

INTERGRID OPERATORS FOR THE CELL CENTERED FINITE DIFFERENCE MULTIGRID ALGORITHM ON RECTANGULAR GRIDS

JUN S. LEE

ABSTRACT. We introduce intergrid operator recently developed for the cell centered finite difference multigrid on rectangular grids. The main idea of operator construction is based on flux continuity and certain kind of interpolation. This operator works well for solving diffusion equations both with discontinuous coefficient and with smooth coefficient. We discuss on the construction of prolongation operators and compare this operator with the weight prolongation and trivial injection both theoretically and numerically. We present various numerical results to show that multigrid algorithm with our prolongation works well for various interface problems.

1. INTRODUCTION

The cell-centered finite difference (CCFD) is a finite volume type of method and has been used by many engineers for solving elliptic problems with discontinuous diffusion due to its simplicity and local conservation. Of many such problems, a few examples are flows through porous media with different porosity, electric currents through material of different conductivities and heat flows through heterogeneous materials, *etc.* [7, 18]. The numerical methods treating such problems are important areas of research. It also arises from the saddle point formulation of the Raviart-Thomas mixed method by taking certain quadrature.

On the other hand, the multigrid (MG) algorithms has proven to be very effective for a large class of problems [6, 15] and has been the subject of extensive research [2, 3, 4, 13]. The multigrid algorithm for CCFD was considered by a few authors [2, 9, 12, 13, 17] and its W -cycle convergence for Laplace equation was first proved by Bramble *et al.* in [2]. However, their use of natural injection as a prolongation operator is not an optimal choice on triangular and rectangular grids.

On rectangular grid, the V -cycle multigrid convergence with the natural injection is slow. As is shown by Kwak [13], certain weighted prolongation works much better and guarantees V -cycle convergence. Still, it is restricted to problems with smooth coefficients. It does not work well when the jump of the diffusion coefficient is severe. In early years, there have been some efforts to handle discontinuous

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coefficient problems using Galerkin coarse grid approximation[12, 17]. In the finite difference case, Alcouffe *et al.*[1] and Kettler[11] suggested a prolongation operator based on the continuity of flux at finer grid points. However, in this method stiffness matrices of the coarse grids no longer have 5-point structure and the prolongation is nontrivial. Hence extra cost is needed to generate stiffness matrices and its implementation is difficult. Recently, Kwak and Lee developed new prolongation depending on the diffusion and shows W -cycle multigrid convergence[14]. In numerical test, the multigrid with this prolongation yields better convergence than with the natural injection.

The rest of this paper is as follows. In section 2, we briefly describe CCFD for elliptic problem and multigrid algorithm. In section 3, we consider natural injection and modified bilinear prolongation of multigrid method for CCFD and estimate energy norms to get V -cycle convergence and W -cycle convergence. Finally, we give some numerical results for two prolongations in section 4. We report maximum, minimum eigenvalues, condition numbers and contraction numbers of two multigrid algorithms.

2. MULTIGRID AND CELL-CENTERED FINITE DIFFERENCE METHOD

In this section, we briefly describe CCFD on rectangular grid and multigrid algorithm to solve the resulting system of linear equations. We first consider the following model problem :

$$(2.1) \quad -\nabla \cdot p \nabla \tilde{u} = f \quad \text{in } \Omega,$$

$$(2.2) \quad \tilde{u} = 0 \quad \text{on } \partial\Omega.$$

For simplicity, we assume that Ω is the unit square and $\tilde{u} \in H^{1+\alpha}(\Omega)$ for some $0 < \alpha \leq 1$. For $k = 1, 2, \dots, J$, divide Ω into $n \times n$ axis-parallel subsquares, where $n = 2^k$. Such triangulations are denoted by $\{\mathcal{E}_k\}$. Each subsquare in \mathcal{E}_k is called a cell and denoted by E_{ij}^k , $i, j = 1, \dots, 2^k$ and has $u_{i,j}$ as its value at center. For $k = 1, 2, \dots, J$, let V_k denote the space of functions which are piecewise constant on each cell. Integrating by parts (2.1) on each cell and replacing the normal derivative $\partial\tilde{u}/\partial n$ on the edges by difference quotient of function u in V_k , we have the finite difference equations as follows:

$$\begin{aligned} & -p_{i-1/2,j}(u_{i-1,j} - u_{i,j}) - p_{i+1/2,j}(u_{i+1,j} - u_{i,j}) \\ & -p_{i,j-1/2}(u_{i,j-1} - u_{i,j}) - p_{i,j+1/2}(u_{i,j+1} - u_{i,j}) = f_{i,j}h^2, \end{aligned}$$

where $h = h_k = 1/2^k$ and $p_{i-1/2,j} = p(x_{i-1/2}, y_j)$, *etc.* When p is discontinuous along the interface, we take $p_{i-1/2,j}$ as

$$p_{i-1/2,j} = \frac{2p_{i-1,j}p_{i,j}}{p_{i-1,j} + p_{i,j}}.$$

Similarly, $p_{i+1/2,j}$, $p_{i,j-1/2}$ and $p_{i,j+1/2}$ are defined. When one of the edge coincides with the boundary of Ω , we assume a fictitious value by reflection. For example, $u_{0,j}$ is taken as $-u_{1,j}$ and thus $\frac{\partial\tilde{u}}{\partial n}$ at $x = 0$ is approximated by $-2u_{1,j}/h$. Similar

rules apply to the other parts of the boundary of Ω . After dividing the resulting equation by h^2 , we obtain a system of linear equation of the form

$$(2.3) \quad A_k u = f,$$

where A_k is the typical sparse, $n^2 \times n^2$ symmetric, positive definite matrix similar to those arising in the vertex finite difference method, and u is the vector whose entries are $u_{i,j}$ and f is the vector whose entries are $f(x_i, y_j)$. We identify the vector u, v in V_k with their matrix representation in R^{2^k} . For analysis, we define a quadratic form $A_k(\cdot, \cdot)$ on $V_k \times V_k$ in an obvious manner by

$$(2.4) \quad A_k(u, v) = h_k^2 \sum_{i,j} (A_k u)_{i,j} v_{i,j}, \quad \forall u, v \in V_k.$$

Then (2.3) is equivalent to the following problem: Find $u \in V_k$ satisfying

$$(2.5) \quad A_k(u, \phi) = (f, \phi), \quad \forall \phi \in V_k,$$

where (\cdot, \cdot) is the L^2 inner product. The error analysis of the cell centered finite difference method is well known (cf. [5],[8],[10]). Let $Q_k : L^2(\Omega) \rightarrow V_k$ denote the usual $L^2(\Omega)$ projection. If u is the solution of (2.5), then

$$A_k(u - Q_k \tilde{u}, u - Q_k \tilde{u}) \leq Ch_k^2 \|f\|^2,$$

where $\|\cdot\|$ is the usual L^2 norm. Given a certain prolongation operator $I_{k-1}^k : V_{k-1} \rightarrow V_k$, we define the restriction operator $I_k^{k-1} : V_k \rightarrow V_{k-1}$ as its adjoint with respect to $(\cdot, \cdot)_k$:

$$(I_k^{k-1} u, v) = (u, I_{k-1}^k v), \quad \forall u \in V_k, \forall v \in V_{k-1}.$$

Since the space V_k can be viewed as the space of vectors having entries $u_{i,j}$, we also use I_{k-1}^k and I_k^{k-1} to denote their matrix representations. Now multigrid algorithm for solving (2.3) is defined as follows :

Multigrid Algorithm

Set $B_0 = A_0^{-1}$. For $1 < k \leq J$, assume that B_{k-1} has been defined and define $B_k f$ for $f \in V_k$ as follows:

- (1) Set $x^0 = 0$ and $q^0 = 0$.
- (2) Define x^l for $l = 1, \dots, m(k)$ by

$$x^l = x^{l-1} + R_k^{(l+m(k))} (f - A_k x^{l-1}).$$

- (3) Define $y^{m(k)} = x^{m(k)} + I_{k-1}^k q^s$, where q^i for $i = 1, \dots, s$ is defined by

$$q^i = q^{i-1} + B_{k-1} \left[P_{k-1}^0 (f - A_k x^{m(k)}) - A_{k-1} q^{i-1} \right]$$

where $P_{k-1}^0 : V_k \rightarrow V_{k-1}$ is the L^2 -projection.

- (4) Define y^l for $l = m(k) + 1, \dots, 2m(k)$ by

$$y^l = y^{l-1} + R_k^{(l+m(k))} (f - A_k y^{l-1}).$$

- (5) Set $B_k f = y^{2m(k)}$.

If $s = 1$, we obtain $V(m(k), m(k))$ -cycle and if $s = 2$, we obtain $W(m(k), m(k))$ -cycle. If $m(k)$ varies with k , we call it variable V -cycle. Here $R_k^{(l)}$ is a smoother on V_k which alternates between R_k and its adjoint R_k^t and $m(k)$ is the number of smoothings which can vary depending on k . The smoother R_k , as usual, can be taken as the Jacobi or Gauss-Seidel relaxation.

3. PROLONGATION OPERATORS ON RECTANGULAR GRIDS

In this section, first, we briefly introduce trivial injection and bilinear prolongation. Multigrid with bilinear prolongation operator works well for smooth problems but not for nonsmooth problems. On the other hand, multigrid with trivial injection converges for nonsmooth problems but not for smooth problems. Next, we consider how to extend bilinear prolongation operator to modified one, which works well both for smooth and nonsmooth problems. Let $E_{i,j}^{k-1}$ be a cell at level $k - 1$, and $v_{i,j}$ be the values at the center of each cells. Let $u_{I,J}$ be the value of upper right subcell of $E_{i,j}^{k-1}$ and $u_{I,J_1}, u_{I_1,J}$ and u_{I_1,J_1} be those of lower right, upper left, lower left subcells of $E_{i,j}^{k-1}$. (See Figure 1(b)) Here we use the notation $I^1 = I + 1, J_1 = J - 1, etc.$ We first define trivial injection. For $v \in V_{k-1}$, define $u = I_{k-1}^t v$ as follows:

$$(3.1) \quad u_{I,J} = v_{i,j}, \quad u_{I^1,J} = v_{i+1,j}, \quad u_{I,J^1} = v_{i,j+1}, \quad u_{I^1,J^1} = v_{i+1,j+1}.$$

Next, we define bilinear prolongation operator. For $v \in V_{k-1}$, define $u = I_{k-1}^b v$ as follows:

$$(3.2) \quad \begin{aligned} u_{I,J} &= \frac{9v_{i,j} + 3v_{i,j+1} + 3v_{i+1,j} + v_{i+1,j+1}}{16}, \\ u_{I^1,J} &= \frac{9v_{i+1,j} + 3v_{i,j} + 3v_{i+1,j+1} + v_{i,j+1}}{16}, \\ u_{I,J^1} &= \frac{9v_{i,j+1} + 3v_{i,j} + 3v_{i+1,j+1} + v_{i+1,j}}{16}, \\ u_{I^1,J^1} &= \frac{9v_{i+1,j+1} + 3v_{i+1,j} + 3v_{i,j+1} + v_{i,j}}{16}. \end{aligned}$$

For trivial injection, we have the following energy norm estimation:

Proposition 3.1. *We assume that p is piecewise constant on each cell. We have*

$$(3.3) \quad A_k(I_{k-1}^t v, I_{k-1}^t v) \leq 2A_{k-1}(v, v), \text{ for } v \in V_{k-1}.$$

This result together with the regularity and approximation property, which can be shown exactly the same way as in [13], we can deduce that W -cycle multigrid algorithm converges when p is piecewise constant. Note that even when p is discontinuous, W -cycle multigrid convergence is guaranteed.

On the other hand, by the technique in [13, 14], we have the following energy norm estimation for bilinear prolongation:

Proposition 3.2. *We have*

$$(3.4) \quad A_k(I_{k-1}^b v, I_{k-1}^b v) \leq C_* A_{k-1}(v, v), \text{ for } v \in V_{k-1}.$$

where $C_* = 1$ if p is constant, and $1 + C(p)h_k$ if p is Lipschitz continuous.

In this case, we can deduce that V -cycle multigrid algorithm converges when p is constant and is a good preconditioner when p is smooth. But the V -cycle convergence is not guaranteed when p is nonsmooth. In fact, the energy norm of I_{k-1}^b goes to infinity as the jump of p grows and numerical result shows that it diverges when p is not smooth. So we want to devise a new prolongation whose energy norm remains bounded even if the jump ratio of the coefficients approaches ∞ . For a motivation, we consider the following one dimensional diffusion equation on $[0, 1]$.

$$\begin{cases} -\frac{d}{dx} \left(p \frac{du(x)}{dx} \right) = f & \text{in } (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$

Let $x_i, i = 1, 2, \dots, n$ be the center of equally spaced grid on $[0, 1]$ and h be the mesh size. When this grid is refined, new center will have locations at $x_{i+1/4}, x_{i+3/4}, \text{ etc.}$ Now, we consider how to define the values at $x_{i+1/4}$. A natural way is to impose the flux continuity at $x_{i+1/4}$, so that the following equality holds:

$$p_i \frac{v_{i+1/4} - v_i}{\frac{h}{4}} = p_{i+1} \frac{v_{i+1} - v_{i+1/4}}{\frac{3h}{4}}.$$

Solving, we get

$$v_{i+1/4} = \frac{3p_i v_i + p_{i+1} v_{i+1}}{3p_i + p_{i+1}}.$$

This can be viewed as the linear interpolation of neighboring point with diffusion weights. Motivated by this, we define a new prolongation I_{k-1}^m for two dimensional problem as follows:

$$\begin{aligned} u_{I,J} &= \frac{9p_{i,j}v_{i,j} + 3p_{i,j+1}v_{i,j+1} + 3p_{i+1,j}v_{i+1,j} + p_{i+1,j+1}v_{i+1,j+1}}{9p_{i,j} + 3p_{i+1,j} + 3p_{i,j+1} + p_{i+1,j+1}}, \\ u_{I^1,J} &= \frac{9p_{i+1,j}v_{i+1,j} + 3p_{i,j}v_{i,j} + 3p_{i+1,j+1}v_{i+1,j+1} + p_{i,j+1}v_{i,j+1}}{9p_{i+1,j} + 3p_{i,j} + 3p_{i+1,j+1} + p_{i,j+1}}, \\ u_{I,J^1} &= \frac{9p_{i,j+1}v_{i,j+1} + 3p_{i,j}v_{i,j} + 3p_{i+1,j+1}v_{i+1,j+1} + p_{i+1,j}v_{i+1,j}}{9p_{i,j+1} + 3p_{i,j} + 3p_{i+1,j+1} + p_{i+1,j}}, \\ u_{I^1,J^1} &= \frac{9p_{i+1,j+1}v_{i+1,j+1} + 3p_{i+1,j}v_{i+1,j} + 3p_{i,j+1}v_{i,j+1} + p_{i,j}v_{i,j}}{9p_{i+1,j+1} + 3p_{i+1,j} + 3p_{i,j+1} + p_{i,j}}. \end{aligned}$$

It is easy to check that this prolongation operator reduces to the bilinear one when the diffusion p is constant. For modified bilinear prolongation, we have the following:

Proposition 3.3. *Let $p = p_1$ for $x < 1/2$ and $p = p_2$ for $x > 1/2$. Then for any pair of positive constants p_1 and p_2 , we have*

$$(3.5) \quad A_k(I_{k-1}^m v, I_{k-1}^m v) \leq \frac{11}{8} A_{k-1}(v, v), \quad \forall v \in V_{k-1}.$$

Now we consider ‘‘regularity and approximation property’’. Since p is discontinuous, it is not trivial to prove this property. Fortunately, by adopting a nearby

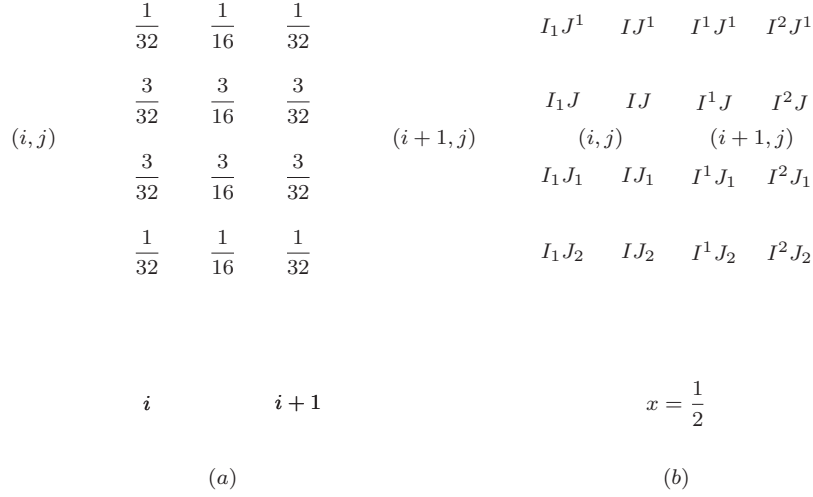


FIGURE 1. *Bilinear contributions and indices* ($I_1 = I - 1, J^1 = J + 1$)

smooth problem which has enough regularity, Kwak and Lee showed “regularity and approximation property” of discontinuous problem(See for the detail[14]).

Lemma 3.1. *Let the operator P_{k-1} be elliptic projection. Then*

$$A_k((I - I_{k-1}P_{k-1})u, u) \leq C_\alpha \left(\frac{\|A_k u\|}{\lambda_k} \right) A_k(u, u)^{1-\alpha}, \quad \forall u \in V_k$$

holds for $\alpha = \frac{1}{2}$. Here, λ_k is the largest eigenvalue of A_k and I_{k-1} is I_{k-1}^t, I_{k-1}^b or I_{k-1}^m .

This together with (3.3), we obtain the following result for I_{k-1}^t .

Theorem 3.1. *Let $E_k = I - B_k A_k$ in multigrid algorithm with $s = 2$ (W-cycle algorithm). Then, for any m , we have*

$$(3.6) \quad A_k(E_k u, E_k u) \leq \delta_k A_k(u, u), \quad \forall u \in V_k,$$

where $\delta_k = \frac{M}{M + \sqrt{m}}$. In particular, W-cycle works with one smoothing.

By the energy norm estimates (3.4) and (3.5), we have the following results for I_{k-1}^m .

Theorem 3.2. *First, let p be constant and let $E_k = I - B_k A_k$ in multigrid algorithm with $s = 1$ (V-cycle algorithm). Then, for any m , we have*

$$0 \leq A_k(E_k u, u) \leq \delta_k A_k(u, u), \quad \forall u \in V_k,$$

where $\delta_k = \frac{CM}{CM+\sqrt{m}}$. Second, let p has a simple discontinuity and let $E_k = I - B_k A_k$ in multigrid algorithm with $s = 2$ (W -cycle algorithm). Then, for any m , we have (3.6).

Remark 3.1. In case of the modified prolongation I_{k-1}^m , more general discontinuity can be handled similarly as long as the discontinuity arise as jumps along some line segments parallel to the axes.

4. NUMERICAL RESULTS

We consider the following problem on the unit square (rectangular grid):

$$(4.1) \quad \begin{aligned} -\nabla \cdot p \nabla \tilde{u} &= f & \text{in } \Omega = (0, 1)^2, \\ \tilde{u} &= 0 & \text{on } \partial\Omega. \end{aligned}$$

In all problems, we use one pre-post Gauss-Seidel relaxation, i.e., $V(1, 1)$. We report the eigenvalues, condition numbers, and reduction factors of trivial, bilinear and modified bilinear operators.

Problem 1. The domain is the unit square and the diffusion coefficient p is 1. In this case, the modified bilinear prolongation is similar to the weighted prolongation and works better than the trivial injection.

TABLE 1. $V(1, 1)$ -cycle with natural injection, Figure 2 (a).

| h_J | λ_{min} | λ_{max} | K | δ |
|-------|-----------------|-----------------|-------|----------|
| 1/32 | 0.784 | 1.444 | 1.842 | 0.218 |
| 1/64 | 0.783 | 1.534 | 1.960 | 0.309 |
| 1/128 | 0.784 | 1.618 | 2.060 | 0.403 |

TABLE 2. $V(1, 1)$ -cycle with modified bilinear prolongation, Figure 2 (a).

| h_J | λ_{min} | λ_{max} | K | δ |
|-------|-----------------|-----------------|-------|----------|
| 1/32 | 0.693 | 0.999 | 1.444 | 0.075 |
| 1/64 | 0.688 | 0.999 | 1.454 | 0.081 |
| 1/128 | 0.683 | 0.999 | 1.463 | 0.083 |

Problem 2. The domain is still the unit square and has simple discontinuity on the line $x = 1/2$ and $0 < y < 1$ (See Figure 2 (b)). We observed that multigrid with standard bilinear prolongation diverges. Multigrid algorithms with trivial and modified bilinear prolongation converge. But the modified bilinear prolongation converges faster and the contraction number is independent of number of levels.

TABLE 3. $V(1, 1)$ -cycle with natural injection, $p_1 = 1, p_2 = 10^3$, Figure 2 (b).

| h_J | λ_{min} | λ_{max} | K | δ |
|-------|-----------------|-----------------|-------|----------|
| 1/32 | 0.769 | 1.491 | 1.938 | 0.210 |
| 1/64 | 0.770 | 1.574 | 2.045 | 0.298 |
| 1/128 | 0.779 | 1.649 | 2.116 | 0.389 |

TABLE 4. $V(1, 1)$ -cycle with bilinear prolongation, $p_1 = 1, p_2 = 10^3$, Figure 2 (b).

| h_J | λ_{min} | λ_{max} | K | δ |
|-------|-----------------|-----------------|--------|----------|
| 1/32 | 0.405 | 24.258 | 59.823 | > 1 |
| 1/64 | 0.737 | 30.457 | 41.304 | > 1 |
| 1/128 | 0.876 | 35.969 | 41.065 | > 1 |

TABLE 5. $V(1, 1)$ -cycle with modified bilinear prolongation, $p_1 = 1, p_2 = 10^3$, Figure 2 (b).

| h_J | λ_{min} | λ_{max} | K | δ |
|-------|-----------------|-----------------|-------|----------|
| 1/32 | 0.628 | 0.999 | 1.591 | 0.191 |
| 1/64 | 0.605 | 0.999 | 1.652 | 0.221 |
| 1/128 | 0.592 | 0.999 | 1.690 | 0.243 |

 $p = 1$ p_1 p_2

(a)smooth

(b)nonsmooth

FIGURE 2. test problems

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DEPARTMENT OF MATHEMATICS, KAIST, TAEJON, KOREA 305-701

E-mail address: jslee@math.kaist.ac.kr