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C^1-STABLY EXPANSIVE SETS FOR FLOWS

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Abstract. Let X be a C^1 vector field on a closed C^∞ manifold M. We introduce the concept of C^1 stable expansivity for compact X_t-invariant set, and use a flow-version of Mane's results (Lemma II.3 in *"Mane, R. An ergodic closing lemma. Ann. of Math. (2) 116 (1982), no. 3, 503–540"*) on *uniformly hyperbolic families of periodic linear differential equations* in order to get the hyperbolicity for the C^1 stably expansive homoclinic class $H_X(\gamma)$ of a hyperbolic periodic orbit γ.

1. Main results

Let X be a C^1 vector field on a closed C^∞ manifold M, and $\mathcal{X}^1(M)$ be the set of C^1 vector field on M endowed with C^1 topology. Denoted $\mathcal{E}(M)$ by the set of expansive vector fields on M. Given a vector field $X \in \mathcal{X}^1(M)$, we denote $\mathrm{Sing}(X)$ by the set of singularities of X; $PO(X_t)$ by the set of periodic orbits of X_t; $R(X)$ by the chain recurrent set of X. Let γ be a hyperbolic periodic orbit of X_t. $W^s(\gamma)$ and $W^u(\gamma)$ denote the stable and unstable manifolds of γ; $H_X(\gamma)$ denotes the transversal homoclinic class of X associated with γ, i.e., $H_X(\gamma) = \overline{W^s(\gamma) \pitchfork W^u(\gamma)}$; $C_X(\gamma)$ denotes the chain component of X containing γ; P_t denotes the linear Poincaré flow defined on the normal bundle to X over $M - \mathrm{Sing}(X)$. For these definitions as well as hyperbolicity and dominated splitting of the linear Poincaré flow, see [2, 19]. The following result is from [12].

For $X \in \mathcal{X}^1(M)$*,* $X \in \operatorname{int}\mathcal{E}(M)$ *if and only if* $\operatorname{Sing}(X) = \emptyset$*, and* X *satisfies Axiom A and quasi transversality condition.*

Our aim is to generalize the above result to some subsystems.

Let d be the distance on M induced from a Riemanian metric $\|\cdot\|$ on the tangent bundle TM. Every $X \in \mathcal{X}^1(M)$ generates a C^1 flow $X_t : M \times \mathbb{R} \to M$, that is a C^1 map such that $X_t : M \to M$ is a diffeomorphism satisfying $X_0(x) = x$ and $X_{t+s}(x) = X_t(X_s(x))$ for all $s, t \in \mathbb{R}$ and $x \in M$. Throughout of this paper we assume that $X \in \mathcal{X}^1(M)$ has no singularity.

Let $\Lambda \subset M$ be a compact X_t-invariant subset. The set Λ is called *expansive for* X_t if for any $\varepsilon > 0$ there is $\delta > 0$ with the property that if $d(X_s(x), X_{\alpha(s)}(y)) \leqslant \delta$ for all $s \in \mathbb{R}$, for a pair of points $x, y \in \Lambda$, and for a continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, then $y = X_s(x)$ where $|s| \leqslant \varepsilon$. The constant δ is called an *expansive constant* corresponding to ε (with respect to X_t). When Λ is expansive for X_t, we also say that the subsystem $X|_\Lambda$ is expansive.

It is known that expansivity is a consequence of hyperbolicity. Moreover expansive systems share many like-hyperbolic properties. Hyperbolicity is open in the sense that all nearby systems of hyperbolic systems are hyperbolic. This is not true for expansivity: they are sensative to small perturbations. A simple counterexample is a rational rotation on the unit circle. With those in mind, we introduce the following.

Definition 1.1. Let $\Lambda \subset M$ be a compact X_t-invariant subset. The set Λ is called C^1 *stably expansive for* X_t if there exist a compact set U containing Λ and a C^1 neighbourhood $\mathcal{U}(X)$ of X such that

(i) $\Lambda = \bigcap_{t\in\mathbb{R}} X_t(U)$, i.e., Λ is locally maximal (with isolating block U);
(ii) For all $Y \in \mathcal{U}(X)$, $Y|_{\Lambda_Y}$ is expansive, where $\Lambda_Y = \bigcap_{t\in\mathbb{R}} Y_t(U)$ is the continuation of Λ.

Note that M is C^1 stably expansive for X_t if and only if $X \in \operatorname{int}\mathcal{E}(M)$. In that case, we say X_t is a C^1 stably expansive flow. Suspension of the Smale's horseshoe is an example of C^1 stably expansive flow. We cite the following fact to illustrate how natural the conditon (i) is (for more details, see [15, Theorem 7.4]).

Fact: Let Λ be a locally maximal hyperbolic set for X_t (with isolating block U). Then for every $\varepsilon > 0$ there exists a C^1 neighbourhood $\mathcal{U}(X)$ of X such that for all $Y \in \mathcal{U}(X)$,

(i) $\Lambda_Y = \bigcap_{t\in\mathbb{R}} Y_t(U)$ is hyperbolic for Y,

(ii) $X|_\Lambda$ is topologically conjugate to $Y|_{\Lambda_Y}$ (with a conjugate map $h_Y : \Lambda \to \Lambda_Y$),

(iii) $d_0(h_Y, 1_{\Lambda_Y}) < \varepsilon$.

Proposition 1.2. Suppose that Λ is C^1 stably expansive for X_t. Then there is a C^1 neighbourhood $\mathcal{U}(X)$ of X such that for $Y \in \mathcal{U}(X)$, every $q \in \Lambda_Y \cap \mathrm{PO}(Y_t)$ is hyperbolic.

As a consequence of Proposition 1.2, we obtain a characterization of C^1 stable expansivity on $R(X)$.

Corollary 1.3. Let $X \in \mathcal{X}^1(M)$. Then the followings are mutually equivalent:

(i) $R(X)$ is C^1 stably expansive;

(ii) $R(X)$ is hyperbolic.

Now we are in position to state our main theorem.

Theorem 1.4. *Let $X \in \mathcal{X}^1(M)$. Then $H_X(\gamma)$ is C^1 stably expansive if and only if $H_X(\gamma)$ is hyperbolic.*

The case of diffeomorphisms for Theorem 1.4 was proved in [5]. To prove Theorem 1.4, our main tool is a flow-version of Mane's results on *uniformly hyperbolic families of periodic linear differential equations* on Euclidean spaces (see [9]). As a consequence of Proposition 1.2 with that flow-version of Mane's results, we obtain a dominated splitting of the linear Poincaré flow over $H_X(\gamma)$. Note that results by Doering ([3]) and Liao ([8]) said that the hyperbolicity of $H_X(\gamma)$ for P_t is equivalent to the hyperbolicity of $H_X(\gamma)$ for the underlying flow X_t.

2. Uniformly hyperbolic families of periodic linear differential equations

Denoted $PEP(\mathbb{R}^N)$ by the set of all invertible periodic matrix function $U = U(\cdot) : \mathbb{R} \to GL(\mathbb{R}^N)$ such that $\|U(t)\| \leqslant Ne^{\beta t}$ for every $t \in \mathbb{R}$ and some positive constants N, β (depend on U), i.e., the space of all exponentially bounded periodic evolution processes on $\mathbb{R}^N$. Every U can be considered as *fundamental matrix* (or *evolution matrix*, or *evolution process*) of some linear differential equation $x' = A(t)x, \quad x \in \mathbb{R}^N$, with periodic continuous matrix coefficient $A(\cdot)$.

$PEP(\mathbb{R}^N)$ becomes a Banach space with the following norm:

$$\|U - V\| := \sup_{t \in \mathbb{R}, \alpha \in I} \left\{ \mu > 0 : \exists C > 0, \|U^{(\alpha)}(t) - V^{(\alpha)}(t)\| \leqslant Ce^{\mu t} \right\}.$$

Matrix function $U(\cdot) \in PEP(\mathbb{R}^N)$ is called *hyperbolic* if $\mathbb{R}^N = E^s(U) \oplus E^u(U)$, where

$$E^s(U) = \{v \in \mathbb{R}^N : \sup_{t \geqslant 0} \|U(t)v\| < +\infty\}$$
$$E^u(U) = \{v \in \mathbb{R}^N : \sup_{t \geqslant 0} \|U(-t)v\| < +\infty\}.$$

In case of $U(\cdot) \in PEP(\mathbb{R}^N)$, Floque's Theorem states that there is $L(\cdot) \in GL(\mathbb{R}^N)$ (same period with $U(\cdot)$) such that

$$U(t) = L(t)e^{tB},$$

where B is a constant matrix.

A simple example shows that the limit of sequence of hyperbolic matrix functions is not necessarily hyperbolic. A stronger concept, uniform hyperbolicity, will guarantee hyperbolicity of the limit matrix function.

Definition 2.1. Family $\{U^{(\alpha)}, \alpha \in I\} \subset PEP(\mathbb{R}^N)$ is called *uniformly hyperbolic* if there exist $N, \beta, \varepsilon > 0$ such that

(i) $U^{(\alpha)}$ is hyperbolic for every $\alpha \in I$;
(ii) $\|U^{(\alpha)}\| \leqslant Ne^{\beta t}$, for all $\alpha \in I, t \in \mathbb{R}$;
(iii) if a family $\{V^{(\alpha)}, \alpha \in I\} \subset PEP(\mathbb{R}^N)$ satisfies $\|U - V\| \leqslant \varepsilon$ then $V^{(\alpha)}$ is hyperbolic for every $\alpha \in I$.

Note that in condition (iii) we do not assume that period of U and V are the same. Now we state a flow-version of [9, Lemma II.3].

Theorem 2.2. *Let $\{U^{(\alpha)}, \alpha \in I\} \subset PEP(\mathbb{R}^N)$ be a uniformly hyperbolic family of periodic evolution processes. Then there exist constants $K, T, \lambda > 0$ such that*

(i) *if $U^{(\alpha)}$ has minimum period $\pi(U^{(\alpha)}) \geqslant T$ then for any $T' \in \mathbb{R}$ and any partition $0 < t_1 < t_2 < \cdots < t_k = \pi(U^{(\alpha)})T'$, $t_{i+1} - t_i \geqslant T$, we have*

$$\prod_{i=0}^{k-1} \left\| U^{(\alpha)}(t_{i+1} - t_i)|_{E^s(U^{(\alpha)})} \right\| \leqslant Ke^{-\lambda(t_{i+1} - t_i)},$$

$$\prod_{i=0}^{k-1} \left\| U^{(\alpha)}(-t_{i+1} + t_i)|_{E^u(U^{(\alpha)})} \right\| \leqslant Ke^{-\lambda(t_{i+1} - t_i)};$$

(ii) *For $\alpha \in I, t \in \mathbb{R}$,*

$$\left\| U^{(\alpha)}(t)|_{E^s(U^{(\alpha)})} \right\| \cdot \left\| \left(U^{(\alpha)}(t)\right)^{-1}|_{E^s(U^{(\alpha)})} \right\| \leqslant e^{-2\lambda t};$$

(iii) *For* $\alpha \in I$, $T' \in \mathbb{R}$ *and any partition* $0 < t_1 < t_2 < \cdots < t_k = \pi(U^{(\alpha)})T'$, $t_{i+1} - t_i \geqslant T$, *we have*

$$\frac{1}{\pi(U^{(\alpha)})T'} \sum_{i=0}^{k-1} \log \left\| U^{(\alpha)}(t_{i+1} - t_i)|_{E^s(U^{(\alpha)})} \right\| < -\lambda,$$

$$\frac{1}{\pi(U^{(\alpha)})T'} \sum_{i=0}^{k-1} \log \left\| U^{(\alpha)}(-t_{i+1} + t_i)|_{E^u(U^{(\alpha)})} \right\| < -\lambda.$$

Remark 2.3. Some parts of Theorem 2.2 can be proved using the similar arguments as in [9], but some parts need another techniques to prove them, e.g. *Uniform Boundedness Principle* from Functional Analysis.

Remark 2.4. If $X \in \mathcal{X}^*(M)$ (i.e., every singular and periodic point is stably hyperbolic), then we can apply Theorem 2.2 with evolution matrices is the linear Poincaré flow P_t, and obtain a dominated splitting for P_t on $PO(X_t) - \mathrm{Sing}(X)$. This was proved by Liao (see [8], [18]) with *completely different techniques.*

Theorem 2.2 has its own interest. Next corollary is used in this paper.

Corollary 2.5. If $H_X(\gamma)$ is C^1 stably expansive then the linear Poincaré flow P_t admits a dominated splitting on $H_X(\gamma)$.

Similarly we have:

Corollary 2.6. If $C_X(\gamma)$ is C^1 stably expansive then the linear Poincaré flow P_t admits a dominated splitting on $C_X(\gamma)$.

It seems that it is more convernient to see Theorem 2.2 from point of view of Ordinary Linear Differential Equations (bounded solutions, exponentially bounded evolution process, ordinary dichotomy, exponential dichotomy,. . .).

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