimensional torus.

In [17, Theorem 2] and [18], by using the technique developed in [15] the author characterized the subset of diffeomorphisms in int$\mathcal{WS}(M^2)$ whose all of attractors and repellers are trivial (that is, they are periodic orbits) as the set of diffeomorphisms satisfying both Axiom A and the strong transversality condition. Finally, it was shown in [14] that if the condition of triviality of attractors or repellers assumed above is violated, then there exist diffeomorphisms in int$\mathcal{WS}(M^2)$ possessing various types of non-transversality of stable and unstable manifolds. Therefore, we have completed the study of int$\mathcal{WS}(M^2)$.

In this paper, we study the dynamics of diffeomorphisms of a closed C^∞ three-dimensional manifold having the weak shadowing property under the C^1-open condition. More precisely, we introduce the notion of C^1-stable weak shadowing property for f-invariant compact sets, and by applying the techniques developed in [5] we prove that C^1-generically, for an aperiodic chain component C_f of f isolated in the chain recurrent set, if $f_{|C_f}$ is C^1-stably weak shadowing, then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and an open and dense subset $\mathcal{V}$ of $\mathcal{U}(f)$ such that for any

$g \in \mathcal{V}$, there is a chain component (of g nearby C_f) which is partially hyperbolic. Furthermore, we show analogous but slightly stronger results than above also hold for the surface diffeomorphisms.

2. Statement of Results.

Let M be as before, and denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^b \subset M$ $(-\infty \le a < b \le \infty)$ is called a *δ-pseudo-orbit* of $f \in \mathrm{Diff}(M)$ if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$.

Denote by $f_{|A}$ the restriction of f to a subset A of M. Let $\Lambda \subset M$ be a closed f-invariant set. We say that $f_{|\Lambda}$ has the *weak shadowing property* if for every $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=a}^b \subset \Lambda$ of f $(-\infty \le a < b \le \infty)$, there is $y \in M$ satisfying $\{x_i\}_{i=a}^b \subset B_\epsilon(\{f^i(y)\}_{i=a}^{b-1})$ (see [4]). Here $B_\epsilon(A) = \{x \in M : d(x, A) < \epsilon\}$. Notice that only δ-pseudo-orbits of f contained in Λ can be weakly ϵ-shadowed, but shadowing point $y \in M$ is not necessarily contained in Λ. We say that f has the *weak shadowing property* if $M = \Lambda$ in the above definition. This property does not depend on the metric used, and it is easy to see that $f_{|\Lambda}$ has the weak shadowing property if and only if $f^n_{|\Lambda}$ has the weak shadowing property for every $n \in \mathbb{Z}$.

Let $P(f)$ be the set of periodic points, and denote by $\pi(p) > 0$ the minimum period of $p \in P(f)$. Let $p \in P(f)$ be a hyperbolic saddle. Then the stable manifold $W^s(p)$ and the unstable manifold $W^u(p)$ of p are defined as usual. Denote by $H(p)$ the closure of the set of all transverse homoclinic points $x \in W^s(p) \cap W^u(p)$, and set

$$H(\mathcal{O}_f(p)) = H(p) \cup H(f(p)) \cup \cdots \cup H(f^{\pi(p)-1}(p)).$$

This set is called the *homoclinic class* of p. It is known that the set is closed, f-invariant and $f_{|H(\mathcal{O}_f(p))}$ is topologically transitive.

For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo-orbit $\{x_i\}_{i=0}^{b_\delta}$ $(b_\delta > 0)$ of f such that $x_0 = x$ and $x_{b_\delta} = y$. The set of points $\{x \in M : x \rightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{R}(f)$. It is easy to see that the set is closed and $f(\mathcal{R}(f)) = \mathcal{R}(f)$. Clearly, $P(f) \subset \Omega(f) \subset \mathcal{R}(f)$. Here $\Omega(f)$ is the non-wandering set of f.

Write $x \leftrightsquigarrow y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The relation $\leftrightsquigarrow$ induces on $\mathcal{R}(f)$ an equivalence relation, whose classes are called *chain components* (or *chain recurrence classes*) of f (see [3]). For dynamical systems satisfying Axiom A, hyperbolic basic sets are really basic subsystems possessing lots of important dynamical properties and are investigated very well in view of both stability theory and ergodic theory. In view of shadowing theory of dynamical systems, chain components are one of the basic objects in the investigation and are the natural candidates to replace hyperbolic basic sets. Actually, it is known by [1] that, in the C^1-generic context, every chain component with a periodic point is a homoclinic class.

Let C_f be a (non-trivial) chain component of f which is not aperiodic; that is, there is a periodic point in C_f. It is easy to see that there are no sink and source (hyperbolic) periodic points in C_f. We say that the chain component C_f is *isolated* in $\mathcal{R}(f)$ if there is a neighborhood U of C_f such that $\overline{U}\cap\mathcal{R}(f) = C_f$. Observe that if the number of the chain components of $\mathcal{R}(f)$ is finite, then each of the components is isolated in there.

Let U be a compact subset of M, and put

$$\Lambda_f(U) = \bigcap_{n\in\mathbb{Z}} f^n(U).$$

This notation will be used when U is a compact neighborhood of some f-invariant set, so that the set $\Lambda_f(U) \neq \emptyset$ in general.

If $\mathcal{P}$ is some property of diffeomorphisms, let us denote by $\mathcal{P}(U)$ the set of diffeomorphisms such that $f_{|\Lambda_f(U)}$ has the property $\mathcal{P}$. Let us denote by $\mathrm{int}\mathcal{P}(U)$ the C^1-interior of $\mathcal{P}(U)$. Thus, $f \in \mathrm{int}\mathcal{P}(U)$ if and only if there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $g_{|\Lambda_g(U)}$ has the property $\mathcal{P}$. The set $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is called the *continuation* of $\Lambda_f(U)$.

Let Λ be a closed f-invariant set. We say that Λ is *locally maximal in* U if there is a compact neighborhood U of Λ such that $\Lambda = \Lambda_f(U)$. Note that if Λ is locally maximal in U and V $(\subset U)$ is a compact neighborhood of Λ, then $\Lambda = \Lambda_f(U) = \Lambda_f(V)$.

We say that $f_{|\Lambda}$ is C^1*-stably weak shadowing* (*in* U) if there is a compact neighborhood U of Λ such that Λ is locally maximal in U and $f \in \mathrm{int}\mathcal{WS}(U)$.

A splitting $T_\Lambda M = E\oplus F$ defined on Λ is *dominated* if E and F are Df-invariant and there are constants $m > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^m_{|E_x}\| \cdot \|D_{f^m(x)} f^{-m}_{|F_{f^m(x)}}\| < \lambda$$

for all $x \in \Lambda$. A Df-invariant bundle E defined on Λ is *uniformly contracting* (resp. *expanding*) if there are constants $C > 0$ and $0 < \lambda < 1$ such that for every $n > 0$, one has

$$\|D_x f^n(v)\| \le C\lambda^n\|v\| \quad (\text{resp. } \|D_x f^{-n}(v)\| \le C\lambda^n\|v\|)$$

for all $x \in \Lambda$ and $v \in E$.

The set Λ is *hyperbolic* if there is a Df-invariant splitting $T_\Lambda M = E \oplus F$ such that E is uniformly contracting and F is uniformly expanding. The splitting $E\oplus F$ is called *hyperbolic.*

The set Λ is *partially hyperbolic* if there is a dominated splitting $T_\Lambda M = E \oplus F$ such that either E is uniformly contracting or F is uniformly expanding. In the first case we write $T_\Lambda M = E^s \oplus E^{cu}$, otherwise we write $E^u \oplus E^{cs}$. Notice that we can have simultaneously both types of splittings $T_\Lambda M = E^s \oplus E^{cu} = E^u \oplus E^{cs}$. Then, taking $E^c = E^{cu} \cap E^{cs}$, one has a Df-invariant splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$ with three nontrivial directions.

Denote by M^3 a three-dimensional C^∞ closed manifold. The purpose of this paper is to prove the following results.

Theorem A. *There is a residual subset $\mathcal{G} \subset \mathrm{Diff}(M^3)$ such that for $f \in \mathcal{G}$, if C_f is a chain component of f which is not aperiodic and isolated in $\mathcal{R}(f)$, and if $f_{|C_f}$ is C^1-stably weak shadowing in U, then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and an open and dense subset $\mathcal{V} \subset \mathcal{U}(f)$ such that for any $g \in \mathcal{V}$, the continuation $\Lambda_g(U)$ of $\Lambda_f(U)$ is a chain component and partially hyperbolic.*

In the same split, an analogous result was proved by [5] in the context of the so-called robust transitivity. We explain little more about the result. Let Λ be a compact invariant set of $f \in \mathrm{Diff}(M^3)$, and let U be a compact neighborhood of Λ. As before, denote by $\mathcal{T}(U)$ the set of diffeomorphisms g of M^3 such that $g_{|\Lambda_g(U)}$ is topologically transitive. We say that $f_{|\Lambda_f(U)}$ is *robust transitive* if $f \in \mathrm{int}\mathcal{T}(U)$; that is, for every g C^1-nearby f, $g_{|\Lambda_g(U)}$ is topologically transitive. It was proved there that there is an open and dense subset $\mathcal{A}(U) \subset \mathrm{int}\mathcal{T}(U)$ such that for any $g \in \mathcal{A}(U)$, $\Lambda_g(U)$ is partially hyperbolic. Observe that the converse is not true in general. Indeed, the standard so-called *DA*-diffeomorphism φ of $\mathbb{T}^3$ is partially hyperbolic (on the whole space), but φ is not topologically transitive. Theorem A is proved by modifying the techniques developed in [5, Theorem A] (recall that topological transitivity is stronger than the weak shadowing property).

Owing to Hayashi Connecting Lemma ([8]), study of the global dynamics of C^1-generic diffeomorphisms has been rapidly and intensively developed, for instance, see [1] and [2]. We also apply such the latest developments to the proof of Theorem A.

Remark. Let $f \in \mathrm{Diff}(M^3)$, and let Λ be a closed f-invariant set. Even though Λ is partially hyperbolic, $f_{|\Lambda}$ is not necessarily C^1-stably weak shadowing.

Let M be as before. It was known in [2] that there exists a residual set $\mathcal{G} \subset \mathrm{Diff}(M)$ such that for any $g \in \mathcal{G}$, if the number of the chain components are finite, then the chain components coincide with the homoclinic classes. Thus we have the following corollary of Theorem A.

Corollary B. *Let $\mathcal{G}$ be as in Theorem A, and suppose that $f \in \mathcal{G}$. If $f_{|\mathcal{R}(f)}$ is C^1-stably weak shadowing, and the number of the chain components are finite, then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and an open and dense subset $\mathcal{V} \subset \mathcal{U}(f)$ such that for any $g \in \mathcal{V}$, each of the continuations of the chain components of f is a chain component of g and partially hyperbolic.*

Let $f \in \mathrm{Diff}(M)$. We say that a closed f-invariant set Λ is *strong partially hyperbolic* if there are a Df-invariant splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$ (the bundles E^s and E^u are non-trivial and uniformly contracting and expanding, respectively) and constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n(v^s)\| \cdot \|D_{f^n(x)} f^{-n}(v^c)\| \le C\lambda^n \|v^s\| \cdot \|v^c\|,$$

$$\|D_x f^n(v^c)\| \cdot \|D_{f^n(x)} f^{-n}(v^u)\| \le C\lambda^n \|v^c\| \cdot \|v^u\|$$

for all $n \ge 0$, $v^\sigma \in E^\sigma$ $(\sigma = s, c, u)$.

Let U be a closed subset of M^3. We say that $\Lambda_f(U)$ has *robustly real eigenvalues* ([5]) if there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for every $g \in \mathcal{U}(f)$ and every $p \in P(g_{|\Lambda_g(U)})$, all the eigenvalues of $D_p g^{\pi(p)}$ are real. Denote by $\mathcal{E}(U)$ the set of diffeomorphisms f such that the set $\Lambda_f(U)$ has robustly real eigenvalues and has hyperbolic periodic points with different indices (that is, $\Lambda_g(U)$ is "not" uniformly hyperbolic for all g C^1-nearby f).

It was shown in [5, Theorem C] that there is an open and dense subset $\mathcal{A}(U)$ of $\mathrm{int}\mathcal{T}(U)$ such that for any $g \in \mathcal{A}(U) \cap \mathcal{E}(U)$, $\Lambda_g(U)$ is strong partially hyperbolic. By modifying the proof of this result, we have the following.

Theorem C. *Let $\mathcal{G}$ be as in Theorem A, and let $f \in \mathcal{G}$. Suppose that C_f is a chain component of f which is not aperiodic and isolated in $\mathcal{R}(f)$. If $f_{|C_f}$ is C^1-stably weak shadowing in U, then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and an open and dense subset $\mathcal{V} \subset \mathcal{U}(f)$ such that for any $g \in \mathcal{V} \cap \mathcal{E}(U)$, the continuation $\Lambda_g(U)$ of $\Lambda_f(U)$ is a chain component and strong partially hyperbolic.*

Recall Mañé's C^1-open set $\Theta \subset \mathrm{Diff}(\mathbb{T}^3)$. We see that each $f \in \Theta$ satisfies all the assumptions of Theorem C. Note that $C_f = \mathbb{T}^3$.

Let us consider the corresponding results for two-dimensional dynamical systems to the above theorems. More precisely, here we consider the C^1-stable weak shadowing property for two-dimensional semi-local systems, namely chain recurrent sets and chain components of diffeomorphisms of a closed C^∞ surface M^2 (as we stated before, we completed the study on $\mathrm{int}\mathcal{WS}(M^2)$ where we assumed the weak shadowing property in the whole space).

Let $f \in \mathrm{Diff}(M^2)$, and let Λ be a closed f-invariant set. If $f_{|\Lambda}$ is C^1-stably weak shadowing, then, it is easy to see that there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, every $p \in P(g_{|\Lambda_g(U)})$ is hyperbolic. Thus, the reasoning used in the proof of [10, Theorem 1.2] shows the next theorem asserting a stronger result than Corollary B.

Theorem D. *Let $f \in \mathrm{Diff}(M^2)$. Then, $f_{|\mathcal{R}(f)}$ is C^1-stably weak shadowing if and only if f satisfies both Axiom A and the no-cycle condition.*

Indeed, since the correspondence $f \mapsto \mathcal{R}(f)$ is upper semi-continuous, there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $g_{|\mathcal{R}(g)}$ has the weak shadowing property. Thus, as in the proof of [17, Theorem 1] we see that $f \in \mathcal{F}(M^2)$ so that f satisfies both Axiom A and the no-cycle condition, and hence, $\mathcal{R}(f) = \Omega(f)$ is hyperbolic (since there are no-cycles). Here $\mathcal{F}(M)$ is the C^1-interior of diffeomorphisms of M whose any periodic points are hyperbolic (see [7]).

Moreover, since $\dim M^2 = 2$ we see that the indices of periodic points is constant in a homoclinic class. Thus, by applying the proof of [12, Theorem B] we have the following (see the proof of [10, Theorem 1.3]).

Theorem E. *Let $f \in \mathrm{Diff}(M^2)$, and let $H(\mathcal{O}_f(p))$ be the homoclinic class of a hyperbolic periodic point p. Then, $f_{|H(\mathcal{O}_f(p))}$ is C^1-stably weak shadowing if and only if $H(\mathcal{O}_f(p))$ is hyperbolic.*

It was known in [1] and [2, Theorem 10.15] that C^1-generically, every chain component is a homoclinic class (this result does not depend on the dimension of the manifold). Thus we have the following corollary of Theorem E asserting a stronger result than Theorem A.

Corollary F. *There exist a residual set $\mathcal{G} \subset \mathrm{Diff}(M^2)$ such that for any $f \in \mathcal{G}$ and any chain component C_f of f which is not aperiodic, $f_{|C_f}$ is C^1-stably weak shadowing if and only if C_f is a hyperbolic homoclinic class.*

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