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QUASI-NORM TECHNIQUES FOR FINITE ELEMENT APPROXIMATION OF p-LAPLAICAN

KWANG-YEON KIM

ABSTRACT. The p-Laplacian problem is one of the typical examples of degenerate nonlinear systems arising from nonlinear diffusion and filtration, power-law materials and quasi-Newtonian flows. In this article we give a survey of quasi-norm techniques to establish optimal error estimates for finite element approximation of p-Laplacian.

1. INTRODUCTION

For an open bounded domain Ω in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, the p-Laplacian problem is given by

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f \in (W_0^{1,p}(\Omega))'$ and $g \in W^{1-1/p,p}(\partial\Omega)$ are the given data, and $1 < p < \infty$. In what follows we will use the standard notation $W^{s,p}(\Omega)$ for the Sobolev space on Ω, with its norm and seminorm denoted by $\|\cdot\|_{s,p}$ and $|\cdot|_{s,p}$.

Problem (1.1) is one of the typical examples of degenerate nonlinear systems arising from, for example, nonlinear diffusion and filtration, power-law materials and quasi-Newtonian flows. For $p = 2$, this problem corresponds to the familiar Poisson equation. For $p \neq 2$, the Banach space $W^{1,p}(\Omega)$ is the natural space for the variational formulation of problem (1.1) which reads as follows:

find $u \in W_g^{1,p}(\Omega) := \{v \in W^{1,p}(\Omega) : v|_{\partial\Omega} = g\}$ such that

$$a(u, v) = (f, v) \qquad \forall v \in W_0^{1,p}(\Omega), \tag{1.2}$$

where

$$a(u, v) = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx$$

and (f, v) denotes the duality paring between $W_0^{1,p}(\Omega)$ and its dual space.

The variational formulation (1.2) is equivalent to the minimization problem

$$J(u) \le J(v) \qquad \forall v \in W_g^{1,p}(\Omega), \tag{1.3}$$

where $J : W^{1,p}(\Omega) \to \mathbb{R}$ is the strictly convex functional defined by

$$J(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \, dx - \int_\Omega f v \, dx.$$

More precisely, if $J' : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ is the Gâteaux derivative of J, then we have for all $u, v \in W_0^{1,p}(\Omega)$

$$(J'(u), v) = a(u, v) - (f, v).$$

We refer the interested readers to [3, 4] for the existence and uniqueness of a solution of problem (1.2) or (1.3).

2. Finite Element Approximation

In order to avoid technical difficulties, it is assumed throughout the paper that Ω is a polygonal domain and $g = 0$.

For construction of a finite element approximation based on the weak form (1.2), we partition the domain Ω into triangular elements whose collection is denoted by $\mathcal{T}_h$ (h denotes the maximal element size). As is usually done, $\mathcal{T}_h$ is assumed to be shape-regular in the sense of Ciarlet [4]: there exists a constant $c > 0$ independent of the mesh size satisfying

$$\frac{h_K}{\rho_K} \le c \qquad \forall K \in \mathcal{T}_h,$$

where h_K is the diameter of K, and ρ_K is the radius of the largest ball contained in K.

Due to the limited regularity of the solution u of problem (1.1) (cf. [6]), we restrict ourselves to the continuous piecewise linear element

$$\mathcal{V}_h = \{v \in W_0^{1,p}(\Omega) : v|_K \in \mathbb{P}_1(K) \ \ \forall K \in \mathcal{T}_h\}.$$

Let $\pi_h : W_0^{1,p}(\Omega) \cap C(\overline{\Omega}) \to \mathcal{V}_h$ be the standard Lagrange interpolation, that is,

$$\pi_h v(z) = v(z) \qquad \forall \text{vertex } z \text{ of } \mathcal{T}_h.$$

The following optimal estimates for π_h are well known from the standard approximation theory: for all $1 \le r \le \infty$, $m = 0, 1$, $l = 1, 2$, and $K \in \mathcal{T}_h$,

$$|v - \pi_h v|_{m,r,K} \le C h_K^{l-m} |v|_{l,r,K}. \tag{2.1}$$

Here and in what follows, C (with or without a subscript) will denote a generic positive constant independent of the mesh size h (but dependent on p) which may take different values at different places.

Now the finite element method for problem (1.1) is defined as follows:
find $u_h \in \mathcal{V}_h$ such that

$$a(u_h, v_h) = (f, v_h) \qquad \forall v_h \in \mathcal{V}_h. \tag{2.2}$$

It can be shown in the same way as in the continuous case that problem (2.2) is equivalent to the minimization problem

$$J(u_h) \le J(v_h) \qquad \forall v_h \in \mathcal{V}_h, \tag{2.3}$$

and thus problem (2.2) admits a unique solution which can be computed by, e.g., the Polak-Ribiére conjugate gradient method. Moreover, taking $v_h = u_h$, we can obtain a uniform bound for $\|\nabla u_h\|_{0,p}$.

Theorem 2.1. *Let $u_h \in \mathcal{V}_h$ be a solution of problem* (2.2). *Then we have*

$$\|\nabla u_h\|_{0,p} \le \|f\|_*^{1/(p-1)},$$

where the dual norm $\|f\|_*$ is defined by

$$\|f\|_* = \sup_{v \in W_0^{1,p}(\Omega)} \frac{(f,v)}{\|\nabla v\|_{0,p}}.$$

3. Error Estimates in $W^{1,p}(\Omega)$

As a first step towards error estimation of problem (2.2), we note the following error equation: for all $v_h \in \mathcal{V}_h$,

$$a(u, u-u_h) - a(u_h, u-u_h) = a(u, u-v_h) - a(u_h, u-v_h). \tag{3.1}$$

This is a straightforward consequence of the Galerkin orthogonality relaionship

$$a(u, v_h) - a(u_h, v_h) = 0 \qquad \forall v_h \in \mathcal{V}_h.$$

To proceed further, the early approaches in [3, 4, 5] were based on the fact that the form $a(\cdot,\cdot)$ is strongly monotone and Lipschitz continuous on bounded sets in the conventional norm $\|\cdot\|_{1,p}$. These results are stated in the following propositions.

Proposition 3.1. *For $1 < p < 2$, there exist constants $C_1, C_2 > 0$ such that, for all $u, v \in W^{1,p}(\Omega)$,*

$$a(u, u-v) - a(v, u-v) \ge \frac{\|\nabla(u-v)\|_{0,p}^2}{C_1 + C_2(\|\nabla u\|_{0,p} + \|\nabla v\|_{0,p})^{2-p}}.$$

For $2 < p < \infty$, there exists a constant $C_1 > 0$ such that, for all $u, v \in W^{1,p}(\Omega)$,

$$a(u, u-v) - a(v, u-v) \ge C_1 \|\nabla(u-v)\|_{0,p}^p.$$

Proposition 3.2. *For $1 < p < 2$, there exists a constant $C_1 > 0$ such that, for all $u, v, w \in W^{1,p}(\Omega)$,*

$$a(u, w) - a(v, w) \le C_1 \|\nabla(u-v)\|_{0,p}^{p-1} \|\nabla w\|_{0,p}.$$

For $2 < p < \infty$, there exist constants $C_1, C_2 > 0$ such that, for all $u, v, w \in W^{1,p}(\Omega)$,

$$a(u, w) - a(v, w) \le (C_1 + C_2(\|\nabla u\|_{0,p} + \|\nabla v\|_{0,p})^{p-2}) \|\nabla(u-v)\|_{0,p} \|\nabla w\|_{0,p}.$$

Propositions 3.1–3.2 can be derived from sharper results given in the next section, and so their proofs are omitted.

Combining Theorem 2.1 and Propositions 3.1–3.2 with the error equation (3.1), we can derive the following error estimate for finite element approximation (2.2).

Theorem 3.3. *Let u and u_h be the solutions of problems* (1.2) *and* (2.2), *respectively. Then there exists a constant $C > 0$ such that*

$$\|\nabla(u-u_h)\|_{0,p} \le C \inf_{v_h \in \mathcal{V}_h} \|\nabla(u-v_h)\|_{0,p}^s$$

with

$$s = \begin{cases} 1/(3-p) & \text{if } 1 < p < 2, \\ 1/(p-1) & \text{if } 2 < p < \infty. \end{cases}$$

Therefore, we obtain for $u \in W^{2,p}(\Omega)$

$$\|\nabla(u-u_h)\|_{0,p} \le \begin{cases} Ch^{1/(3-p)} & \text{if } 1<p<2, \\ Ch^{1/(p-1)} & \text{if } 2<p<\infty. \end{cases}$$

Proof. Take $v = u_h$ and $w = u - v_h$ in Propositions 3.1–3.2 to obtain the first result. For the second result, set $v_h = \pi_h u$ and use the estimate (2.1). □

An improved estimate may be obtained by using the minimization property of u and u_h (cf. [3]). From the mean value theorem

$$J(v) - J(u) = \int_0^1 (J'(u + s(v-u), v-u)\, ds$$

and the fact that $(J'(u), v) = 0$ for all $v \in W_0^{1,p}(\Omega)$, it follows that

$$\begin{aligned} J(v) - J(u) &= \int_0^1 (J'(u + s(v-u) - J'(u), v-u)\, ds \\ &= \int_0^1 (a(u+s(v-u), v-u) - a(u, v-u))\, ds \\ &= \int_0^1 \frac{a(u+s(v-u), s(v-u)) - a(u, s(v-u))}{s}\, ds \end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$. By Proposition 3.1 this gives

$$J(v) - J(u) \ge \begin{cases} \dfrac{\|\nabla(u-v)\|_{0,p}^2}{C_1 + C_2(\|\nabla u\|_{0,p} + \|\nabla v\|_{0,p})^{2-p}} & \text{if } 1<p<2, \\ C_1\|\nabla(u-v)\|_{0,p}^p & \text{if } 2<p<\infty. \end{cases}$$

Similarly, we obtain by Proposition 3.2

$$J(v) - J(u) \le \begin{cases} C_1\|\nabla(u-v)\|_{0,p}^p & \text{if } 1<p<2, \\ (C_1 + C_2(\|\nabla u\|_{0,p} + \|\nabla v\|_{0,p})^{p-2})\|\nabla(u-v)\|_{0,p}^2 & \text{if } 2<p<\infty. \end{cases}$$

Since the minimization property implies that

$$J(u_h) - J(u) \le J(v_h) - J(u) \qquad \forall v_h \in \mathcal{V}_h,$$

these two-sided bounds establish that Theorem 3.3 is valid with the improved order of convergence

$$s = \begin{cases} p/2 & \text{if } 1<p<2, \\ 2/p & \text{if } 2<p<\infty. \end{cases}$$

Before closing this section, let us point out that all the estimates obtained above are suboptimal unless $p = 2$.

4. Quasi-norm techniques

Barrett and Liu developed the quasi-norm techniques which take the possible degeneracy of the p-Laplacian into consideration (cf. [1, 2, 7, 8, 9]) and obtained sharper error estimates than those of the previous section.

Let us first introduce the quasi-norm for $v \in W^{1,p}(\Omega)$

$$|v|^2_{(u,p)} = \int_\Omega (|\nabla u| + |\nabla v|)^{p-2} |\nabla v|^2 \, dx$$

where u is the solution of problem (1.2). Although $|\cdot|_{(u,p)}$ is not a norm itself, it possesses many norm-like properties.

Proposition 4.1. *We have for all $v \in W^{1,p}(\Omega)$*

$$|v|_{(u,p)} \geq 0 \quad \textit{with equality if and only if} \quad v = 0.$$

Furthermore, the following triangle inequality holds for all $v, w \in W^{1,p}(\Omega)$

$$|v + w|_{(u,p)} \leq C(|v|_{(u,p)} + |w|_{(u,p)})$$

with $C = \max(2, 2^{p-1})^{1/2}$.

Proof. The first result is trivial.. The second result follows immediately from the inequality $(a \geq 0, 1 < p < \infty, \sigma_1, \sigma_2 \in \mathbb{R}^2)$

$$(a + |\sigma_1 + \sigma_2|)^{p-2} |\sigma_1 + \sigma_2|^2 \leq \max(2, 2^{p-1})[(a + |\sigma_1|)^{p-2} |\sigma_1|^2 + (a + |\sigma_2|)^{p-2} |\sigma_2|^2].$$

This inequality is a consequence of Jensen's inequality applied to the function $\phi(t) = (a + t)^{p-2} t^2$ which is increasing and convex on $t \geq 0$. □

The relationship between the quasi-norm $|\cdot|_{(u,p)}$ and the standard Sobolev norm $\|\cdot\|_{1,p}$ is stated in the following proposition.

Proposition 4.2. *Let $v \in W^{1,p}(\Omega)$. Then we have for $1 < p < 2$*

$$|v|^2_{(u,p)} \leq \|\nabla v\|^p_{0,p} \leq |v|^p_{(u,p)} D(u, v), \tag{4.1}$$

and for $2 < p < \infty$

$$\|\nabla v\|^p_{0,p} \leq |v|^2_{(u,p)} \leq \|\nabla v\|^2_{0,p} D(u, v), \tag{4.2}$$

where we set

$$D(u, v) := \left(\int_\Omega (|\nabla u| + |\nabla v|)^p \, dx \right)^r \leq C(\|\nabla u\|_{0,p} + \|\nabla v\|_{0,p})^{rp}$$

with

$$r = \begin{cases} (2 - p)/2 & \textit{if } 1 < p < 2, \\ (p - 2)/p & \textit{if } 2 < p < \infty. \end{cases}$$

Proof. Let us first deal with the case $1 < p < 2$. It is easy to check the left inequality, as we have

$$|v|^2_{(u,p)} = \int_\Omega (|\nabla u| + |\nabla v|)^{p-2} |\nabla v|^2 \, dx \le \int_\Omega |\nabla v|^p \, dx = \|\nabla v\|^p_{0,p}.$$

To derive the right inequality, we use the Hölder inequality with $\frac{p}{2} + r = 1$ to obtain

$$\begin{aligned}
\|\nabla v\|^p_{0,p} &= \int_\Omega |\nabla v|^p (|\nabla u| + |\nabla v|)^{(p-2)p/2} (|\nabla u| + |\nabla v|)^{-(p-2)p/2} \, dx \\
&\le \left(\int_\Omega (|\nabla u| + |\nabla v|)^{(p-2)} |\nabla v|^2 \, dx \right)^{p/2} \left(\int_\Omega (|\nabla u| + |\nabla v|)^{-(p-2)p/2r} \, dx \right)^r \\
&= |v|^p_{(u,p)} D(u,v).
\end{aligned}$$

The proof for $2 < p < \infty$ can be done in a similar way. For the left inequality we note that

$$|v|^2_{(u,p)} = \int_\Omega (|\nabla u| + |\nabla v|)^{p-2} |\nabla v|^2 \, dx \ge \int_\Omega |\nabla v|^p \, dx = \|\nabla v\|^p_{0,p}.$$

For the right inequality we use the Hölder inequality with $\frac{2}{p} + r = 1$ to obtain

$$\begin{aligned}
|v|^2_{(u,p)} &= \int_\Omega (|\nabla u| + |\nabla v|)^{p-2} |\nabla v|^2 \, dx \\
&\le \left(\int_\Omega (|\nabla u| + |\nabla v|)^{(p-2)/r} \, dx \right)^r \left(\int_\Omega |\nabla v|^p \, dx \right)^{2/p} \\
&= D(u,v) \|\nabla v\|^2_{0,p}.
\end{aligned}$$

This completes the proof. □

In order to derive error estimates in the quasi-norm $|\cdot|_{(u,p)}$, we need some results on upper and lower bounds of $a(\cdot,\cdot)$ with respect to the quasi-norm which are similar to Propositions 3.1–3.2. Before going further, we present two lemmas which will be crucially used for this purpose.

Lemma 4.3. *For all* $1 < p < \infty$ *and* $\xi, \eta \in \mathbb{R}^2$*, we have*

$$| \, |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \, | \le C(|\xi| + |\eta|)^{p-2} |\xi - \eta| \tag{4.3}$$

and

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta)_{\mathbb{R}^2} \ge C(|\xi| + |\eta|)^{p-2} |\xi - \eta|^2, \tag{4.4}$$

where $(\cdot,\cdot)_{\mathbb{R}^2}$ *denotes the Euclidean inner product in* $\mathbb{R}^2$.

Lemma 4.4. *For all* $a, \sigma_1, \sigma_2 \ge 0, p > 1, \theta > 0$*, we have*

$$(a + \sigma_1)^{p-2} \sigma_1 \sigma_2 \le \theta^{-\gamma} (a + \sigma_1)^{p-2} \sigma_1^2 + \theta (a + \sigma_2)^{p-2} \sigma_2^2,$$

where

$$\gamma = \begin{cases} 1 & 1 < p \le 2, \ \theta \in [1,\infty) \ \text{or} \ 2 < p < \infty, \ \theta \in (0,1), \\ \dfrac{1}{p-1} & 1 < p \le 2, \ \theta \in (0,1) \ \text{or} \ 2 < p < \infty, \ \theta \in [1,\infty). \end{cases}$$

Now we are ready to prove the following proposition.

Proposition 4.5. *Let u be a solution of problem* (1.2) *and let* $1 < p < \infty$. *Then there exist constants* $C_1, C_2 > 0$ *such that, for all* $v, w \in W^{1,p}(\Omega)$,

$$a(u, u - v) - a(v, u - v) \geq C|u - v|^2_{(u,p)}$$

and

$$a(u, w) - a(v, w) \leq C\theta^{-\gamma}|u - v|^2_{(u,p)} + C\theta|w|^2_{(u,p)},$$

where $\theta > 0$ *is any given constant and* $\gamma > 0$ *is given in Lemma* 4.4.

Proof. The first result is a direct consequence of the inequality (4.4). To derive the second result, we note that

$$\frac{1}{2}(|\nabla u| + |\nabla(u - v)|) \leq |\nabla u| + |\nabla v| \leq 2(|\nabla u| + |\nabla(u - v)|),$$

which gives by (4.3)

$$\begin{aligned} a(u, w) - a(v, w) &\leq C \int_\Omega (|\nabla u| + |\nabla v|)^{p-2} |\nabla(u - v)|\, |\nabla w|\, dx \\ &\leq C \int_\Omega (|\nabla u| + |\nabla(u - v)|)^{p-2} |\nabla(u - v)|\, |\nabla w|\, dx. \end{aligned}$$

The proof is finished by applying Lemma 4.4 with $a = |\nabla u|$, $\sigma_1 = |\nabla(u - v)|$ and $\sigma_2 = |\nabla w|$. □

Proposition 4.5 implies that the quasi-norm $|u - v|^2_{(u,p)}$ is equivalent to the total energy difference $a(u, u - v) - a(v, u - v)$.

Now it is a simple matter to deduce the following theorem from Proposition 4.5 and the error equation (3.1),

Theorem 4.6. *Let u and u_h be the solutions of problems* (1.2) *and* (2.2), *respectively. Then there exists a constant* $C > 0$ *such that*

$$|u - u_h|_{(u,p)} \leq C \inf_{v_h \in \mathcal{V}_h} |u - v_h|_{(u,p)}.$$

Proof. Set $v = u_h$ and $w = u - v_h$ in Proposition 4.5, and choose $\theta > 0$ sufficiently small. □

Finally, we establish some sharper error estimates by setting $v_h = \pi_h u$ and assuming a suitable regularity on the solution u.

Theorem 4.7. *Let u and u_h be the solutions of problems* (1.2) *and* (2.2), *respectively. Then we have for* $1 < p < 2$

$$|u - u_h|_{(u,p)} \leq \begin{cases} Ch^{p/2} & \text{if } u \in W^{2,p}(\Omega), \\ Ch & \text{if } u \in C^{2,2/p-1}(\overline{\Omega}) \cap W^{3,1}(\Omega), \end{cases}$$

and for $2 < p < \infty$

$$|u - u_h|_{(u,p)} \leq Ch^{s/2} \qquad \text{if } u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega),\ s \in [1, 2].$$

Therefore, the following error estimates in the convectional norm $\|\cdot\|_{1,p}$ *holds: for* $1 < p < 2$

$$\|\nabla(u-u_h)\|_{0,p} \le \begin{cases} Ch^{p/2} & \text{if } u \in W^{2,p}(\Omega), \\ Ch & \text{if } u \in C^{2,2/p-1}(\overline{\Omega}) \cap W^{3,1}(\Omega), \end{cases}$$

and for $2 < p < \infty$

$$\|\nabla(u-u_h)\|_{0,p} \le Ch^{s/p} \qquad \text{if } u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega),\ s \in [1,2].$$

Proof. We prove the first three estimates in the quasi-norms, as the other estimates are straightforward consequences of these estimates and Proposition 4.2.

The first estimate ($1 < p < 2$) is easily proved by using Proposition 4.2 and the estimate (2.1):

$$|u-\pi_h u|_{(u,p)} \le \|\nabla(u-\pi_h u)\|_{0,p}^{p/2} \le Ch^{p/2}.$$

The second estimate ($1 < p < 2$) is rather technical, and thus we only sketch the proof here (see [1] for details). By the regularity assumption on u it follows that, for all $K \in \mathcal{T}_h$ and $x \in \overline{K}$,

$$|\nabla(u-\pi_h u)(x)| \le Ch|H[u]|_{0,\infty,T} \le ChH[u](x) + Ch^{2/p}$$

and that

$$\int_\Omega |\nabla u|^{p-2} H[u]^2 \, dx < \infty,$$

where $H[u]$ denotes the absolute value of the Hessian of u. Using the fact that $\phi(t) = (a+t)^{p-2}t^2$ ($a \ge 0$) is increasing on $t \ge 0$ and $\phi(|t_1+t_2|) \le 2(\phi(|t_1|)+\phi(|t_2|))$, we obtain

$$\begin{aligned} |u-\pi_h u|_{(u,p)}^2 &\le Ch^2 \int_\Omega (|\nabla u| + ChH[u])^{p-2} H[u]^2 \, dx \\ &\quad + Ch^{4/p} \int_\Omega (|\nabla u| + Ch^{2/p})^{p-2} \, dx \\ &\le Ch^2 \int_\Omega |\nabla u|^{p-2} H[u]^2 \, dx + Ch^2 \le Ch^2. \end{aligned}$$

For the third estimate ($2 < p < \infty$) we note that, for $u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega)$ and $s \in [1,2]$,

$$\begin{aligned} |u-\pi_h u|_{(u,p)}^2 &= \int_\Omega (|\nabla u| + |\nabla(u-\pi_h u)|)^{p-2} |\nabla(u-\pi_h u)|^2 \, dx \\ &\le \int_\Omega (|\nabla u| + |\nabla(u-\pi_h u)|)^{p-s} |\nabla(u-\pi_h u)|^s \, dx \\ &\le C\|\nabla(u-\pi_h u)\|_{0,s}^s \le Ch^s \end{aligned}$$

with C depending on $\|u\|_{1,\infty}$ only. □

Let us end with some remarks which compares the results in this section with those in the previous section.

- For $1 < p < 2$ we have the optimal error estimate
$$\|\nabla(u - u_h)\|_{0,p} \le Ch,$$
provided u belongs to $C^{2,2/p-1}(\overline{\Omega}) \cap W^{3,1}(\Omega)$.
- for $2 < p < \infty$ we have the (suboptimal) error estimate
$$\|\nabla(u - u_h)\|_{0,p} \le Ch^{2/p}$$
under the weaker regularity condition $u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega)$.

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Department of Mathematics KAIST, Taejon, KOREA 305-701