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MEASURABLE AND TOPOLOGICAL ENTROPY DIMENSIONS

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ABSTRACT. We state a couple of theorems on topological or metric entropy dimension which was introduced to measure the complexity of entropy zero systems. Entropy dimension measures the superpolynomial, but subexponential growth rate of orbits. It is the notion to classify the complexity of entropy zero systems.

1. INTRODUCTION

Entropy has played an important role since it has been introduced by Kolmogorov from information sciences. It measures the randomness or sensitivity to the initial conditions. That is, minor differences in the initial conditions will lead to dramatic differences in the long term behavior. In the case of differentiable maps, it is associated with positive Lyapunov exponents.

Metric entropy as an isomorphism invariant was introduced to classify the Bernoulli automorphisms. The notion of entropy in ergodic theory, information theory, statistical mechanics as well as thermodynamics share the same meaning. Topological entropy is defined also to measure the orbit growth rate of a system. Recently Ornstein and Weiss [11] show that any finitely observable isomorphism invariant for an ergodic system is a function of the entropy.

In topological setting, entropy computed by the number of open sets of the iterated cover to cover most of the space can be regarded as a measurement of independence [9].

We would like to investigate the "randomness" or the "complexity" of entropy zero systems. They are called deterministic systems in the sense that the past determines the future. The following is the list of the reasons to consider these systems.
1. Although entropy zero systems make up a dense G_δ subset of all homeomorphisms, not much study has been done on the complexity of the systems. In particular, we need to develop the properties to distinguish entropy zero systems.
2. Many interesting bigger group actions, like Z^n-actions have entropy zero and their noncocompact subgroups do exhibit complexity and they have sometimes positive entropy.[12] [10] [1]
3. There are various orbit growth rates, ranging from polynomial to various degrees of subexponential growth rate. [6]
4. There are many physical models showing intermediate chaotic behavior.[13, 14]

2. DEFINITIONS AND RESULTS

To capture the "independence" or the "randomness" of the entropy zero systems, we introduce the notion of entropy dimension in topological systems. ([5] and [4]).

Let (X, T) be a topological dynamical systems(TDS) and $\mathcal{U}$ be a finite open cover of X. For $\alpha \geq 0$, we define

$$\overline{D}(T, \alpha, \mathcal{U}) = \limsup_{n\to\infty} \frac{\log \mathcal{N}(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U})}{n^\alpha} \text{ and } \underline{D}(T, \alpha, \mathcal{U}) = \liminf_{n\to\infty} \frac{\log \mathcal{N}(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U})}{n^\alpha}.$$

It is clear that $\overline{D}(T, \alpha, \mathcal{U}) \leq \overline{D}(T, \alpha', \mathcal{U})$ if $\alpha \geq \alpha' \geq 0$ and $\overline{D}(T, \alpha, \mathcal{U}) \notin \{0, +\infty\}$ for at most one $\alpha \geq 0$. We define *the upper entropy dimension of* $\mathcal{U}$ by

$$\overline{D}(T, \mathcal{U}) = \inf\{\alpha \geq 0 : \overline{D}(T, \alpha, \mathcal{U}) = 0\} = \sup\{\alpha \geq 0 : \overline{D}(T, \alpha, \mathcal{U}) = \infty\}.$$

Similarly, $\underline{D}(T, \alpha, \mathcal{U}) \leq \underline{D}(T, \alpha', \mathcal{U})$ if $\alpha \geq \alpha' \geq 0$ and $\underline{D}(T, \alpha, \mathcal{U}) \notin \{0, +\infty\}$ for at most one $\alpha \geq 0$. We define *the lower entropy dimension of* S by

$$\underline{D}(T, \mathcal{U}) = \inf\{\alpha \geq 0 : \underline{D}(T, \alpha, \mathcal{U}) = 0\} = \sup\{\alpha \geq 0 : \underline{D}(T, \alpha, \mathcal{U}) = \infty\}.$$

If $\overline{D}(T, \mathcal{U}) = \underline{D}(T, \mathcal{U}) = \alpha$, then we say $\mathcal{U}$ has topological entropy dimension α. Clearly $0 \leq \underline{D}(T, \mathcal{U}) \leq \overline{D}(T, \mathcal{U}) \leq 1$ and if $h(T, \mathcal{U}) > 0$, then the entropy dimension of $\mathcal{U}$ is equal to 1.

Metric entropy dimension can be analogously defined with careful modification so that it is isomorphism invariant [8] [5]. Both of them, topological or metric, measure the superpolynomial, but subexponential growth rate of the number of open sets or the partitions under the iterations. Clearly the irrational rotations or the Toeplitz systems of polynomial growth rate have entropy dimension 0.

Theorem 2.1. *For any given $\alpha \geq 0$, there exists a topological*[2] *and a measurable*[5] *system of a given entropy dimension α.*

Let (X,T) be a TDS and $\mathcal{U} \in \mathcal{C}_X^o$. We say an increasing sequence of integers $S = \{s_1 < s_2 < \cdots\}$ is an *entropy generating sequence* of $\mathcal{U}$ [3] if

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathcal{N}(\bigvee_{i=1}^{n} T^{-s_i}\mathcal{U}) > 0.$$

Denote by $\mathcal{E}(T,\mathcal{U})$ the set of all entropy generating sequences of $\mathcal{U}$.

We also introduced the notion of the dimension(upper and lower) of a subset of $\mathbb{Z}$ which may have density 0. Let $S = \{s_1 < s_2 < \cdots\}$ be a sequence of increasing integers. For $\tau \geq 0$, we define

$$\overline{D}(S,\tau) = \limsup_{n\to\infty} \frac{n}{(s_n)^\tau} \text{ and } \underline{D}(S,\tau) = \liminf_{n\to\infty} \frac{n}{(s_n)^\tau}.$$

It is clear that $\overline{D}(S,\tau) \leq \overline{D}(S,\tau')$ if $\tau \geq \tau' \geq 0$ and $\overline{D}(S,\tau) \notin \{0,+\infty\}$ for at most one $\tau \geq 0$. We define *the upper dimension of S* by

$$\overline{D}(S) = \inf\{\tau \geq 0 : \overline{D}(S,\tau) = 0\} = \sup\{\tau \geq 0 : \overline{D}(S,\tau) = \infty\}.$$

Similarly, $\underline{D}(S,\tau) \leq \underline{D}(S,\tau')$ if $\tau \geq \tau' \geq 0$ and $\underline{D}(S,\tau) \notin \{0,+\infty\}$ for at most one $\tau \geq 0$. We define *the lower dimension of S* by

$$\underline{D}(S) = \inf\{\tau \geq 0 : \underline{D}(S,\tau) = 0\} = \sup\{\tau \geq 0 : \underline{D}(S,\tau) = \infty\}.$$

Clearly $0 \leq \underline{D}(S) \leq \overline{D}(S) \leq 1$. When $\overline{D}(S) = \underline{D}(S) = \tau$, we say the sequence S has dimension τ.

Theorem 2.2. *If a system (X,T) has positive topological entropy dimension, then there exists an entropy generating sequence $S \subset \mathbb{Z}_+$ which is a union of disjoint finite sets along which the dynamics are "independent". Moreover, the entropy dimension of the system is the supremum of the dimensions of the entropy generating sequences.*

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