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ALMOST PERIODIC SOLUTIONS OF NONLINEAR DISCRETE VOLTERRA EQUATIONS WITH UNBOUNDED DELAY

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Abstract. In this paper we study the existence of almost periodic solution for nonlinear discrete Volterra equation with unbounded delay, as a discrete version of the results for integro-differential equations.

1. Almost periodic sequences and difference equations

Bohr's theory of almost periodic functions has been extensively studied, especially in connection with differential equations.

Almost periodic solutions of ordinary differential systems are vector valued functions defined on the set $\mathbb{R}$ of real numbers. But the notion of almost periodicity makes sense on any additive group other than $\mathbb{R}$. Indeed, the Bohr definition for an almost periodic function is valid for vector doubly infinite sequences defined on the set $\mathbb{Z}$ of integers. This is important since infinite sequences are candidate solutions of difference equations. Also, the generalizations of almost periodic functions-asymptotic almost periodicity by Frechet, pseudo almost periodicity by Zhang can be defined on sequences.

A sequence $x : \mathbb{Z} \to \mathbb{R}^n$ is said to be *almost periodic* if for any $\varepsilon > 0$ there exists an integer $l(\varepsilon) > 0$ such that each interval of length l contains an integer τ for

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which

$$|f(n+\tau) - f(n)| < \varepsilon,\ n \in \mathbb{Z}.$$

Note that in the process of discretization a periodic function such as $f(t) = \sin t$ over $\mathbb{R}$ does not lead to a periodic sequence in the sense that the sequence $(\sin nh), n \in \mathbb{Z}, h \in (0, \infty), h \neq 2\pi$, is not periodic with integer period. However, such a sequence is almost periodic. For example, $(\sin n)$ is almost periodic.

If $f : \mathbb{R} \to \mathbb{R}^n$ is an almost periodic function, then $(f(n))$ is an almost periodic sequence. Conversely, if x is an almost periodic sequence, then there exists an almost periodic function $f : \mathbb{R} \to \mathbb{R}^n$ such that $f(n) = x(n)$ for $n \in \mathbb{Z}$.

Bochner's criterion: x is an almost periodic sequence if and only if for any integer sequence (k_i') there exists a subsequence $(k_i) \subset (k_i')$ such that $x(n + k_i)$ converges uniformly on $\mathbb{Z}$ as $i \to \infty$.

Furthermore, the limit sequence is also an almost periodic sequence.

Recall that $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be *almost periodic* in n uniformly for $x \in \mathbb{R}^n$, or uniformly almost periodic if for any $\varepsilon > 0$ and any compact set $K \subset \mathbb{R}^n$, there exists a positive integer $l = l(\varepsilon, K)$ such that any interval of length l contains an integer τ for which

$$|f(n+\tau, x) - f(n, x)| < \varepsilon,\ n \in \mathbb{Z}, x \in K.$$

The *hull* of f, denoted by $H(f)$, is defined to be the set of all $g : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ such that there exists an integer sequence (h_k) and $\lim f(n + h_k, x) = g(n, x)$ uniformly on $\mathbb{Z} \times K$, where K is any compact set in $\mathbb{R}^n$.

Difference equations are more appropriate that their continuous counter parts in cases when processes evolve in stages. For example population may grow or decline through non-overlapping generations. Besides, difference equations are well known for simulating continuous models for numerical purposes. Indeed, many discrete models in population dynamics and neural dynamic systems are proposed and well studied with respect to their stability, permanence, bifurcation, chaotic behavior, oscillation, periodicity, etc.

2. Discrete Volterra equations

There are many papers which study the existence of almost periodic solutions of integro-differential equation

$$x'(t) = f(t, x(t)) + \int_{-\infty}^{0} F(t, s, x(t+s), x(t))ds,$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and is almost periodic in t uniformly for $x \in \mathbb{R}^n$, and $F : \mathbb{R} \times (-\infty, 0] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and is almost periodic in t uniformly for $(s,x,y) \in (-\infty,0] \times \mathbb{R}^n \times \mathbb{R}^n$. Hamaya discussed the relationship between some stabilities in the above integro-differential equation.

A discrete version of the above integro-differential equations is Volterra difference equation with unbounded delay

$$x(n+1) = f(n,x(n)) + \sum_{j=-\infty}^{n} B(n,j,x(j),x(n)), \tag{E}$$

where $f : \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $x \in \mathbb{R}^n$ for every $n \in \mathbb{Z}$ and is almost periodic in $n \in \mathbb{Z}$ uniformly for $x \in \mathbb{R}^n$, $B : \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous for $x, y \in \mathbb{R}^n$ for any $j \leq n \in \mathbb{Z}$ and is almost periodic in n uniformly for $j \in \mathbb{Z}^-, x, y \in \mathbb{R}^n$. Song and Tian studied periodic and almost periodic solutions under some suitable conditions.

3. Stabilities

We consider two kinds of stability-total stability and stability under disturbances from hull.

Total stability introduced by Malkin in 1944 requires that the solution of $x'(t) = f(t,x)$ be "stable" not only with respect to the small perturbations of the initial conditions, but also with respect to the perturbations, small in a suitable sense, of the right hand side of the equation.

The bounded solution $x(n)$ of (E) is said to *totally stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $n_0 \geq 0, \rho(x_{n_0}, y_{n_0}) \leq \delta$ and $p(n)$ is a sequence such that $|p(n)| \leq \delta$ for all $n \geq n_0$, then

$$\rho(x_n, y_n) < \varepsilon,\ n \geq n_0,$$

where $y(n)$ is any solution of

$$x(n+1) = f(n,x(n)) + \sum_{j=-\infty}^{n} B(n,j,x(j),x(n)) + p(n) \tag{P}$$

such that $y_{n_0}(j) \in K$ for all $j \in \mathbb{Z}^-$, and K is any compact set in $\mathbb{R}^n$, and

$$\begin{aligned} \rho(\phi,\psi) &= \sum_{q=0}^{\infty} \frac{\rho_q(\phi,\psi)}{2^q[1+\rho_q(\phi,\psi)]}, \\ \rho_q(\phi,\varphi) &= \max_{-q \leq m \leq 0} |\phi(m) - \psi(m)|, q \geq 0 \end{aligned}$$

and $x_n : \mathbb{Z}^- \to \mathbb{R}^n$ is defined by $x_n(j) = x(n+j)$ for $x : \{n \in \mathbb{Z} : n \leq k\} \to \mathbb{R}^n$.

The concept of stability under disturbances from hull was introduced by Sell in 1967 for the ordinary differential equation. Hamaya proved that Sell's definition is equivalent to Hamaya's definition. Also, he showed that total stability implies stability under disturbances from hull.

The bounded solution $x(n)$ of (E) is said to be *stable under disturbances from* $H(f,B)$ *with respect to* K if for any $\varepsilon > 0$ there exists an $\eta = \eta(\varepsilon) > 0$ such that if $\pi((f,B),(g,D)) \le \eta$ and $\rho(x_{n_0}, y_{n_0}) \le \eta$ for some $n_0 \ge 0$, then

$$\rho(x_n, y_n) < \varepsilon,\ n \ge n_0,$$

where $y(n)$ is any solution of the limiting equation

$$x(n+1) = g(n, x(n)) + \sum_{j=-\infty}^{n} D(n, j, x(j), x(n)),\ (g,D) \in H(f,B) \tag{LE}$$

of (E) which passes through (n_0, y_{n_0}) such that $y_{n_0}(j) \in K$ for all $j \in \mathbb{Z}^-$, and $(g,D) \in H(f,B)$ means that for any sequence $(n'_k) \subset \mathbb{Z}^+$ with $n'_k \to \infty$ as $k \to \infty$, there exists a subsequence $(n_k) \subset (n'_k)$ such that

$$f(n + n_k, x) \to g(n, x)$$

uniformly on $\mathbb{Z} \times S$ for any compact set $S \subset \mathbb{R}^n$,

$$B(n + n_k, n + l + n_k, x, y) \to D(n, n + l, x, y)$$

uniformly on $\mathbb{Z} \times S^*$ for any compact set $S^* \subset \mathbb{Z}^- \times \mathbb{R}^n \times \mathbb{R}^n$. Here, for $(p,P) \in H(f,B)$ and $(q,Q) \in H(f,B)$,

$$\begin{aligned} \pi(p,q) &= \sup\{|p(n,x) - q(n,x)| : n \in \mathbb{Z}, x \in K\}, \\ \pi(p,Q) &= \sum_{j=1}^{\infty} \frac{\pi_j(p,Q)}{2^j[1 + \pi_j(p,Q)]}, \end{aligned}$$

$$\begin{aligned} \pi_j(p,Q) &= \sup\{|p(n,j,x,y) - Q(n,j,x,y)| : n \in \mathbb{Z}, j \in [-s,0], x,y \in K\} \\ \pi((p,P),(q,Q)) &= \max\{\pi(p,q), \pi(p,Q)\}. \end{aligned}$$

4. Results

Throughout this paper we assume that

(H1) For any $\varepsilon > 0$ and any $\tau > 0$, there exists an integer $M = M(\varepsilon, \tau) > 0$ such that

$$\sum_{j=-\infty}^{n-M} |B(n, j, x(j), x(n))| < \varepsilon, n \in \mathbb{Z},$$

whenever $|x(j)| < \tau$ for all $j \le n$,

(H2) (E) has a bounded solution $x(n) = x(n, \phi)$, that is, $|x(n)| \le c$ for some $c \ge 0$, passing through $(0, \phi)$, where ϕ is any bounded sequence on $\mathbb{Z}^-$.

Under the hypotheses $(H1)$ and $(H2)$ we obtain the following results:

1. If the bounded solution of (E) is asymptotically almost periodic, then (E) has an almost periodic solution.

(2) If the bounded solution $x(n)$ is totally stable, then it is stable under disturbances from $H(f, B)$ with respect to K.

(3) If the bounded solution $x(n)$ is stable under disturbances from $H(f, B)$ with respect to K, then $x(n)$ is asymptotically almost periodic.

(4) Hence (E) has an almost periodic solution if the bounded solution $x(n)$ is stable under disturbances from $H(f, B)$ with respect to K.

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