ENTROPY AND MEASURE DEGENERACY FOR FLOWS

WENXIANG SUN* AND EDSON VARGAS**

ABSTRACT. In discrete dynamical systems topological entropy is a topological invariant and a measurement of the complexity of a system. In continuous dynamical systems, in general, topological entropy defined as usual by the time one map does not work so well in what concerns these aspects. The point is that the natural notion of equivalence in the discrete case is topological conjugacy which preserves time while in the continuous case the natural notion of equivalence is topological equivalence which allow reparametrizations of the orbits. The main issue happens in the case that the system has fixed points and will be our subject here.

1. INTRODUCTION

Two continuous flows defined on a compact metric space are *topologically equivalent* if there exists a homeomorphism of this space that maps each orbit of one onto an orbit of the other and preserves the time orientation. The topological entropy (and also the measure-theoretic entropy) of a flow are defined using its time one map, see [3] [7] and [21]. Then, due to possible reparametrizations of orbits, these entropies may assume different values in a class of topological equivalence. However, if a flow has no fixed points and its topological entropy is 0 or ∞ then it is

constant in the corresponding class of topological equivalence, see [9], [16], [19] and [20]. But in the presence of fixed points this fact can be false as shown in [9] in the case of continuous flows and in [15] in the case of smooth flows. So, against what happens for dynamics with discrete time, in the case of flows with fixed points, the topological entropy is neither a measurement of topological complexity nor a topological invariant. Note that the topological entropy of a smooth flow on a compact manifold is always finite, see Theorem 7.15 in [22].

In [19] and [20] it was introduced a new concept of topological entropy for flows and in [13] and [16] it was introduced a new concept of measure-theoretic entropy. These new concepts of entropy take into consideration all possible reparametrizations of a flow. In the case of flows without fixed points, each one of these new entropies coincides with the corresponding entropy defined using the time one map. But still, in the case of flows with fixed points, they do not have the invariance properties wanted. Nevertheless these new concepts of entropies were used successfully in [16] to prove that an expansive flow without fixed points has the same topological entropy as any of its symbolic extension. This was a problem posed by Bowen and Walters in [2]. In the case of the action induced by a flow on the bundle of n-frames ocer a manifold a concept of entropy introduced in [14] was used in [17] to solve a problem posed by Liao in [8].

It is not known yet whether 0 or $+\infty$ entropy (using the new concepts of entropy from [19], [20], [13] and [16]) are preserved or not when we change from one continuous flow to a topologically equivalent flow.

To improve the understanding of the effect of reparametrizations on the complexity of a flow it was introduced in [18] two concepts of degeneracy. One is the concept of *topological entropy degeneracy* which means that one flow with positive topological entropy is topologically equivalent to a flow such that this entropy vanishes. Other is the concept of *measure degeneracy* which means that one flow having invariant and ergodic measures of positive measure-theoretic entropy is topologically equivalent to a flow such that all invariant and ergodic measures are degenerate to atomic measures supported on fixed points. In [18] it is proved that the topological entropy degeneracy for a flow happens iff the measure degeneracy also happens for it and vice-versa.

2. Basic concepts and notations

Throughout this article, unless differently stated, (X, d) denotes a compact metric space with a metric d and ϕ denotes a continuous flow (for simplicity, just flow) on (X, d), that is: ϕ is a continuous map from $X \times \mathbb{R}$ to X satisfying $\phi(x, 0) = x$

and $\phi(x, s + t) = \phi(\phi(x, s), t)$, for $x \in X$ and $s, t \in \mathbb{R}$. For a fixed $t \in \mathbb{R}$ the homeomorphism $\phi_t : X \to X$ defined by $\phi_t(x) = \phi(x, t)$ is called the time t map of ϕ.

A Borel probability measure (measure for short) μ is called ϕ_t-invariant if for any Borel set B it holds that $\mu(\phi_t(B)) = \mu(B)$. A ϕ_t-invariant measure is called ϕ_t-ergodic if any Borel set which is ϕ_t-invariant has measure 0 or 1. A measure is called ϕ-invariant if it is ϕ_t-invariant for all t. A measure (not necessarily ϕ-invariant) is called ϕ-ergodic if for any Borel set which is ϕ-invariant (that is, it is ϕ_t-invariant for any t) has measure 0 or 1. Here we remark that the ϕ-ergodicity of a measure does not assume its ϕ-invariance nor its ϕ_t-ergodicity. The set of all measures which are both ϕ_t-invariant and ϕ_t-ergodic is denoted by $\mathcal{M}_{erg,\phi_t}$. The set of all measures which are both ϕ-invariant and ϕ-ergodic is denoted by $\mathcal{M}_{erg,\phi}$.

The following result is well known, see [3].

Theorem 2.1. *Let $(X, \mathcal{B}(X), \mu)$ be a Borel probability space where μ is invariant by the flow $\phi : X \times \mathbb{R} \to X$. Then, for any given $s, t \in \mathbb{R} \setminus \{0\}$, it holds that $\frac{1}{|s|} h_\mu(\phi_s) = \frac{1}{|t|} h_\mu(\phi_t)$, where $h_\mu(\phi_t)$ denotes the measure-theoretic entropy of ϕ_t with respect to the measure μ. For the topological entropy a similar formula holds, that is: $\frac{1}{|s|} h(\phi_s) = \frac{1}{|t|} h(\phi_t)$, where $h(\phi_t)$ denotes topological entropy of ϕ_t.*

Observe that in the above theorem μ is ϕ_t-invariant and $h_\mu(\phi_t)$ is well defined. Based on this theorem one may define the measure-theoretic and the topological entropy of a flow ϕ using its time one map, they are respectively:

$$h_\mu(\phi) := h_\mu(\phi_1) \quad \text{and} \quad h(\phi) := h(\phi_1),$$

where μ is a ϕ_t-invariant measure.

The variational principle in Equation 1 below holds for flows, see [9] and [14]. Observe that it is slightly different from the variational principle for homeomorphisms due to the settings where the supremum is taken.

$$(1) \qquad h(\phi) = \sup_{\mu \in \mathcal{M}_{erg,\phi_1}} h_\mu(\phi_1) = \sup_{m \in \mathcal{M}_{erg,\phi}} h_m(\phi_1).$$

Definition 2.2. *Let $\phi, \psi : X \times \mathbb{R} \to X$ be two flows on the compact metric space (X, d). They are said topologically equivalent, if there is a homeomorphism $\pi : X \to X$ which maps orbits of ϕ onto orbits of ψ and preserves their time orientation. The homeomorphism π is called a topological equivalence between ϕ and ψ.*

The following lemma is from [12].

Lemma 2.3. *Let* $\pi : X \to X$ *be a topological equivalence between the flows* ϕ, ψ : $X \times \mathbb{R} \to X$. *If* X_0 *is the set of fixed points of* ϕ *then there is a continuous function* $\theta : (X \setminus X_0) \times \mathbb{R} \to \mathbb{R}$ *such that for* $x \in X$ *and* s, $t \in \mathbb{R}$ *the following hold:*

(1) $\theta(x, 0) = 0$ *and* $\theta_x = \theta(x, \cdot) : \mathbb{R} \to \mathbb{R}$ *is strictly increasing;*

(2) $\theta_x(s + t) = \theta_x(s) + \theta_{\phi_{s(x)}}(t)$;

(3) $\pi \circ \phi_t(x) = \psi_{\theta(x,t)} \circ \pi(x)$.

3. Entropy degeneracy for flows

In this section we introduce an example by Ohno [9] which shows that there exist continuous flows with positive topological entropy which are topologically equivalent to continuous flows for which this entropy vanishes. In this case we say that the topological entropy degenerates. Another example, by Sun, Young and Zhou [15] shows that the topological entropy may degenerate even in the class of smooth flows.

3.1. **The case of continuous flows.** Let (Σ_2, d) be the compact metric space where Σ_2 is the set $\prod_{i=-\infty}^{+\infty} \{0, 1\}$ of bi-infinite sequences $x = (\ldots, x_{-1}, x_0, x_1, \ldots)$ with $x_i \in \{0, 1\}$ and d is the metric given by

$$d(x, y) := \sum_{i=-\infty}^{\infty} \frac{\delta(x_i, y_i)}{2^{|i|}},$$

where $\delta(x_i, y_i)$ is 0 or 1 depending, respectively, on x_i being different or equal to y_i. Also consider the shift map $T : \Sigma_2 \to \Sigma_2$ given by $(T(x))_i = x_{i+1}$.

The construction of the example of Ohno [9] is divided in 3 steps.

Step 1. A special bi-infinite sequence.

Let $x = (x_i)_i \in \Sigma_2$ satisfy the two properties below:

- For any given $i \in \mathbb{Z}$ and $n \geq 1$, the word $x_i x_{i+1} \ldots x_{i+4 \cdot 3^n}$ contains the number 1 in at least $2n - 1$ consecutive positions;
- For $n \geq 2$ set $p_n := \frac{3^{n-1}+1}{2}$. The sequence x contains at least 2^{p_n} subsequences of length $2 \cdot 3^{n-1}$.

Step 2. The topological entropy of a special subshift.

Set X to be the closure of the orbit of x from Step 1 by the shift map T. Then X is a T-invariant compact set and $\mathcal{M}_{erg,T}(X) \neq \emptyset$. If $\mu \in \mathcal{M}_{erg,T}(X)$ the Birkhoff ergodic theorem implies that there is $y \in X$ such that

$$\mu(I_n) = \int_X \chi_{I_n} d\mu = \lim_{m \to \infty} \frac{1}{m} \sum_{j=0}^{m-1} \chi_{I_n}(T^j y)$$

for any $n \geq 1$, where $I_n = \{x \in X \mid x_i = 1, -n + 1 \leq i \leq n - 1\}$ and χ_{I_n} is the characteristic function of I_n.

Since $y \in X$ is an accumulation point of $\{T^n x \,|\, n \in \mathbb{Z}\}$ it follows that

$$\mu(I_n) \geq \frac{1}{4 \cdot 3^n}.$$

On the other hand,

$$h(T) = \lim_{n \to \infty} \frac{1}{n} \log \sigma_n(X) \geq \lim_{n \to \infty} \frac{1}{2 \cdot 3^{n-1}} \log 2^{p_n} = \frac{1}{4} \log 2 > 0,$$

where

$$\sigma_n(X) = \#\{[i_0 \ldots i_{n-1}] \,|\, x_0 = i_0, \ldots, x_{n-1} = i_{n-1}, \text{ for some } x \in X\}.$$

Step 3. The suspension flow of a special subshift.

Consider suspension flow of (X, T). The infinite sequence with 1 at all positions, denoted by p, is the unique fixed point of $T : X \to X$. So, $X_* := X \setminus \{p\}$ is a local compact space. By using a positive continuous function $\gamma : X \to (0, +\infty)$ and the map $T : X \to X$ we define the quotient space $X_*^\gamma := X_* \times \mathbb{R}/ \cong$ where $\cong$ identifies $(x, \gamma(x))$ with $(T(x), 0)$. The suspension flow $\varphi^\gamma : X_*^\gamma \times \mathbb{R} \to X_*^\gamma$ is defined by $\varphi^\gamma(x, t + s) = \varphi^\gamma(\varphi^\gamma(x, t), s)$, where $0 \leq t + s < \gamma(x)$ and $x \in X_*$.

Let $X^\gamma = X_*^\gamma \cup \{\Delta\}$ be the compactification of X_*^γ by adding one point Δ. Then X^γ is a compact metric space and it is easy to see that $(x_n, u_n) \to \Delta$ on X^γ iff $x_n \to p$ on X. Setting $\varphi_t^\gamma(\Delta) = \Delta$, for all $t \in \mathbb{R}$ we get a extension of the flow $\{\varphi_t^\gamma\}$ on X_*^γ to a continuous flow Φ^γ on X^γ.

Lemma 3.1. *For any two positive continuous functions $\gamma, \gamma' : X_* \to (0, +\infty)$ the continuous flows Φ^γ and $\Phi^{\gamma'}$ constructed as above are topologically equivalent.*

Proof. Define

$$\begin{aligned} \pi : \quad X^\gamma \quad &\to \quad X^{\gamma'} \\ (x, u) \quad &\mapsto \quad (x, u\tfrac{\gamma'(x)}{\gamma(x)}), \quad 0 \leq u < \gamma(x). \end{aligned}$$

Then π is a homeomorphism from X^γ to $X^{\gamma'}$ which maps the orbit of Φ^γ starting on $(x, 0)$ to the orbit of $\Phi^{\gamma'}$ starting on the same point $\pi(x, 0) = (x, 0)$. $\square$

Lemma 3.2. [9] *Let $\gamma : X_* \to \mathbb{R}$ be a positive continuous function. If $\int \gamma \, d\mu = \infty$ for all non atomic measure $\mu \in \mathcal{M}_{erg,T}$, then the Dirac measure δ_Δ is the unique measure which is both Φ^γ-invariant and Φ^γ-ergodic.*

We will choose two positive functions and construct two suspension flows as in the above procedure. The two flows obtained are equivalent but for one of them the topological entropy is positive and for the other it vanishes.

In fact we choose $\gamma'(x) = 1$, and choose

$$\gamma(x) = \begin{cases} n \cdot 4 \cdot 3^n & x \in I_n \setminus I_{n+1}, \quad n \geq 1, \\ 1 & x \in X_* \setminus I_1. \end{cases}$$

$\gamma(x)$ depends continuously on x and the two flows (X^γ, Φ^γ) and $(X^{\gamma'}, \Phi^{\gamma'})$ are topologically equivalent by Lemma 3.1. Note $T = \Phi_1^{\gamma'}$, $h(\Phi^{\gamma'}) = h(T) > 0$. On the other hand, for any non atomic measure $\mu \in \mathcal{M}_{erg,T}$, it follows that

$$\int \gamma \, d\mu > \int_{I_n} n \cdot 4 \cdot 3^n d\mu > n \cdot 4 \cdot 3^n \mu(I_n) > n.$$

So $\int \gamma d\mu = \infty$. By Lemma 3.2, δ_Δ is the unique measure which are both Φ^γ-invariant and Φ^γ-ergodic. So, $h(\Phi^\gamma) = 0$ by (1).

The two flows are suspension flows and have the same orbits. Therefore they are topologically equivalent. One suspension flow just keeps the entropy of the discrete system (X, T). To construct the other suspension flow we choosen an exponentially increasing function γ, this implies that the flow is slowed down exponentially near the unique fixed point Δ. As a result, the unique Φ^γ-invariant and Φ^γ-ergodic measure is degenerate to δ_Δ and thus its topological entropy vanishes.

3.2. **The case of smooth flows.** The example by Ohno [9] mentioned above is on the setting of continuous flows. The flow Φ^γ with topological entropy zero constructed depends on the exponential growth of the function $\gamma : X_* \to (0, +\infty)$, which is continuous only in the totally disconnect metric space (X_*, d). Next, the existence or not of a smooth flows with positive topological entropy which are topologically equivalent to smooth flows for which this entropy vanishes. becomes a natural question, it was posed by Ohno in [9]. More precisely, if the topological entropy vanishes for a smooth flow it also vanishes for all smooth flows in the same class of topological equivalence? Sun, Young and Zhou construct in [15] two C^r (for any $r \geq 1$) topologically equivalent smooth flows on a compact smooth Riemannian manifold M of dimension at least 3 such that the topological entropy is positive for one of them and vanishes for the other. This answered the question of Ohno negatively. Observe that flows on compact manifold of dimension 1 or 2 always have zero topological entropy.

Let us comment briefly the construction in [15]. Let us choose a compact smooth manifold M with dimension ≥ 2 and a C^r (for any $r \geq 1$) diffeomorphism $f : M \to M$ which is minimal and preserves a measure μ of positive measure-theoretic entropy. From [6] such a manifold and such a diffeomorphism exist. By the variational principle f has positive topological entropy and by using ergodicity decomposition we can assume that μ is an ergodic measure. Let Ω be the standard suspension manifold (by the constant function 1) with the standard smooth suspension flow $\psi : \Omega \times \mathbb{R} \to \Omega$. Denote by V the C^r vector field on Ω which induces ψ

and fix a point p in Ω. Then we choose two function $\widehat{\alpha}$, $\alpha \in C^r(\Omega, [0, 1])$ such that $\widehat{\alpha}(p) = \alpha(p) = 0$ and $\widehat{\alpha}(x) > \alpha(x) > 0$, for all $x \neq p$.

Choosing appropriate $\widehat{\alpha}$ and α as in [15] the C^r vector fields $V_{\widehat{\alpha}}$ and V_α on Ω obtained by multiplying V by $\widehat{\alpha}$ and α, respectively, are topologically equivalent. Moreover, the topological entropy is positive for the flow generated by $V_{\widehat{\alpha}}$ and vanishes for the flow generated by V_α. This happens because α is taken much smaller that $\widehat{\alpha}$ near p. So, the speed of V_α is much smaller than the speed of $V_{\widehat{\alpha}}$ near their unique fixed point p. In this way all the non atomic measures which are invariant and ergodic for the flow of $V_{\widehat{\alpha}}$ degenerate to the Dirac measure δ_p which becomes the unique invariant and ergodic measure for the flow of V_α.

4. Characterization of entropy degeneracy

The phenomenon as in [9] and [15] that one flow with positive topological entropy is topologically equivalent to a flow such that this entropy vanishes is called *entropy degeneracy*. The phenomenon that one flow ϕ which has invariant and ergodic measures of positive measure-theoretic entropy is topologically equivalent to a flow such that all invariant measures are degenerate to atomic measures supported on fixed points is called *measure degeneracy*. Next we will comment on the work of Sun and Vargas in [18] where it is proved that the phenomenon of entropy degeneracy implies the phenomenon of measure degeneracy and vice-versa.

Let us start introducing more precisely the above concepts and some additional notation. Given a flow ϕ we set

$$\mathcal{M}^+_{erg,\phi} := \{\mu \in \mathcal{M}_{erg,\phi} \mid h_\mu(\phi) > 0\}.$$

Let $\pi : X \to X$ be a topological equivalence between two continuous flows ϕ, $\psi : X \times \mathbb{R} \to X$. In particular we have that

$$\pi(Orb(x, \phi)) = Orb(\pi(x), \psi) \quad \text{for all} \quad x \in X.$$

If $h(\phi) > 0$ and $h(\psi) = 0$ hold we say that the positive entropy of ϕ degenerates to 0 or that the phenomenon of *entropy degeneracy* happens for the pair ϕ and ψ. If

$$\mathcal{M}^+_{erg,\phi} \neq \emptyset, \quad \mathcal{M}^+_{erg,\psi} = \emptyset \quad \text{and} \quad \pi_*(\mathcal{M}^+_{erg,\phi}) \cap \mathcal{M}_{erg,\psi} = \emptyset,$$

we say that all measures in $\mathcal{M}^+_{erg,\phi}$ degenerate or that the phenomenon of *measure degeneracy* happens for the pair ϕ and ψ.

Theorem 4.1. [18] *Let $\phi, \ \psi : X \times \mathbb{R} \to X$ be two topologically equivalent continuous flows. Then the phenomenon of entropy degeneracy happens for the pair ϕ and ψ iff the phenomenon of measure degeneracy happens for this pair.*

From (1) the corresponding orbits of topologically equivalent flows are time reparametrizations one of the other. The following two propositions from [18] explain the relation between these reparameterization and measure degeneracy in Theorem 4.1.

Proposition 4.2. [18] *Denote by X_0 the set of fixed points a flow ϕ on X and assume $\theta : (X \setminus X_0) \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following properties:*

- $\theta(x, 0) = 0$ and $\theta_x = \theta(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is strictly increasing;
- $\theta_x(s + t) = \theta_x(s) + \theta_{\phi_{s(x)}}(t)$.

Then, for a given $\mu \in \mathcal{M}_{erg,\phi}$ we have that for x μ-almost everywhere in X one of the following possibilities holds:

$$\lim_{t \to +\infty} \frac{\theta(x, t)}{t} = \lim_{t \to +\infty} \frac{\int_0^t \theta(\phi_s x, 1) \, ds}{t} = \int \theta(x, 1) \, d\mu$$

or

$$\limsup_{t \to +\infty} \frac{\theta(x, t)}{t} = \infty.$$

Proposition 4.3. [18] *Let $\pi : X \to X$ be a topological equivalence between the flows $\phi, \ \psi : X \times \mathbb{R} \to X$. Suppose that $\lim_{t \to +\infty} \frac{\theta(x,t)}{t} = \infty$ for x μ-almost everywhere in X , where μ is a non atomic and ϕ-invariant measure such that $Supp(\mu) \cap \{fixed\ points\ of\ \phi\} = \{p\}$. Then $\nu := \pi_* \mu$ is a ψ-ergodic and non invariant.*

Let $\pi : X \to X$ be a topological equivalence between two flows $\phi, \ \psi : X \times \mathbb{R} \to X$. There exist by Lemma 2.3 a function $\theta(x, t)$ meeting the assumption of Proposition 4.2 such that $\pi \circ \phi_t(x) = \psi_{\theta(x,t)} \circ \pi(x), \ \forall x \in X \setminus X_0, \ t \in \mathbb{R}$. For each given measure $\mu \in \mathcal{M}^+_{erg,\phi}$ there are two posibilities by Proposition 4.2: the limit of $\frac{\theta(x,t)}{t}$ is finite or not for x ϕ-almost everywhere in X. If the later case happens and the limit is infinte for every measure in $\mathcal{M}^+_{erg,\phi}$, then by Proposition 4.3 the measure degeneracy happens.

5. Topological chaotic and measure degenerated flows

Phenomena of entropy and measure degeneracy for transitive flows lead to transitive flows which are trivial from the statistical point of view in the sense that

$\mathcal{M}_{erg,\phi} = \{$ fixed points $\}$. We say that these flows are *topologically chaotic* and *measure degenerated*. The analogous concepts exist for homeomorphisms.

Using different approaches in different areas the examples of topologically chaotic but measure degenerated systems were constructed by Ohno [9] in the case of continuous flows, by Sun-Young-Zhou [15] in the case of C^r $(r \geq 1)$ flows and by He-Zhou[5] in the case of C^0 homeomorphisms. Taking the time one map from the example of Sun-Young-Zhou [15] yields a C^r $(r \geq 1)$ diffeomorphism which is topologically chaotic and measure degenerated.

For $r \geq 1$ we denote by $\mathcal{X}^r(M)$ the set of all C^r vector fields on a compact smooth Riemannian manifold M. We also denote by $\mathcal{T}^r(M)$ the subset of $\mathcal{X}^r(M)$ consisting of all C^r vector fields which are topologically chaotic and measure degenerated. One conjecture of Palis [10] claims that there exists a C^1 open and dense subset $\mathbb{O} \subset \mathcal{X}^1(M)$ such that any vector field in $\mathbb{O}$ is either Morse-Smale or has a transverse homoclinic intersection. The diffeomorphism version of this conjecture has been proved by Pujals and Sambarino [11] in dimension 2, by Bonatti, Gan and Wen [1] in dimension 3 and by Crovisier [4] in dimension ≥ 4. Transverse homoclinic intersection implies the existence of Smales horseshoes and thus positive topological entropy. The topological chaotic and measure degenerated flows generated by the vector fields in $\mathcal{T}^1(M)$ is neither Morse-Smale nor has a transverse homoclinic intersection. Usually these flows have some degree of non hyperbolicity, degeneracy and can reveal many new interesting phenomenons yet not explored. On the other hand it is clear from [18] that this set of flows is still not negligible since it contains infinite dimensional Banach submanifolds.

References

1. Ch. Bonatti, S. Gan, L. Wen, On the existence of non-trivial homoclinic classes, Erg. Th. Dyn. Syst., **27**(2007), 1473-1508

2. R. Bowen, P. Walters, On expansive one-parameter flows, J. Diff. Eq., **12** (1972),180-193.

3. P. Cornfeld, V. Fomin, Ya. Sinai, Erogodic theory, Springer-Verlag, New York, 1982

4. S. Crovisier, Birth of homoclinic intersections: a model for the central dynamics of partially hyperbolic systems, preprint, 2005

5. W. He, Z. Zhou, A topologically mixing system with its measure center being a singleton, Acta. Math. Sinica(in Chinses), **45** (2002), 929-934

6. M.R. Herman, Construction d'un diffeomorphisme minimal d'entropie topologique non nulle, *Erg. Th. Dyn. Systems*, **1** (1981), 65-76.

7. A. Katok, B. Hasselblatt, Introduction to the modern theory of dynamical systems, With a supplementary chapter by Katok and Leonardo Mendoza, Encyclopedia of Mathematics and its Applications, 54, Cambridge University Press, Cambridge, 1995

8. S. T. Liao, Standard systems of differential equations and obstruction sets— from linearity to perturbations, *System Researches, Proceedings dedicated to the 85th anniversary of Qian Xue-Sen,* Zhejiang Education Press, Hangzhou, China (1996), 279-290(in Chinese)

9. T. Ohno, A weak equivalence and topological entropy, *Publ. RIMS, Kyoto Univ.* **16** (1980), 289-298.

10. J. Palis, A global view of dynamics and a conjecture on the denseness of finitude of attractors. Gomerie complexe et systmes dynamiques (Orsay, 1995). Astrisque **261** (2000), xiii–xiv, 335–347.

11. E. Pujals, M. Sambarino, Homoclinic tangencies and hyperbolicity for surface diffeomorphisms, Ann. of Math. **151** (2000), no. 3, 961–1023.

12. V. A. Rohlin, Entropy of metric autonormophism, Dokl. Acad. Nauk USSR, **124**(1959)(in Russian)

13. W. Sun, Measure-theoretic entropy for flows, Sci. China A, **40**(1997), 725-731

14. W. Sun, Entropy of orthonormal n-frame flows, *Nonlinearity*, **14** (2001),829-842

15. W. Sun, T.Young , Y. Zhou, Topological entropies of equivalent smooth flows, to appear in Trans. Amer. Math. Soci.

16. W. Sun, E. Vargas, Entropy of flows, revisited, *Bol. Soc. Bra. Mat.* **30** (1999), 313-333

17. W. Sun, E. Vargas, Entropy and ergodic probability for smooth systems and their bundle extentions, Top. Appl. **154** (2007), 683-697

18. W. Sun, E. Vargas, Topological chaos and measure degeneracy, Preprint 2008

19. R. Thomas, Topological entropy of fixed-point free flows, *Trans. Amer. Math. Soc.*, **319** (1985), 601-618

20. R. Thomas, Entropy of expansive flows, *Erg. Th. Dyn. Systems*, **7** (1987), 611-625

21. H. Totoki, Time changes of flows, *Mem. Fac. Sci. Kyushu Univ., Ser. A*, **20** (1966), 27-55.

22. P. Walters, *An introduction to ergodic theory*, Springer Verlag, 1982

SCHOOL OF MATHEMATICAL SCIENCES - PEKING UNIVERSITY, CHINA
E-mail address: `sunwx@math.pku.edu.cn`

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA - UNIVERSIDADE DE SÃO PAULO, BRAZIL
E-mail address: `vargas@ime.usp.br`