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C^1-STABLY SHADOWABLE CHAIN COMPONENTS ARE HYPERBOLIC

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We call a property is a stable (or robust) property if it holds for a system as well as all nearby systems. In the literature of robust dynamics, stably shadable always means hyperbolicity.

Theorem 1 (K. Sakai). *Stably shadowable diffeomorphism satisfies Axiom A and the strong transversality.*

Theorem 2 (K. Moriyasu). *If the chain recurrent set is stably shadowable, then the chain recurrent set is hyperbolic.*

In this article, we consider a single chain component and prove the following theorem.

Theorem 3. *Let p be a hyperbolic periodic point of f and $C_f(p)$ be the chain component of f containing p. If $C_f(p)$ is C^1-stably shadowable, then $C_f(p)$ is hyperbolic.*

As usual, let M be a compact C^∞ Riemannian manifold without boundary, and $\mathrm{Diff}^1(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from the Riemannian metric on the tangent bundle.

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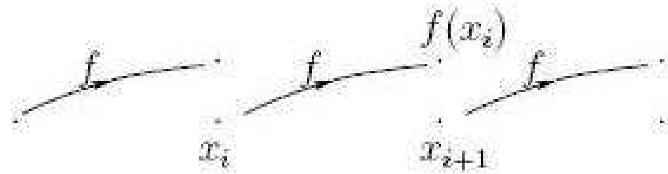
Definition 1. *For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b} \subset M$ $(-\infty \le a < b \le \infty)$ is called a δ-pseudo-orbit or a δ-chain of $f \in \mathrm{Diff}^1(M)$ if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$.*



Definition 2.
- *We call a point $x \in M$ is a Chain recurrent point , if for any $\delta > 0$, there exist δ-pseudo orbit $\{x_i\}_{i=0}^{n} (n > 0)$ such that $x_0 = x$ and $x_n = x$*
- *The set of chain recurrent points of f is called the chain recurrent set of f, denoted by $\mathrm{CR}(f)$.*
- *For given $x, y \in M$, we write $x \rightsquigarrow y$ if for every $\delta > 0$ there exists a δ-chain $\{x_i\}_{i=0}^{n}$ $(n > 0)$ such that $x_0 = x$ and $x_n = y$. We write $x \sim y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The equivalence class of $\sim$ are called chain component of f.*

Definition 3. *Let $\Lambda \subset M$ be a compact f-invariant set. We say Λ is shadowable, if for every $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=a}^{b} \subset \Lambda$ of f $(-\infty \le a < b \le \infty)$, there is $y \in M$ satisfying $d(f^i(y), x_i) < \epsilon$ for all $a \le i \le b$.*

Definition 4. *Let p be a hyperbolic periodic point of f. Denote $C_f(p)$ the chain component of f containing p. We say that $C_f(p)$ is C^1-stably shadowable if there exists a neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $g|_{C_g(p_g)}$ has the shadowing property, where p_g is the continuation of p.*

Our conclusion is based on the following theorem:

Theorem 4 (K. Sakai)**.** *Let $C_f(p)$ be C^1-stably shadowable. Then there exists a neighborhood $\mathcal{U}(f)$ of f and constants $m > 0$, $C > 0$ and $0 < \lambda < 1$ satisfying:*

1. *For every $g \in \mathcal{U}(f)$, $C_g(p_g)$ coincides with $H_f(O(p_g))$ and admit a dominated splitting $T_{C_g(p_g)}M = E(g) \oplus F(g)$ with $\dim E(g) = \mathrm{ind}(p_g)$.*
2. *For every $g \in \mathcal{U}(f)$, if $q \in C_g(p_g) \cap P(g)$ is hyperbolic, then $\mathrm{ind}(q) = \mathrm{ind}(p_g)$ and*

$$\prod_{i=0}^{k-1} \|Dg^m|_{E^s(g^{im}(q))}\| < C\lambda^k,$$

$$\prod_{i=0}^{k-1} \|Dg^{-m}|_{E^u(g^{-im}(q))}\| < C\lambda^k,$$

where $k = [\pi(q)/m]$ ($\pi(q)$ *represents the minimal period of* q *and* $[\cdot]$ *represents the integer part*).

Actually, we prove the following proposition.

Proposition 5 (Main proposition)**.** *Let* p *be a hyperbolic periodic point, and* $C_f(p)$ *be the chain component of* f *containing* p. *Let* $0 < \lambda < 1$, $L \geq 1$ *be given. Assume* Λ *satisfies the following properties* $(P1)$ *to* $(P4)$*:*

$(P1)$. $C_f(p)$ *equals the homoclinic class* $H_f(O(p))$.

$(P2)$. *There exist a continuous* Df*-invariant splitting* $T_\Lambda M = E \oplus F$ *with* $\dim E = \mathrm{ind}(p)$ *such that for every* $x \in \Lambda$,

$$\|Df|_{E(x)}\|/m(Df|_{F(x)}) < \lambda^2.$$

$(P3)$. *For any* $q \in P(f) \cap \Lambda$, *if* q *is hyperbolic and* $\pi(q) > L$, *then* $\mathrm{ind}(q) = \mathrm{ind}(p)$ *and*

$$\prod_{i=0}^{\pi(q)-1} \|Df|_{E^s(f^i(q))}\| < \lambda^{\pi(q)}$$

$$\prod_{i=0}^{\pi(q)-1} \|Df|_{E^u(f^{-i}(q))}\| < \lambda^{\pi(q)}.$$

$(P4)$. Λ *is shadowable.*

Then Λ *is hyperbolic for* f.

In our proof, the following notion of quasi hyperbolic string plays the most important role.

Definition 5. *Let* $f \in \mathrm{Diff}^1(M)$, Λ *be a compact invariant set of* f, *and* $T_\Lambda M = E \oplus F$ *be a continuous* Df*-invariant splitting. Let* $\lambda \in (0,1)$. *An orbit string* $(x, f^n x) = \{x, fx, f^2x, ..., f^nx\}$ *in* Λ *is called a* λ*-quasi hyperbolic string with respect to the splitting of* $E \oplus F$ *if the following conditions are satisfied:*

(1) $\|Df|_{E(f^ix)}\|/m(Df|_{F(f^ix)}) \leq \lambda^2$ *for every* $i = 0, ..., n-1$.
(2) $\prod_{i=0}^{k-1} \|Df|_{E(f^ix)}\| \leq \lambda^k$, *for* $k = 1, 2, ..., n$.
(3) $\prod_{i=k-1}^{n-1} m(Df|_{F(f^ix)}) \geq \lambda^{k-n-1}$ *for* $k = 1, ..., n$.

Remark 6. *The first condition just correspond to dominated splitting.*

The most important property of a quasi-hyperbolic string is that it can be shadowed by a periodic point if the two end points of the strings are sufficiently close:

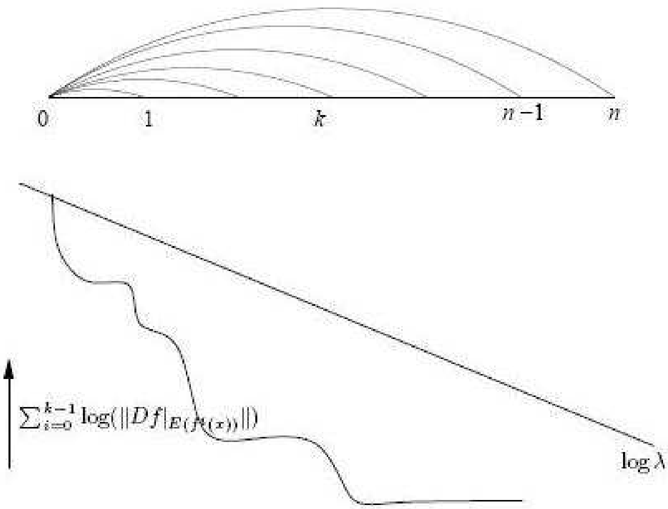


Figure 1. The second and third inequalities in quasi hyperbolic strings

Theorem 7 (Liao)**.** *Let $T_\Lambda M = E \oplus F$ be a continuous Df-invariant splitting. For any $0 < \lambda < 1$ and any $\epsilon > 0$, there is $\delta > 0$ such that for any λ-quasi hyperbolic string $(x, f^n x)$ of f in Λ with $d(f^n x, x) \le \delta$, there is a periodic point $p \in M$ of f such that $f^n(p) = p$ and $d(f^i p, f^i x) \le \epsilon$ for all $0 \le i \le n-1$.*

The main issue in the application of quasi hyperbolic string is finding a quasi-hyperbolic string with two end points sufficiently close. quasi hyperbolic strings. We construct the following lemma in this article.

Lemma 8. *Let $\{a_i\}_{i=0}^{\infty}$ be an infinite sequence with all $|a_i| < K$ for a uniform constant K. Assume*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} a_i = \xi$$

and

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} a_i = \xi',$$

where $\xi' < \xi$.

Then for any ξ_1, ξ_2 with $\xi_1 < \xi < \xi_2$, there is an infinite sequence $\{m_i\}_{i=1}^{\infty} \subset \mathbb{N}$ such that for every $i = 1, 2, \ldots$ and every $k = 1, \ldots, m_{i+1} - m_i$,

$$\frac{1}{k}\sum_{j=m_i}^{m_i+k-1} a_j \le \xi_2,$$

$$\frac{1}{k}\sum_{j=m_{i+1}-k}^{m_{i+1}-1} a_j \geq \xi_1.$$

Remark 9. *Note that the last two inequalities are just the conditions for what we call a (ξ_1, ξ_2)-"Liao string".*

Applying above lemma, we can get the following proposition.

Proposition 10. *Let $0 < \lambda < 1$ be given. Let Λ be a compact invariant set of f with a continuous Df-invariant splitting $T_\Lambda M = E \oplus F$ such that*

$$\frac{\|Df|_{E(x)}\|}{m(Df|_{F(x)})} < \lambda^2$$

for any $x \in \Lambda$. Assume there is a point $a \in \Lambda$ satisfying

$$\log\lambda < \log\lambda_1 = \limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log(\|Df|_{E(f^i a)}) < 0$$

and

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log(\|Df|_{E(f^i a)}) < \log\lambda_1.$$

Then for any λ_2 and λ_3 with $\lambda < \lambda_2 < \lambda_1 < \lambda_3 < 1$, and any neighborhood U of Λ, there exist a hyperbolic period point q of index $\dim(E)$ such that its orbit $O(q)$ is entirely contained in U and the derivatives along $O(q)$ satisfies

$$\prod_{i=0}^{k-1}\|Df|_{E^s(f^i q)}\| < \lambda_3^k,$$

$$\prod_{i=k-1}^{\pi(q)-1}\|Df|_{E^s(f^i q)}\| > \lambda_2^{\pi(q)-k+1},$$

for all $k = 1, 2, \cdots, \pi(q)$. Furthermore, q can be chosen such that $\pi(q)$ is arbitrarily large.

Remark 11. *By the assumption of dominated splitting,*

$$\frac{\|Df|_{E(x)}\|}{m(Df|_{F(x)})} < \lambda^2,$$

the second inequalities imply

$$\prod_{i=0}^{k-1}\|Df^{-1}|_{E^u(f^{-i}q)}\| < (\lambda^2/\lambda_2)^k,$$

for every $k > 0$.

We will prove the main proposition by reduction to contradiction. Assume $C_f(p)$ is not hyperbolic, then either E is not contracting or F is not expanding. Without lost of generality, we assume E is not contracting.

Claim 1. Since E is not contracting, we can find a "bad" point b in $C_f(p)$ such that

$$\prod_{j=0}^{n-1} \|Df|_{E(f^j b)}\| \geq 1$$

for any $n > 0$.

We construct a pseudo orbit, which connected by the segments hyperbolic periodic orbits and the orbit segments of b. By shadowing these pseudo orbits, we can get the following claim.

Claim 2. For any $\lambda < \eta < \eta' < 1$ and any neighborhood V of $C_f(p)$, there exists $a \in V$ with $O(a) \subset V$ such that

$$\liminf_{n\to+\infty} \sum_{j=0}^{n-1} \log(\|Df|_{E(f^j a)}\|) < \eta < \limsup_{n\to+\infty} \sum_{j=0}^{n-1} \log(\|Df|_{E(f^j a)}\|) < \eta'$$

By the above claim and proposition 10, we can obtain: **Claim 3.** For any $\lambda < \eta < \eta' < 1$ and any neighborhood V of Λ, there exist a hyperbolic period point q of index $\dim(E)$ such that its orbit $O(q)$ is entirely contained in V and the derivatives along $O(q)$ satisfies

$$\prod_{i=0}^{k-1} \|Df|_{E^s(f^i q)}\| \leq \eta'^k,$$

$$\prod_{i=k-1}^{\pi(q)-1} \|Df|_{E^s(f^i q)}\| \geq \eta^{\pi(q)-k+1},$$

for all $k = 1, 2, \cdots, \pi(q)$. Furthermore, q can be chosen such that $\pi(q) > L$.

Proof of main proposition. By Claim 3, we can find a sequence of hyperbolic periodic point $q_k \in M$ converge to a point $z \in \Lambda$. For every q_k, we have

$$\prod_{i=0}^{n-1} \|Df|_{E(f^i(q_k))}\| < \eta'^n, \text{ for any } n \geq 1,$$

$$\prod_{i=0}^{n-1} \|Df^{-1}|_{E(f^{-i}(q_k))}\| < (\frac{\lambda^2}{\eta})^n, \text{ for any } n \geq 1.$$

Since q_k have uniform size stably and unstably manifolds, there exist $N \geq 1$ such that for any $k, l \geq N$, q_k and q_l are homoclinicly related, meaning $W^s(q_k) \pitchfork W^u(q_l)$ and $W^u(q_k) \pitchfork W^s(q_l)$. Therefore, $z \in H_f(O(q_k))$ for all $k \geq N$. Hence $q_k \in C_f(p)$ for all $k \geq N$, contradicting hypothesis (P2). This proves the main proposition.

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