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MULTIPLICITY OF SOLUTIONS VIA FORCING TERMS IN ONE DIMENSIONAL SUSPENSION BRIDGE EQUATIONS

Q-HEUNG CHOI AND TACKSUN JUNG

ABSTRACT. First we reveal multiplicity of solutions via source terms in the nonlinear suspension bridge equation. We also investigate the number of solutions of the nonlinear suspension bridge equation when the nonlinearity crosses k eigenvalues. We show by critical point theory that the equation has at least k solutions.

0. INTRODUCTION

We investigate multiplicity of solutions via source terms in a nonlinear suspension bridge equation in an interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, with nonconstant load

$$u_{tt} + u_{xxxx} + bu^+ = f(x) \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \tag{0.1}$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \tag{0.2}$$

$$u \textit{ is } \pi - \textit{periodic in } t \textit{ and even in } x \textit{ and } t, \tag{0.3}$$

where the nonlinearity $-(bu^+)$ crosses an eigenvalue λ_{10}.

Let L be the differential operator, $Lu = u_{tt} + u_{xxxx}$. Then the eigenvalue problem for $u(x,t)$

$$Lu = \lambda u \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times R$$

with (0.2) and (0.3), has infinitely many eigenvalues $\lambda_{mn} = (2n+1)^4 - 4m^2$ $(m, n = 0, 1, 2, \cdots)$ and corresponding eigenfunctions $\phi_{mn}(m, n \geq 0)$ given by $\phi_{mn} = \cos 2mt \cos(2n+1)x$.

Let Q be the square $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and H the Hilbert space defined by

$$H = \left\{u \in L^2(Q) : \quad u \text{ is even in } x \text{ and } t \quad \right\}.$$

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Then the set of eigenfunctions $\{\phi_{mn}\}$ is an orthonormal base in H. Hence equation (0.1) with (0.2) and (0.3) is equivalent to

$$Lu + bu^+ = f \quad \text{in} \quad H. \tag{0.4}$$

In [9], the authors showed by degree theory that equation (0.4) with constant load $1+\epsilon h$ (h is bounded) has at least two solutions. In [5], the authors showed by a variational reduction method that equation (0.4) with constant load $1+\epsilon h$ (h is bounded) has at least three solutions when condition (0.3) is replaced by

$$u \ is \ \pi-periodic \ in \ t \ and \ even \ in \ x. \tag{0.3$'$}$$

Let V be the two dimensional subspace space of H spanned by ϕ_{00} and ϕ_{10}. Let $\Phi : V \to V$ be a map (cf. equation (1.5)) defined by

$$\Phi(v) = Lv + P(b(v+\theta(v))^+), \quad v \in V.$$

In section 1, we investigate the properties of the map Φ (cf. Lemma 1.3, Theorem 1.2). In section 2, we show by critical points theory that equation (0.4) with $f = s\phi_{00}(s>0)$ has a positive solution and at least two change sign solutions (cf. Theorem 2.2). In section 3, we reveal a relation between multiplicity of solutions and source terms in equation (0.4) when f belongs to the two dimensional space V. In section 4, we investigate the number of solutions of the nonlinear suspension bridge equation with Dirichlet boundary condition when the nonlinearity crosses k eigenvalues. We show by critical point theory that the equation has at least k solutions.

1. A Variational Reduction

In this section, we suppose $3 < b < 15$. We have a concern with the multiplicity of solutions of a nonlinear suspension bridge equation

$$Lu + bu^+ = f \quad \text{in} \quad H. \tag{1.1}$$

Here we suppose that f is generated by two eigenfunctions ϕ_{00} and ϕ_{10}.

Let V be the two dimensional subspace of H spanned by $\{\phi_{00}, \phi_{10}\}$ and W be the orthogonal complement of V in H. Let P be an orthogonal projection H onto V. Then every element $u \in H$ is expressed by

$$u = v + w,$$

where $v = Pu$, $w = (I-P)u$. Hence equation (1.1) is equivalent to a system

$$Lw + (I-P)(b(v+w)^+) = 0, \tag{1.2}$$

$$Lv + P(b(v+w)^+) = s_1\phi_{00} + s_2\phi_{10}. \tag{1.3}$$

Here we look on (1.2) and (1.3) as a system of two equations in the two unknowns v and w.

Lemma 1.1. *For fixed* $v \in V$, (1.2) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous (with respect to the L^2 norm) in terms of v.

By Lemma 1.1, the study of the multiplicity of solutions of (1.1) is reduced to the study of the multiplicity of solutions of an equivalent problem

$$Lv + P(b(v+\theta(v))^+ = s_1\phi_{00} + s_2\phi_{10} \tag{1.4}$$

defined on the two dimensional subspace V spanned by $\{\phi_{00}, \phi_{10}\}$.

Since the subspace V is spanned by $\{\phi_{00}, \phi_{10}\}$, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \geq 0, |c_2| \leq c_1\}$$

so that $v \geq 0$ for all $v \in C_1$ and a cone C_3 defined by

$$C_3 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \leq 0, |c_2| \leq |c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$.

Now, we define a map $\Phi : V \to V$ given by

$$\Phi(v) = Lv + P(b(v+\theta(v))^+), \quad v \in V. \tag{1.5}$$

Then Φ is continuous on V, since θ is continuous on V and we have the following lemma.

LEMMA 1.2. *$\Phi(cv) = c\Phi(v)$ for $c \geq 0$.*

The map Φ maps C_1 onto the cone

$$R_1 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, |d_2| \leq \left(\frac{b+\lambda_{10}}{b+\lambda_{00}}\right) d_1 \right\}$$

and the cone C_3 onto the cone

$$R_3 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \leq 0,\ d_2 \leq |\frac{\lambda_{10}}{\lambda_{00}}||d_1| \right\}.$$

THEOREM 1.1. *(i) If f belongs to R_1, then equation (1.1) has a positive solution and no negative solution. (ii) If f belongs to R_3, then equation (1.1) has a negative solution.*

Now we set

$$C_2 = \{c_1\phi_{00} + c_2\phi_{10} \mid c_2 \geq 0,\ c_2 \geq |c_1|\},$$
$$C_4 = \{c_1\phi_{00} + c_2\phi_{10} \mid c_2 \leq 0,\ c_2 \leq -|c_1|\}.$$

Then the union of C_1, C_2, C_3, C_4 is the space V.

Lemma 1.2 means that the images $\Phi(C_2)$ and $\Phi(C_4)$ are the cones in the plane V. Before we investigate the images $\Phi(C_2)$ and $\Phi(C_4)$, we set

$$R_2' = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_2 \geq 0, -|\frac{\lambda_{00}}{\lambda_{10}}|d_2 \leq d_1 \leq \left|\frac{b+\lambda_{00}}{b+\lambda_{10}}\right| d_2 \right\},$$
$$R_4' = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_2 \leq 0, |\frac{\lambda_{00}}{\lambda_{10}}|d_2 \leq d_1 \leq \left(\frac{b+\lambda_{00}}{b+\lambda_{10}}\right) d_2 \right\}.$$

Then the union of R_1, R_2', R_3, R_4' is the plane V.

THEOREM 1.2. *For $i = 2, 4$, if we let $\Phi_i(C_i) = R_i$, then R_2 is one of sets $R_1 \cup R_4'$, $R_2' \cup R_3$ and R_4 is one of sets $R_3 \cup R_4'$, $R_1 \cup R_2'$. Furthermore, for each $1 \le i \le 4$, the restriction Φ_i maps C_i onto R_i. In particular, Φ_1 and Φ_3 are bijective.*

2. Positive Load

We investigate the multiplicity of solutions of a nonlinear suspension bridge equation

$$Lu + bu^+ = s\phi_{00} \quad \text{in} \quad H, \tag{2.1}$$

where we $3 < b < 15$ and s is real.

Now we define a subspace H_0 of the Hilbert space H as follows

$$H_0 = \left\{ u \in H : u = \sum h_{mn}\phi_{mn},\ \sum |\lambda_{mn}| h_{mn}^2 < \infty \right\}$$

with norm $|\|u\|| = \left[\sum |\lambda_{mn}| h_{mn}^2\right]^{\frac{1}{2}}$. Then this normed space is complete.

We investigate the existence of solutions of (2.1) in the subspace H_0 of H

$$Lu + bu^+ = s\phi_{00} \ in \ H_0. \tag{2.2}$$

Now we define the functional on H_0

$$I_b(u) = \int_Q [\frac{1}{2}(-|u_t|^2 + |u_{xx}|^2) + \frac{b}{2}|u^+|^2 - s\phi_{00}u] dtdx. \tag{2.3}$$

Then the functional I_b is continuous in H_0 and Fréchet differentiable in H_0. The solutions of (2.2) coincide with the critical points of I_b.

THEOREM 2.1. *Let $3 < b < 15$ and $s > 0$. Then $I_b(v)$ has a critical point in* Int C_1 *and at least one critical point in* Int $C_2 \cap C_4$. *Therefore equation (2.1) has a positive solution and at least one sign changing solution.*

3. Multiplicity of Solutions via Source Terms

We reveal the relation between multiplicity of solutions and source terms in the nonlinear suspension bridge equation (1.1).

For $f \in V$, we establish an *a priori* bound for solutions of

$$Lv + P(b(v + \theta(v))^+) = f \quad \text{in} \quad V. \tag{3.1}$$

LEMMA 3.1. *Let $-1 < b < 15$ and $k(\ge b+1)$ be fixed. Let $f \in V$ with $\|f\| = k$ and $\alpha > 0$ be given. Then there exists $R_0 > 0$ (depending only on k and α) such that for all b with $-1 + \alpha \le b \le 15 - \alpha$ the solutions of (3.1) satisfy $\|v\| < R_0$.*

LEMMA 3.2. *Let $-1 < b < 15$ and $k(\geq b+1)$ be fixed. Let $f \in V$ with $\|f\| = k$. Then we have*

$$d(v - L^{-1}(f - P(b(v + \theta(v))^+)), B_R, 0) = 1$$

for all $R \geq R_0$.

LEMMA 3.3. *Let $3 < b < 15$ and $f = (b+1)\phi_{00}$. Then equation (3.1) has a positive solution in* $\mathrm{Int}C_1$, *at least one sign changing solution in* $\mathrm{Int}C_2$, *and at least one sign changing solution in* $\mathrm{Int}C_4$.

LEMMA 3.4. *Let $3 < b < 15$. For $1 \leq i \leq 4$, let $\Phi(C_i) = R_i$. Then $R_2 = R_1 \cup R_4'$ and $R_4 = R_1 \cup R_2'$, where R_2', R_4' are the same cones as in section 1.*

THEOREM 3.1. *Let $3 < b < 15$. Then we have the following.*
(i) If $f \in \mathrm{Int}\ R_1$, then equation (1.1) has a positive solution and at least two sign changing solutions.
(ii) If $f \in \partial R_1$, then equation (1.1) has a positive solution and at least one sign changing solution.
(iii) If $f \in \mathrm{Int}\ R_i'(i = 2, 4)$, then equation (1.1) has at least one sign changing solution.
(iv) If $f \in \mathrm{Int}\ R_3$, then equation (1.1) has only the negative solution.
(v) If $f \in \partial R_3$, then equation (1.1) has a negative solution.

4. Linking Theorem and its Application

Let X be a real Hilbert space on which the compact Lie group S^1 acts by means of time translations, i.e., for $u \in X$ and $\theta \in [0, \pi]$, set:

$$s_\theta u(x,t) = u(x, t+\theta).$$

Let $Fix(S^1)$ be the set of fixed points of the action, i.e.,

$$\begin{aligned} Fix(S^1) &= \{u \in X |\ s_\theta u = u, \qquad \forall \theta \in [0,\pi]\} \\ &= \{u \in X |\ u \text{ is independent of } t\}. \end{aligned}$$

We call a subset B of X an invariant set if for all $u \in B$, $s_\theta u \in B$ for all $\theta \in [0, \pi]$. A function $f : X \to R^1$ is called S^1-invariant, if $f(s_\theta u) = f(u)$, $\forall u \in X$, for all $\theta \in [0, \pi]$. Let $C(B, X)$ be the set of continuous functions from B into X. If B is an invariant set we say $h \in C(B, X)$ is an equivariant map if $h(s_\theta u) = s_\theta h(u)$ for all $\theta \in [0, \pi]$ and $u \in B$. Let S_r be the sphere centered at the origin of radius r. Let $f : X \to R$ be a functional of the form

$$f(x) = \frac{1}{2}(Lx|x) - \psi(x),$$

where $L : X \to X$ is linear, continuous, symmetric and equivariant, $\psi : X \to R$ is of class C^1 and invariant and $D\psi : X \to X$ is compact. The following result follows from [4].

THEOREM 4.1. *Assume that $f \in C^1(X, R^1)$ is S^1-invariant and there exist two closed invariant linear subspaces V, W of X and $r > 0$ with the following properties:*
(a) $V + W$ is closed and of finite codimension in X;
(b) $Fix(S^1) \subseteq V + W$;
(c) $L(W) \subseteq W$;
(d) $\sup_{S_r \cap V} f < +\infty$ and $\inf_W f > -\infty$;
(e) $u \notin Fix(S^1)$ whenever $Df(u) = 0$ and

$$\inf_W f \le f(u) \le \sup_{S_r \cap V} f;$$

(f) f satisfies $(PS)_c$ whenever $\inf_W f \le c \le \sup_{S_r \cap V} f$.
Then f possesses at least

$$\frac{1}{2}(dim(V \cap W) - codim_X(V + W))$$

distinct critical orbits in $f^{-1}([\inf_W f, \sup_{S_r \cap V} f])$.

Our aim is to prove the multiplicity rsult for solutions of (0.1) with (0.2) and (0.3′) by Theorem 4.1. Let $\{\mu_k^+ | k \in N\}$ and $\{\mu_k^- | k \in N\}$ the sequence of positive and negative eigenvalues in $\{\mu_k | k \in Z\}$ respectively. That is,

$$\ldots < \mu_3^- < \mu_2^- < \mu_1^- < 0 < \mu_1^+ < \mu_2^+ < \mu_3^+ < \ldots. \tag{4.1}$$

Let X and $|||\cdot|||$ be the space and norm in X introduced in section 3. Now we state our main results:

THEOREM 4.2. *Let $h \in X$, $|||h||| = 1$. Let $-\mu_k^- < b < -\mu_{k+1}^-$, $k \ge 1$. Let $f = \epsilon h(x,t)$. Then, for small $\epsilon > 0$, problem (0.1) with (0.2) and (0.3′) has at least k solutions.*

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Q-Heung Choi :Department of Mathematics,Inha University, Incheon 402-751, Korea

E-mail address: `qhchoi@math.inha.ac.kr`

Tacksun Jung :Department of Mathematics, Kunsan N. University, Kunsan 573-360, Korea

E-mail address: `tsjung@kunsan.ac.kr`