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# HYPERBOLICITY OF $C^1$-STABLY EXPANSIVE HOMOCLINIC CLASSES

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Abstract. Let $f$ be a diffeomorphism of a compact $C^\infty$ manifold, and let $p$ be a hyperbolic periodic point of $f$. In this paper we introduce the notion of $C^1$-stable expansivity for a closed $f$-invariant set, and prove that ($i$) the chain recurrent set $\mathcal{R}(f)$ of $f$ is $C^1$-stably expansive if and only if $f$ satisfies both Axiom A and no-cycle condition, ($ii$) the homoclinic class $H_f(p)$ of $f$ associated to $p$ is $C^1$-stably expansive if and only if $H_f(p)$ is hyperbolic, and ($iii$) $C^1$-generically, the homoclinic class $H_f(p)$ is $C^1$-stably expansive if and only if $H_f(p)$ is $C^1$-robustly expansive and the $H_f(p)$-germ of $f$ is expansive.

## 1. Introduction

It has been a problem in differentiable dynamical systems during last decades to understand the influence of a robust dynamic property (i.e. a property that holds for a system and all nearby ones) on the behavior of the tangent map of the system. For instance, Mañé [6] showed that if $M$ is a compact manifold and a diffeomorphism $f$ of $M$ and all $C^1$ nearby ones are expansive then $f$ is a quasi-Anosov system; i.e., any nonzero vector grows exponentially in norm by forward or backward iterations of the tangent map. In this paper, we study the case when the homoclinic class $H_f(p)$ of $f$ associated to a hyperbolic periodic point $p$ is $C^1$-stably expansive. Let us be more precise.

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Let $M$ be a compact $C^\infty$ manifold, and let Diff(M) be the space of diffeomorphisms of $M$ endowed with the $C^1$-topology. Denote by $d$ the distance on $M$ induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle $TM$. Let$f \in$ Diff(M), and let $\Lambda \subset M$ be a compact $f$-invariant set. We say that $f$ restricted to $\Lambda$, $f|_\Lambda$, is *expansive* if there is $\alpha > 0$ such that for any pair of distinct points $x, y \in \Lambda$, $d(f^n(x), f^n(y)) > \alpha$ for some $n \in \mathbb{Z}$ . The number $\alpha > 0$ is called an *expansive constant* for $f|_\Lambda$. We know that if $\Lambda$ is hyperbolic for $f$ then $f|_\Lambda$ is expansive. Note that $f|_\Lambda$ is expansive if and only if $f^n|_\Lambda$ is expansive for some $n \in \mathbb{Z}$. Moreover, we say that the $\Lambda$-*germ of* $f$ *is expansive* if there is $\alpha > 0$ such that if $x \in \Lambda,\ y \in M$ and $d(f^n(x), f^n(y)) \leq \alpha$ for all $n \in \mathbb{Z}$ then $x = y$. Expansiveness is a property shared by a large class of dynamical systems exhibiting chaotic behavior.

It is well known that if $p$ is a hyperbolic periodic point of $f$ with period $k$ then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\} \text{ and}$$
$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are $C^1$-injectively immersed submanifolds of $M$. A point $x \in W^s(p) \cap W^u(p)$ is called a *homoclinic point* of $f$ associated to $p$, and it is said to be a *transversal homoclinic point* of $f$ if the above intersection is transversal at $x$; i.e., $x \in W^s(p) \pitchfork W^u(p)$. The closure of the homoclinic points of $f$ associated to $p$ is called the *homoclinic class* of $f$ associated to $p$, and it is denoted by $H_f(p)$. The closure of the transversal homoclinic points of $f$ associated to $p$ is called the *transversal homoclinic class* of $f$ associated to $p$, and it is denoted by $H_f^T(p)$. It is clear that both $H_f(p)$ and $H_f^T(p)$ are compact and invariant sets. Homoclinic classes are the natural candidates to replace hyperbolic basic sets in nonhyperbolic theory. Several recent papers ([8, 9, 12, 15, 16, 17]) explore their "hyperbolic-like" properties, many of which hold only for generic diffeomorphisms.

Let $q$ be a hyperbolic periodic point of $f$. We say that $p$ and $q$ are *homoclinic related*, and write $p \sim q$ if

$$W^s(p) \pitchfork W^u(q) \neq \phi \text{ and } W^u(p) \pitchfork W^s(q) \neq \phi.$$

It is clear that if $p \sim q$ then index$(p)$ = index$(q)$; i.e., $\dim W^s(p) = \dim W^u(q)$. By the Smale's transeverse homoclinic point theorem, $H_f^T(p)$ coincides with the closure of the set of hyperbolic periodic points $q$ of $f$ such that $p \sim q$. Note that if $p$ is a hyperbolic periodic point of $f$ then there is a neighborhood $U$ of $p$ and a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ such that for any $g \in \mathcal{U}(f)$ there exists a unique hyperbolic periodic point $p_g$ of $g$ in $U$ with the same period as $p$ and index$(p_g)$ = index$(p)$. Such that point $p_g$ is called the *continuation* of $p = p_f$.

We say that a homoclinic class $H_f(p)$ is *uniformly $C^1$-robustly expansive* if there exist $\alpha > 0$ and a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ such that for any $g \in \mathcal{U}(f)$, $g|_{H_g(p_g)}$ is expansive with an expansive constant $\alpha$, where $H_g(p_g)$ is the homoclinic class of $g$ associated to $p_g$. Moreover, we say that a homoclinic class $H_f(p)$ is *$C^1$-robustly expansive* if there is a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ such that for any $g \in \mathcal{U}(f)$, $g|_{H_g(p_g)}$ is expansive ([16]). The difference between the definitions above is that in the latter we do not require a uniform expansivity constant on $H_g(p_g)$ for $g \in \mathcal{U}(f)$. Thus the robust expansiveness is a weaker notion than uniformly robust expansiveness. The same definitions can be applied to $H_f^T(p)$. For instance: $H_f^T(p)$ is uniformly $C^1$-robustly expansive when there is $\alpha > 0$ and a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ such that for any $g \in \mathcal{U}(f)$, $g|_{H_g^T(p_g)}$ is expansive with an expansive constant $\alpha$, where $H_g^T(p_g)$ is the tansversal homoclinic class of $g$ associated to $p_g$ ([8, 9]). Analogously with the definition of $C^1$-robustly expansive.

Recently it was proved in [9] that uniformly $C^1$-robustly expansive transversal homoclinic classes $H_f^T(p)$ of a three dimensional manifold $M$ are generically hyperbolic, and the result was generalized in two ways. First, in [8], they dropped the assumption dim$M$=3 and showed that uniformly $C^1$-robustly expansive codimension one transversal homoclinic classes are hyperbolic; that is, if the transversal homoclinic class $H_f^T(p)$ is uniformly $C^1$-robustly expansive and dim$W^s(p) = 1$, then $H_f^T(p)$ is hyperbolic. Moreover, Sambarino and Vieitez [16] proved that if the homoclinic class $H_f(p)$ is $C^1$-robustly expansive and the $H_f(p)$-germ of $f$ is expansive then $H_f(p)$ is hyperbolic. In this direction, the following problem is still open: *Are the $C^1$-robustly expansive homoclinic classes hyperbolic?*

On the other hand, Sakai [15] showed that if the homoclinic class $H_f(p)$ is $C^1$-stably shadowable (for definition, see below) and the $H_f(p)$-germ of $f$ is expansive then $H_f(p)$ is hyperbolic. In fact, he proved that every $C^1$-stably shadowable chain component $C_f(p)$ coincides with the homoclinic class $H_f(p)$ and $H_f(p)$ is hyperbolic if the $H_f(p)$-germ of $f$ is expansive.

In this paper, we introduce the notion of $C^1$- stable expansivity for a closed $f$-invariant subset of $M$, and show that the chain recurrent set $\mathcal{R}(f)$ of $f$ is $C^1$-stably expansive if and only if $f$ satisfies both Axiom A and no-cycle condition. Moreover we prove that the homoclinic class $H_f(p)$ of $f$ associated to $p$ is $C^1$-stably expansive if and only if $H_f(p)$ is hyperbolic. Finally we claim that $C^1$-generically, the homoclinic class $H_f(p)$ is $C^1$-stably expansive if and only if $H_f(p)$ is both $C^1$-robustly expansive and the $H_f(p)$-germ of $f$ is expansive.

**Definition 1.1.** Let $f \in$ Diff(M). We say that a closed $f$-invariant set $\Lambda \subset M$ is $C^1$-*stably expansive* if there exist a compact neighborhood $U$ of $\Lambda$ and a $C^1$-neighborhood of $\mathcal{U}(f)$ of $f$ such that

(i) $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$; i.e., $\Lambda$ is *locally maximal* in $U$.
(ii) $g|_{\Lambda_g}$ is expansive for $g \in \mathcal{U}(f)$,

where $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$.

Recall that a compact invariant set $\Lambda$ is called *hyperbolic* for $f$ if the tangent bundle $T_\Lambda M$ has a continuous $Df$-invariant splitting $E \oplus F$ and there exist constants $C > 0, 0 < \lambda < 1$ such that

$$\|Df^n|_{E(x)}\| \leq C\lambda^n$$

and

$$\|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

We say that $f$ is *Anosov* if $M$ is hyperbolic for $f$, and $f$ is *quasi-Anosov* if for any nonzero vector $v \in TM$, the set $\{\|(Tf)^n v\| : n \in Z\}$ is not bounded. Note that every Anosov diffeomorphism is quasi-Anosov, but the converse is not true in general. Moreover it is proved in [6] that $f$ is quasi-Anosov if and only if $f$ belongs to the $C^1$-interior of the set of expansive diffeomorphisms in Diff(M). Thus we can restate the above facts as follows.

**Theorem A.** *$M$ is $C^1$-stably expansive for $f$ if and only if $f$ is quasi-Anosov.*

For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^b$ in $M(-\infty \leq a < b \leq \infty)$ is called a $\delta$-*pseudo-orbit* (or $\delta$-*chain*) of $f$ if $d(f(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b-1$. For a $f$-invariant closed subset $\Lambda$ of $M$, we say that $f|_\Lambda$ has the *shadowing property* (or $\Lambda$ is *shadowable*) if for every $\epsilon > 0$, there is $\delta > 0$ such that for any $\delta$-pseudo-orbit $\{x_i\}_{i=a}^b \subset \Lambda$ of $f$ $(-\infty \leq a < b \leq \infty)$, there is $y \in M$ satisfying $d(f^i(y), x_i) < \epsilon$ for all $a \leq i \leq b-1$. The point $y$ is said to be $\epsilon$-*shadowed* by $\{x_i\}_{i=a}^b$. Notice that only $\delta$-pseudo-orbits of $f$ contained in $\Lambda$ can be $\epsilon$-shadowed, but the shadowing point $y \in M$ is not necessary contained in $\Lambda$. It is easy to see that $f|_\Lambda$ has the shadowing property if and only if $f^n|_\Lambda$ has the shadowing property for $n \in \mathbb{Z} - \{0\}$.

For given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a $\delta$-pseudo-orbit $\{x_i\}_{i=a_\delta}^{b_\delta} (a_\delta < b_\delta)$ of $f$ such that $x_{a_\delta} = x$ and $x_{b_\delta} = y$. The set $\{x \in M : x \rightsquigarrow x\}$ is called the *chain recurrent set* of $f$ and is denoted by $\mathcal{R}(f)$. It is easy to see that the set is closed and $f(\mathcal{R}(f)) = \mathcal{R}(f)$. A nice property the chain recurrent set $\mathcal{R}(f)$ holds is that it naturally decomposes into disjoint union of compact invariant sets.

Precisely, define a relation $\sim$ on $\mathcal{R}(f)$ by $x \sim y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. It is clear that $\sim$ is an equivalent relation on $\mathcal{R}(f)$. The equivalence classes are called the *chain components* of $f$. These are compact invariant sets and can not be decomposed into two disjoint compact invariant sets and so serve as "elementary pieces" of the dynamical systems. Denote $C_f(p)$ the chain component of $f$ that contains a hyperbolic periodic point of $f$.

We say $C_f(p)$ is $C^1$-*stably shadowable* if there is a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ such that for every $g \in \mathcal{U}(f)$, $C_g(p_g)$ is shadowable for $g$, where $p_g$ is the continuation of $p$ and $C_g(p_g)$ is the chain component of $g$ containing $p_g$. Recently, Sakai [15] proved that if $C_f(p)$ is $C^1$-stably shadowable and the $C_f(p)$-germ of $f$ is expansive, then $C_f(p)$ coincides with $H_f(p)$ and is hyperbolic. Moreover, Wen *et al.* [18] showed that the assumption of the $C_f(p)$-germ expansivity of $f$ can be dropped in the above result to show the hyperbolicity of the $C^1$ stably shadowable chain component $C_f(p)$.

We say that a closed $f$-invariant set $\Lambda \subset M$ has the $C^1$-*stable shadowing property* if $\Lambda$ is locally maximal in a compact neighborhood $U$ of $\Lambda$ and there is a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ such that for any $g \in \mathcal{U}(f)$, $g_{|\Lambda_g}$ has the shadowing property, where $\Lambda_g = \cap_{n\in\mathbb{Z}} g^n(U)$. In [5], Lee *et al.* proved that $C_f(p)$ has the $C^1$-stable shadowing property if and only if $C_f(p)$ is hyperbolic.

If we denote the set of periodic points of $f$ by $P(f)$, then $P(f) \subset \Omega(f) \subset \mathcal{R}(f)$. Here $\Omega(f)$ is the set of non-wandering points of $f$. It is well known that the map $f \mapsto \mathcal{R}(f)$ is upper semi-continuous. More precisely, for any neighborhood $U$ of $\mathcal{R}(f)$, there is $\delta > 0$ such that if $\rho_0(f, g) < \delta(g \in \text{Diff(M)})$, then $\mathcal{R}(g) \subset U$. Here $\rho_0$ is the usual $C^0$-metric on Diff(M). From this fact, we can prove the first result of this paper based on [3].

**Theorem B.** *The chain recurrent set $\mathcal{R}(f)$ of $f$ is $C^1$-stably expansive if and only if $f$ satisfies both Axiom A and no-cycle condition.*

Let $f$ satisfy Axiom A. Then, it is well known that $\Omega(f) = \mathcal{R}(f)$ if and only if $f$ satisfies no-cycle condition. Hence, the $C^1$-stable expansivity on $\mathcal{R}(f)$ is characterized as the $\Omega$-stability of the system by Theorem B. For dynamical systems satisfying Axiom A, the hyperbolic basic set is a really basic system possessing lots of important dynamical properties and investigated well in view of stability theory and ergodic theory. As stated before, homoclinic classes are the natural candidates to replace hyperbolic basic sets in nonhyperbolic theory.

Let $D^2 \subset \mathbb{R}^2$ be a two disk, and let $f$ be the Smale's hyperbolic horseshoe map on $D^2$ with a (hyperbolic) saddle fixed point $p$. Then the homoclinic class $H_f(p)$

coincides with the hyperbolic horseshoe containing $p$. Since $f$ is $\Omega$-stable, we can see that the homoclinic class $H_f(p)$ is $C^1$-stably expansive by Theorem B. Moreover we can see that the horseshoe with a homoclinic tangency is expansive, but it is not $C^1$-stably expansive (for instance, see [15, Example 2.2]).

The main purpose of this paper is to characterize the homoclinic class $H_f(p)$ containing a hyperbolic periodic point $p$ by making use of the expansivity under $C^1$-open condition.

**Theorem C.** *Let $p$ be a hyperbolic periodic point of $f$. Then the homoclinic class $H_f(p)$ is $C^1$-stably expansive if and only if $H_f(p)$ is hyperbolic.*

Using the above result, we prove the following theorem.

**Theorem D.** *There exists a residual subset $\mathcal{R} \subset \mathrm{Diff(M)}$ such that for any $f \in \mathcal{R}$ and a hyperbolic periodic point $p$ of $f$, the homoclinic class $H_f(p)$ is $C^1$-stably expansive if and only if $H_f(p)$ is $C^1$-robustly expansive and the $H_f(p)$-germ of $f$ is expansive.*

**Corollary.** *There exists a residual subset $\mathcal{R} \subset \mathrm{Diff(M)}$ such that if $H_f(p)$ is $C^1$-stably expansive for $f \in \mathcal{R}$ then there is a $C^1$-neighborhood $\mathcal{U}(f)$ of $f$ such that for all $g \in \mathcal{U}(f)$, $H_g(p_g)$ is hyperbolic for $g$, where $p_g$ is the continuation of $p$.*

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