Trends in Mathematics - New Series
Information Center for Mathematical Sciences
Volume 10, Number 2, 2008, pages 65–69

2008 International Workshop on Dynamical Systems and Related Topics

SOME DYNAMICAL PROPERTIES OF THE TRANSVERSELY HOLOMORPHIC FLOWS ON CLOSED 3-MANIFOLDS

JAEYOO CHOY AND HAHNG-YUN CHU*

Abstract. In this paper, we study some dynamical properties of the orientable transversely holomorphic riemannian foliations on a closed oriented 3-manifolds. As one aspect, we will see that the compact surface induced by the normal bundle has no hyperbolic structure.

1. Introduction

In the study of dynamical systems, numerical simulations generally provide us with approximated data of orbits(or pseudo-orbits) rather than actual orbits of the system. To understand such simulations as tracers of orbits of the dynamical systems, we have to make sure that sufficiently precise computations produce pseudo-orbits which are followed(or shadowed) by true orbits. This property is called the *shadowing property*(or *pseudo-orbit tracing property*). The shadowing property, as an observer of orbit behaviors, holds a fundamental importance in the general theory of dynamical systems, especially in the applications to dynamical systems. In discrete dynamics(or cascade), a basic set of every hyperbolic dynamical system displays the shadowing property, as shown by Anosov[An]. There is a general belief of "equivalence" between shadowing property and hyperbolicity since they are so closely related. This equivalence is, of course, not always true. Indeed,

2000 *Mathematics Subject Classification.* 37F15, 37F75.
Key words and phrases. transversely holomorphic flow, 3-manifold, hyperbolicity, shadowing property.
∗ *Corresponding author.*
This work was supported by Korea Research Foundation Grant(KRF-2008-331-C00015) .

there are some examples of systems which are not hyperbolic, but nonetheless exhibit the shadowing property[YY]. However, it is reasonable to expect that, in most systems, empirically, the shadowing property and hyperbolicity are indeed equivalent.

Let $\mathcal{L}$ be a 1-dimensional foliation on a oriented closed connected 3-manifold M. $\mathcal{L}$ is *transversely holomorphic* if the induced holonomy pseudogroup is given by biholomorphisms between open sets of $\mathbb{C}$ (cf. [BG, pp.126]). We say that $\mathcal{L}$ is *riemannian* if there is a riemannian metric on the normal bundle (which is holonomy-invariant). By Brunella and Ghys' results ([BG],[Gh]) on the classification of the transversely holomorphic flows on 3-manifolds, the flows are under investigation in detail. Specifically when some cohomological condition, say $H^2(M, \mathcal{O}) \neq 0$ for the sheaf $\mathcal{O}$ of germs of functions constant along the leaves and holomorphic in the transverse direction, is fulfilled, Ghys enables the case to fall into the riemannian flows which Carrière has once clarified in [Ca]. His idea in [Gh] uses the harmonic transition data developed by Brunella in [BG]. Consequently, the transition function becomes harmonic in the flow direction and holomorphic in the transverse direction, so that one finds the harmonic conjugate. The upshot in the proof of [Gh], although the holomorphic transition data by adding the harmonic conjugate do not satisfy the cocycle condition, is that the Kähler geometric techniques around DeRham complex, Dolbeault's theorem, Kodaira-Serre's duality and so on are still applicable.[1] Finally, one can obtain a metric on the normal bundle of $\mathcal{L}$ which satisfies a certain rigidity of the distance along the flow direction. We call this metric as *Ghys' metric*. Based upon the result of Ghys, let us extract one dynamical property on hyperbolicity as below, which seeds other dynamical properties appearing in §3.

Theorem (Nonhyperbolicity theorem) Let $\mathcal{L}$ be a transversely holomorphic foliation on a oriented closed connected 3-manifold M. Assume that $H^2(M, \vartheta) \neq 0$ where ϑ denotes the sheaf of germs of functions which are constant along the leaves and holomorphic in the transverse direction. Then there is a (local) surface (of real dimension 2) such that the Poincaré map on the surface is nonhyperbolic.

The theorem assumes Ghys and Carrière's classification results so that one can take the compact surface where the hyperbolicity does not hold. But, if one defines the hyperbolicity in a bit wider sense, we are able to deduce the theorem without

[1] In the case of Brunella, the extended transition forms a cocycle, hence a complex surface. Due to the non-degenerate flow, the Kodaira-Enrique classification with Inoue's theorem is possible ([BG][In]).

their results (also without the statement involving the surface) as we will do in the coming section.

2. Nonhyperbolicity of transversely holomorphic flows

Definition Let X be a finite dimensional riemannian manifold and $f : X \to X$ a diffeomorphism. We say that X has a *hyperbolic structure* with respect to f if there is a continuous splitting of TX into the direct sum of Tf-invariant subbundles E^s and E^u such that for some constants A and λ and for all $v \in E^s$, $w \in E^u$ and $n \geq 0$,

$$\| Tf^n(v) \| \leq A\lambda^n \| v \|, \quad \| Tf^{-n}(w) \| \leq A\lambda^n \| w \|,$$

where $0 < \lambda < 1$.

Thus one may say that Tf is eventually contracting on E^s and eventually expanding on E^u. A *hyperbolic subset* of X with respect to f is a closed invariant subset of X that has a hyperbolic structure.

Let ϕ^t be a flow on Hausdorff topological space X and $q \in X$. We define the *ω-limit set* and *α-limit set* of q by

$$\omega(q) = \{x \in M : x = \lim_{n\to\infty} \phi^{t_n}(q) \text{ for some sequence } t_n \to \infty \text{ as } n \to \infty\},$$

$$\alpha(q) = \{x \in M : x = \lim_{n\to\infty} \phi^{-t_n}(q) \text{ for some sequence } t_n \to \infty \text{ as } n \to \infty\}.$$

There are several useful and natural relations between continuous-time and discrete-time dynamical systems. We consider only the construction for passing from a flow to a map. The most obvious way to associate a discrete-time system to a flow $\{\phi^t\}_{t\in\mathbb{R}}$ is to take the iterates of the map ϕ^{t_0} for some value of t_0, say, $t_0 = 1$. Another more local but also more useful method is the construction of the *Poincaré(first-return) map.*

Definition Let ϕ^t be a flow on a Hausdorff topological space X. A point $x \in X$ is *ω-recurrent* or *positively recurrent* with respect to ϕ^t if $x \in \omega(x)$ and is *α-recurrent* or *negatively recurrent* with respect to ϕ^t if $x \in \alpha(x)$. A point $x \in X$ is *Poincaré recurrent* with respect to ϕ^t if x is positively recurrent and negatively recurrent with respect to ϕ^t

A proof of the theorem is given now (ref. [CC]). Since every flow on a compact manifold has a nonempty recurrent set, there exists a recurrent point p of M. Let U be a neighborhood of p in M. From the definition of the recurrence, there exists some part of the flow from p is contained in U. Observe that the flow along the leaves are constant in the neighborhood U of p, equivalently U, after shrinking if necessary, canonically diffeomorphic to the product of an open subset of $\mathbb{C}$ and an open subset of $\mathbb{R}$. Let us denote by S the image of the open subset $\mathbb{C}$. By the recurrence property, the Poincaré map is well-defined on S. By the construction of Ghy's metric, the Poincaré map preserves the metric, i.e. isometry. This, due to the holonomy-invariance, implies the nonhyperbolicity and S satisfies the theorem.

3. Applications of Nonhyperbolicity Theorem

As we observed in the proof in §2, the proof of Ghys' theorem asserts that with regard to the Ghys metric, the Poincaré mapping is a isometry near a recurrence point. In fact, thanks to the classification of Carrière, we are able to conclude that every point of M is a recurrence point (ref. [CC]).

A sequence of points $\{x_i : a < i < b\}$ of a metric space X with metric d is called a δ*-pseudo orbit* of f if $d(f(x_i), x_{i+1})$ for $i \in (a, b-1)$. Given $\varepsilon > 0$ a δ-pseudo orbit x_i is said to be ε*-traced* by a point $x \in X$ if $d(f^i(x), x_i)$ for $i \in (a, b)$. We say that f has the *shadowing property (pseudo orbit tracing property)* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo orbit of f can be ε-traced by some point of X. For compact metric spaces this shadowing property is independent of the compatible metrics used. And here is another important property of dynamical systems. A homeomorphism $f : X \to X$ is *expansive* if there is a constant $e > 0$ satisfying if $x, y \in X$ with $x \neq y$ then there exists an integer n such that $d(f^n(x), f^n(y)) > e$. Such a constant e is called an *expansive constant* for f. In this case of the definition, when X is compact, the property is also not dependent of the choice of metrics for X.

Lemma Every isometry for a manifold has neither shadowing property nor expansivity.

The above discussion yields the following theorem.

Theorem The surface in the nonhyperbolicity theorem has neither shadowing property nor expansivity for the Poincaré mapping.

References

[An] Anosov, D.V.: *Geodesic flows on closed Riemannian manifolds with negative curvature.* Proc. Steklov Institute of Mathematics **90** (1967).

[Br] Brunella, M., Ghys, É.: *Umbilical foliations and transversely holomorphic flows.* J. Differ. Geom. **41** (1995), no. 1, 1–19.

[BG] Brunella, M.: *On transversely holomorphic flows I.* Invent. Math. **126** (1996), no. 2, 265–279.

[Ca] Carrière, Y.: *Flots riemanniens, in Structures transverses des feuilletages.* Astérisque **116** (1984), 31–52.

[CGSY] Cerveau, D., Ghys, É., Sibony, N., Yoccoz, J.-C.: *Complex Dynamics and Geometry* SMF/AMS Texts and Monographs, 10. American Mathematical Society, Providence, RI; Societe Mathematique de France, Paris, 2003.

[CC] Choy, J., Chu, H.-Y.: *Nonhyperbolicity of the holomorphic flows on closed 3-manifolds* In preparation.

[Gh] Ghys, É.: *On transversely holomorphic flows II*. Invent. Math. **126** (1996), no. 2, 281–286.

[HS] Haefliger, A., Sundararaman, D.: *Complexifications of transversely holomorphic foliations.* Math. Ann. **272** (1985), no. 1, 23–27.

[In] Inoue, M.: *On surfaces of class VII_0*. Invent. Math. **24** (1974), 269–310.

[KH] Katok, A., Hasselblatt, B.: *Introduction to the modern theory of dynamical systems* Cambridge University Press, 1995.

[YY] Yuan, G.-C. and Yorke, J.A.: *An open set of maps for which every point is absolutely nonshadowable.* Proc. Amer. Math. Soc. **128** (2000), no. 3, 909–918.

[ZZ] Zelenko, I., Zhitomirskii, M.: *Rigid paths of generic* 2*-distributions on* 3*-manifolds.* Duke Math. J. **79** (1995), no. 2, 281–307.

Dept. of Mathematics, Kyungpook National University, Sankyuk-dong, Buk-gu, Daegu 702-701, Republic of Korea

E-mail address: choy@knu.ac.kr

School of Mathematics, Korea Institute for Advanced Study, Cheongnyangni 2-dong, Dongdaemun-gu, Seoul 130-722, Republic of Korea

E-mail address: hychu@kias.re.kr