Trends in Mathematics - New Series
Information Center for Mathematical Sciences
Volume 10, Number 2, 2008, pages 83–86

2008 International Workshop
on Dynamical Systems and
Related Topics

CONTINUITY OF CHAIN RECURRENT SETS FOR FLOWS

JONG-SUH PARK AND SE-HYUN KU*

Abstract. In this paper we prove that there is a residual subset R of the space of C^0-flows on a compact metric space X such that the maps $f \mapsto$ chain recurrent set for f and $f \mapsto$ number of chain components for f are continuous on R. Our result is a extended version of the Hurley's result to a continuous dynamical system.

The notions of the chain recurrences in study of flows on compact spaces is introduced by Conley [1, 2]. Hurley [7, 8] described the chain recurrence on spaces which are not compact and obtained some results about the chain recurrence on arbitrary metric spaces. Besides, he introduced the alternative definition of chain recurrence for semiflow and proved that the alternative definition is equivalent to the usual definition. And Chu and Park extended the concepts of chain recurrences to the multi-valued dynamical systems [3, 4]. In recent times, the idea of the chain recurrent set is extended to the space of $\mathbb{Z}^d$-actions by Oprocha [10].

In [9], Newhouse has shown that any residual subset of the set of C^r-diffeomorphisms ($r > 1$) on X must contain diffeomorphisms f with the number of chain components for f is infinite. Conley [2] described chain recurrent set using attractors and showed that the chain recurrence mapping is upper semicontinuous. Moreover, Hurley [5] has shown that there exist a residual subset J of the space of

2000 *Mathematics Subject Classification.* 37B20, 54H20.
Key words and phrases. chain recurrence, residual set, flow.

C^1-diffeomorphisms on any compact riemannian manifold X such that the maps

$$(1) f \mapsto \text{chain recurrent set for } f$$

$$(2) f \mapsto \text{number of chain components for } f$$

are continuous on J. Extending the above result to continuous dynamical systems on a compact metric space X is a goal of this paper. From now on $\Phi(X)$ is a set of all continuous flows on a compact metric space (X, d).

A continuous map $\phi : X \times (-\infty, \infty) \to X$ is a *flow* on X if f satisfying (1) $\phi(x, 0) = x$ for all $x \in X$ and (2) $\phi(\phi(x, s), t) = \phi(x, s + t)$ for all x and for all s, t. Let $\varepsilon > 0$ and $t > 0$. For any $x, y \in X$, we say that a sequence $\{(x_i, t_i)\}_{i=1}^n$ is an (ε, t)-*chain* from x to y for $\phi \in \Phi(X)$ if

$$(1)\, x_1 = x \text{ and } t_i \geq t \quad \text{for all} \quad i = 1, \cdots, n$$

$$(2)\, d(\phi(x_i, t_i), x_{i+1}) < \varepsilon \quad \text{for all} \quad i = 1, \cdots, n-1 \quad \text{and} \quad d(\phi(x_n, t_n), y) < \varepsilon.$$

And in case of $1 \leq t_i \leq 2$, (ε, t)-chain is called an ε-*chain*. Now, we can consider an equivalence relation on X. Two points x and y are *chain equivalent* if and only if for every $\varepsilon > 0$ and every $t > 0$, there exist two (ε, t)-chains from x to y and y to x. A point x in X is called *chain recurrent* with respect to ϕ and d if x is chain equivalent to itself. $CR(\phi)$ denotes the set of chain recurrent points of $\phi \in \Phi(X)$ and is called the *chain recurrent set* of ϕ. It is clearly that the chain equivalence relation is an equivalence relation on $CR(\phi)$. The notion of chain equivalence can be changed more simply, that is, two points x, y in $CR(\phi)$ are chain equivalent if and only if for any ε there exists two ε-chains from x to y and from y to x. An equivalence class under this equivalence relation for ϕ is *chain component* of ϕ. It turns out for flows on compact manifolds that the chain components are exactly the connected components of $CR(\phi)$. See [6].

To prove our theorem, we need some definitions and lemmas. Let X_1 be a metric space and X_2 a compact metric space. We define $F(X_2)$ be the set of all closed nonempty subsets of X_2 with the Hausdorff metric, that is,

$$d_H(A, B) = \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) : A, B \in F(X_2)\right\}.$$

A map $f : X_1 \to F(X_2)$ is called *upper*(*lower*) *semicontinuous* at $x \in X_1$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $f(y) \subset B(f(x), \varepsilon)(f(x) \subset B(f(y), \varepsilon))$ respectively.

In [11], a map $f : X_1 \to F(X_2)$ is lower semicontinuous at $x \in X_1$ if and only if for any open subset U of X_2 with $U \cap f(x) \neq \emptyset$, there exists a neighborhood V of x in X_1 such that $U \cap f(y) \neq \emptyset$ for all $y \in V$, and similarly f is upper semicontinuous at $x \in X_1$ if and only if for any open subset U of $f(x)$, there exists a neighborhood V of x in X_1 such that $f(y) \subseteq U$ for all $y \in V$.

Now, define a function $\rho : \Phi(X) \times \Phi(X) \to \mathbb{R}$ by setting

$$\rho(\phi, \psi) = \sup_{T>0} \Big\{ \min \Big\{ \max\{d(\phi(x,t), \psi(x,t)) : x \in X, -T \le t \le T\}, \frac{1}{T} \Big\} \Big\}$$

for all $\phi, \psi \in \Phi(X)$. Then we can check that ρ is a metric on $\Phi(X)$.

And we can define $C^0(X)$ is given by the set $\Phi(X)$ of continuous flows of X to itself with the metric ρ.

The following lemma is the integral continuity theorem of flow version and is helpfully used to prove the lemma.

Integral continuity theorem of flow version. *Let ϕ be a flow on metric space X. Let $x \in X$ and K be a compact subset of $\mathbb{R}^+$. For every positive number ε, there is a positive δ such that $d(x, y) < \delta$ implies $d(\phi(x,t), \phi(y,t)) < \varepsilon$, for every $t \in K$.*

In 1983, M. Hurley showed that if $f : X_1 \to F(X_2)$ be either upper or lower semicontinuous, then the set of all continuity points of f is a residual subset of X_1. See [5]. Recall that a subset S of a topological space X is *residual* if S can be realized as a countable intersection of open dense subsets of X. Using the above integral continuity theorem we obtain the next lemma.

Lemma. *Assume that a sequence (ϕ_n) converges to ϕ in $C^0(X)$ a sequence (x_n) converges to x and a sequence (y_n) also converges to y in X. Suppose that every positive integer n and positive number ε, there exists an ε-chain for ϕ_n from x_n to y_n. Then for every $\varepsilon > 0$, there is an ε-chain for ϕ from x to y.*

Now, we consider a map CR sending a continuous flow ϕ in $C^0(X)$ to a chain recurrent set $CR(\phi)$ for ϕ. Also we consider a function N from a continuous flow ϕ in $C^0(x)$ to a number $N(\phi)$ of chain components in the extended half line $[0, \infty]$(viewed as the one-point compactification of $[0, \infty)$). The following is our main theorem. A proof of the theorem is given now [12].

Main theorem *There is a residual subset R of $C^0(X)$ such that the maps CR and N are continuous at each point of R.*

References

[1] C. Conley, The gradient structure of a flow, *I.B.M. Res. RC 3932. Yorktown Heights, N.Y.*, 1972.

[2] C. Conley, Isolated Invariant Sets and the Morse Index, *C.M.B.S. 38, Amer.Math.Soc., Providence*, 1978.

[3] H. Chu, Chain recurrence for multi-valued dynamical systems on noncompact space, *Nonlinear Anal.* 61 (2005) 715-723.

[4] H. Chu, J. Park, Attractors for relations in σ-compact spaces, *Topology Appl.* 148 (2005) 201-212.

[5] M. Hurley, Bifurcation and chain recurrence, *Ergodic Theory Dyn. Syst.* 3 (1983) 231-240.

[6] M. Hurley, Fixed points of topologically stable flows, *Trans. Amer. Math. Soc.* 294 (1986) 625-633.

[7] M. Hurley, Chain recurrence and attraction in noncompact spaces, *Ergodic Theory Dyn. Syst.* 11 (1991) 709-729.

[8] M. Hurley, Chain recurrence, semiflow and gradient, *J. Dyn. Differ. Equations* 7 (1995) 437-456.

[9] S. Newhouse, Diffeomorphisms with infinitely many sinks, *Topology* 13 (1974) 9-18.

[10] P. Oprocha, Chain recurrence in multidimensional time discrete dynamical systems, *Discrete Contin. Dynam. Syst.* 20 (2008) 1039-1056.

[11] J. Park, D. Kang and H. Chu, Stabilities in multi-valued dynamical systems, *Nonlinear Anal.* 67 (2007) 2050-2059.

[12] J. Park, S. Ku, Characterizations on chain recurrences, In preparation.

Jong-Suh Park, Department of Mathematics, Chungnam National University, Daejeon 305-764, Korea

E-mail address: jspark@math.cnu.ac.kr

Se-Hyun Ku, Department of Mathematics, Chungnam National University, Daejeon 305-764, Korea

E-mail address: shku@cnu.ac.kr