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FREE ACTIONS OF FINITE GROUPS ON THE 3-DIMENSIONAL NILMANIFOLD

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According to Thurston's conjecture, there are 8 kinds of geometries in dimension 3. They are E^3, H^3, S^3, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2 \mathbb{R}}$, Nil and Sol [12].

A question naturally arisen is the problem of the classification of closed 3-dimensional manifolds with a geometric structure modelled on one of these eight types. Except for the manifolds with the hyperbolic structure modelled on H^3 and the solvmanifolds modelled on Sol, all such manifolds are Seifert fiber spaces. The solvmanifolds have also Seifert fiber structures in the general sense. A Seifert fiber space is a 3-dimensional manifold M with decomposition of M into disjoint circles, called fibers, such that each circle has a neighborhood in M which is a union of fibers and is isomorphic to a fibered solid torus or Klein bottle. Therefore, a Seifert fiber space is a kind of circle bundle. It is well-known that Seifert fiber spaces are homeomorphic if and only if they have isomorphic Seifert bundle structures [12]. A Seifert bundle classification of infra-nilmanifolds of dimension 3 appears in Orlik's book [11] on Seifert manifolds. An infra-nilmanifold is finitely covered by a circle bundle over a torus with non-zero Euler number. Moreover, infra-nilmanifolds are determined uniquely by their fundamental groups, called almost Bieberbach groups. It is known ([3; Proposition 6.1.]) that there are 15 classes of distinct closed 3-dimensional manifolds M with a Nil-geometry up to Seifert local invariant.

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The details of this note will be published somewhere else.

The general question of classifying finite group actions on a closed 3-manifold is very hard. However, the actions on a 3-dimensional nilmanifold can be understood easily by the works of Bieberbach, L. Auslander and Waldhausen([5, 6, 14]). Free actions of finite, cyclic and abelian groups on the 3-torus were studied in [4], [7] and [10], respectively.

A group N is *p-step nilpotent* if $N^{(p)} = 1$, where $N^{(1)} = [N, N]$, the commutator subgroup of N, and $N^{(i+1)} = [N, N^{(i)}]$. Let $\mathcal{H}$ be the 3–dimensional Heisenberg group; i.e. $\mathcal{H}$ consists of all 3×3 real upper triangular matrices with diagonal entries 1. That is,

$$\mathcal{H} = \left\{ \left[\begin{array}{c|c|c} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right] \; : \; x, y, z \in \mathbb{R} \right\}.$$

Thus $\mathcal{H}$ is a simply connected, 2-step nilpotent Lie group, and it fits an exact sequence

$$1 \to \mathbb{R} \to \mathcal{H} \to \mathbb{R}^2 \to 1$$

where $\mathbb{R} = \mathcal{Z}(\mathcal{H})$, the center of $\mathcal{H}$. Hence $\mathcal{H}$ has the structure of a line bundle over $\mathbb{R}^2$. We take a left invariant metric coming from the orthonormal basis

$$\left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

for the Lie algebra of $\mathcal{H}$. This is, what is called, the Nil-geometry and its isometry group is $\mathrm{Isom}(\mathcal{H}) = \mathcal{H} \rtimes O(2)$ [12, 13]. All isometries of $\mathcal{H}$ preserve orientation and the bundle structure.

We say that a closed 3-dimensional manifold M has a Nil-geometry if there is a subgroup π of $\mathrm{Isom}(\mathcal{H})$ so that π acts properly discontinuously and freely with quotient $M = \mathcal{H}/\pi$. The simplest such a manifold is the quotient of $\mathcal{H}$ by the lattice consisting of integral matrices.

Let L be a connected and simply connected nilpotent Lie group. Then $\mathrm{Aff}(L) = L \rtimes \mathrm{Aut}(L)$ is called the affine group of L, where the group operation is given by

$$(g, \alpha)(h, \beta) = (g \cdot \alpha(h), \alpha\beta)$$

and $\mathrm{Aff}(L)$ acts on L by

$$(g, \alpha)x = g \cdot \alpha(x)$$

for $(g, \alpha) \in \mathrm{Aff}(L)$ and $x \in L$. Let K be any maximal compact subgroup of $\mathrm{Aut}(L)$. Then a discrete uniform subgroup E of $L \rtimes K$ is called an *almost crystallographic group*. E is torsion-free if and only if the E-action on L is free. In this case E is called an *almost Bieberbach group* and the coset space $E\backslash L$ is an infra-nilmanifold. (In case $E \subset L$, $E\backslash L$ is called a nilmanifold.) If L is abelian ($\cong \mathbb{R}^n$ for some n), this terminology reduces to a *crystallographic group*, a *Bieberbach group* and a *flat Riemannian manifold*, respectively.

Almost Bieberbach groups are exactly the fundamental groups of compact infra-nilmanifolds. A closed 3–dimensional manifold has a Nil–geometry if it is an infra–nilmanifold. It is well known that infra-nilmanifolds are determined completely (up to affine diffeomorphism) by their fundamental groups.

For each integer $p > 0$, let

$$\Gamma_p = \left\{ \begin{bmatrix} 1 & l & \frac{n}{p} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; l, m, n \in \mathbb{Z} \right\}.$$

Then Γ_1 is the discrete subgroup of $\mathcal{H}$ consisting of all integral matrices and Γ_p is a lattice of $\mathcal{H}$ containing Γ_1 with index p. Clearly

$$\mathrm{H}_1(\mathcal{H}/\Gamma_p; \mathbb{Z}) = \Gamma_p/[\Gamma_p, \Gamma_p] = \mathbb{Z}^2 \oplus \mathbb{Z}_p.$$

Note that these ${\Gamma_p}'$s produce infinitely many distinct nilmanifolds

$$\mathcal{N}_p = \mathcal{H}/\Gamma_p$$

covered by $\mathcal{N}_1$. Now we shall find all possible finite groups acting freely on each $\mathcal{N}_p$.

Let G be a finite group acting freely on the nilmanifold $\mathcal{N}_p$. Then clearly, $M = \mathcal{N}_p/G$ is a topological manifold, and $\pi = \pi_1(M) \subset \mathrm{TOP}(\mathcal{H})$ is isomorphic to an almost Bieberbach group. Let π' be an embedding of π into $\mathrm{Aff}(\mathcal{H})$. Such an embedding always exists. Since any isomorphism between lattices extends uniquely to an automorphism of $\mathcal{H}$, we may assume the subgroup Γ_p goes to itself by the embedding $\pi \to \pi' \subset \mathrm{Aff}(\mathcal{H})$. Then the quotient group $G' = \pi'/\Gamma_p$ acts freely on the nilmanifold $\mathcal{N}_p = \mathcal{H}/\Gamma_p$. Moreover, $M' = \mathcal{N}_p/G'$ is an infra-nilmanifold. Thus, a finite free topological action $(G, \mathcal{N}_p)$ gives rise to an isometric action $(G', \mathcal{N}_p)$ on the nilmanifold $\mathcal{N}_p$. Clearly, $\mathcal{N}_p/G$ and $\mathcal{N}_p/G'$ are sufficiently large, see [5; Proposition 2]. By works of Waldhausen and Heil ([5; Theorem A, 13]), M is homeomorphic to M'.

Let π be an almost Bieberbach group and N be a normal nilpotent subgroup of π with $G = \pi/N$ finite. For each almost Bieberbach group π, we calculate the normalizer $N_{\mathrm{Aff}(\mathcal{H})}(\pi)$ and we find all normal nilpotent subgroups N of π, and classify (N, π) up to affine conjugacy.

For $\mathcal{N}/G$ and $\mathcal{N}/G'$ being homeomorphic implies that the two actions $(G,\mathcal{N})$ and $(G',\mathcal{N})$ are topologically conjugate. Consequently, a finite free action $(G,\mathcal{N})$ is topologically conjugate to an isometric action $(G',\mathcal{N})$. Such a pair $(G',\mathcal{N})$ is not unique. However, by the result obtained by Lee and Raymond ([9]), all the others are topologically conjugate.

It is interesting that if a finite group acts freely on the 3-dimensional nilmanifold with the first homology $\mathbb{Z}^2$, then it is cyclic [2]. Free actions of finite abelian groups on the 3-dimensional nilmanifold with the first homology $\mathbb{Z}^2 \oplus \mathbb{Z}_p$ were classified in [1].

In this talk we study free actions of finite groups on 3-dimensional nilmanifolds by utilizing the method used in [1] and classify all such group actions, up to topological conjugacy. This classification problem is reduced to classifying all normal nilpotent subgroups of almost Bieberbach groups of finite index, up to affine conjugacy.

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