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RECURRENCE TIMES AND LARGE DEVIATIONS

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Abstract. We give a criterion to determine the large deviation rate functions for abstract dynamical systems on towers. As an application of this criterion we show the level 2 large deviation principle for some class of smooth interval maps with nonuniform hyperbolicity.

1. Introduction

Let I be a compact metric space with a finite Borel measure m as a reference measure. Unless otherwise stated, m will be normalized Lebegue measure if I is a manifold. We denote by $\mathscr{M}$ the space of the Borel probability measures on I equipped with the weak* topology. For a nonsingular transformation $f : I \to I$, not necessary invariant for m, we say that it satisfies *the (level 2) large deviation principle* if there is an upper semicontinuous function $q : \mathscr{M} \to [-\infty, 0]$, called *the rate function,* satisfying

$$\liminf_{n\to\infty} \frac{1}{n} \log m\left(\{x \in I : \delta_x^n \in \mathscr{G}\}\right) \geq \sup_{\mu\in\mathscr{G}} q(\mu)$$

for each open set $\mathscr{G} \subset \mathscr{M}$, and

$$\limsup_{n\to\infty} \frac{1}{n} \log m\left(\{x \in I : \delta_x^n \in \mathscr{C}\}\right) \leq \max_{\mu\in\mathscr{C}} q(\mu)$$

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for each closed set $\mathscr{C} \subset \mathscr{M}$, respectively, where

$$\delta_x^n := \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)} \in \mathscr{M}$$

denotes the empirical distribution along the orbit of f through $x \in I$. We refer to Ellis' book [9] for a general theory of large deviations and its background in statistical mechanics.

It is well-known that for a uniformly or partially hyperbolic dynamical system on a manifold with a specification property it satisfies the large deviation principle and the rate function is represented as the difference between the metric entropy and the sum of positive Lyapunov exponents [18, 27]. A similar result as above is also known for piecewise expanding maps, and then the rate function q coincides with the free energy function F given by

$$F(\mu) := \begin{cases} h_\mu(f) - \displaystyle\int \log|f'|d\mu, & \text{for } \mu \in \mathscr{M}_f, \\ -\infty & \text{otherwise,} \end{cases}$$

where $\mathscr{M}_f$ denotes the set of f-invariant Borel probability measures, and $h_\mu(f)$ the metric entropy of $\mu \in \mathscr{M}_f$ for f [24]. The results above on the large deviation principle include Ruelle's inequality, Pesin's and Rohlin's formulas for entropy [12, 19, 22].

Some of the large deviations estimates are also known for nonuniformly hyperbolic dynamical systems. Keller and Nowicki [13] gave a large deviations theorem for a nonrenormalizable unimodal map $f : I \to I$ satisfying the Collet-Eckmann condition that: for any continuous function φ of bounded variation with positive variance

$$\alpha(\varepsilon) := \lim_{n\to\infty}\frac{1}{n}\log m\left(\left\{x \in I : |\frac{1}{n}S_n\varphi(x) - \int \varphi d\mu_0| \geq \varepsilon\right\}\right) < 0 \tag{1}$$

exists for small $\varepsilon > 0$, where

$$S_n\varphi(x) := \varphi(x) + \varphi(f(x)) + \cdots + \varphi(f^{n-1}(x)),$$

and μ_0 denotes the absolutely continuous invariant probability measure. A result correponding to that of Keller and Nowicki above was obtained by Araújo and Pacifico [1] in more general setting of nonuniformly hyperbolic dynamical systems. Melbourne and Nicol [16] gave an induced scheme approach for estimates on the rate functions in dynamical systems modelled by Young towers [29, 30] with summable decay of correlations. All of the results above for nonuniformly hyperbolic dynamical systems are obtained under the assumption of the existence of absolutely continuous invariant probability measures. But the case that the absolutely continuous invariant probability measures do not exist has not been considered. Also, it is unknown yet neither the criteria to satisfy the large deviation principle nor the expressions of the rate functions for nonuniformly hyperbolic dynamical systems.

The purpose of this paper is to consider the large deviation principle for dynamical systems from the view point of recurrence times. We offer a little different description of a tower from those already known to consider a kind of specification property for large deviations estimates. The topology on a tower in this paper is slightly coarser than but almost same as that Young [29, 30] introduced, and in which we give a sufficient condition on the shape of a tower to have a property that any orbit not recurrent to the base for arbitrarily long time can be approximated by another one recurrent quickly on a tower. Then a criterion is obtained to ensure the large deviation rate functions for abstract dynamical systems. We show that if a tower satisfies the nonsteep condition mentioned in the next section, then the rate function is explicitly represented by a quantity concerning the difference between the metric entropy and the Jacobian function. The notion of nonsteepness is independent of the decay rate of the tail. In fact, it is possible to have the large deviation rate function for an abstract dynamical system with no absolutely continuous invariant probability measures. On the other hand, there is a tower on which we cannot determine the rate function for a dynamical system although the decay of the tail is exponentially fast. Some of those examples are provided in the third section. Combining the argument on large deviations for abstract dynamical systems with a theory for hyperbolic measures [6, 10, 11] we establish the large deviation principle for some class of smooth interval maps with nonuniform hyperbolicity. It is shown the rate function coincides with the upper regularization of the free energy function. The class of maps for which we can apply the estimates in this paper contains both of Manneville-Pomeau maps [20, 21] and Collet-Eckmann unimodal maps [3, 4, 8, 13, 17, 28]. The author thinks that our result is applicable to a large class of smooth dynamical systems modelled by towers such as considered in [5]. He also thinks that a theory of multifractal analysis is developed from the large deviations estimates of this paper. It will be treated in the forthcoming paper [7].

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2. Results

Let $(X, \mathscr{B}, m)$ be a finite measure space and consider a decreasing sequence $\{X_k\}_{k=0}^{\infty}$ of subsets of $X_0 = X$ with positive measures. Assume that for each integer $k \geq 0$ there is a finite measurable partition $\mathscr{I}_k$ of X_k satisfying the following properties:

(1) each $J \in \mathscr{I}_k$ has positive measure;
(2) if $J \in \mathscr{I}_k$ intersects with X_{k+1} then $J \subset X_{k+1}$;
(3) for any $K \in \mathscr{I}_{k+1}$ there is $J \in \mathscr{I}_k$ such that $K \subset J$.

Then we call the pair $(Z, \mathscr{A})$ which consists of the space

$$Z := \sqcup_{k=0}^{\infty} X_k \times \{k\} \subset X \times \boldsymbol{Z}^+$$

and its countable partition

$$\mathscr{A} := \{J \times \{k\} : J \in \mathscr{I}_k, k = 0, 1, 2, \ldots\}$$

a *tower*. A natural σ-finite measure m_Z on the tower Z is defined by $m_Z(A) = \sum_{k=0}^{\infty} m(A_k)$ for $A = \sqcup_{k=0}^{\infty} A_k \times \{k\} \subset Z$ with $A_k \in \mathscr{B}$ $(k = 0, 1, 2, \ldots)$. It is obvious that the measure m_Z is finite iff $\sum_{k=0}^{\infty} m(X_k) < \infty$. For a tower $(Z, \mathscr{A})$ put $\mathscr{D} := \sqcup_{k=1}^{\infty} \mathscr{D}_k$ where

$$\mathscr{D}_k := \{J \in \mathscr{I}_{k-1} : J \cap X_k = \emptyset\} \ \ (k = 1, 2, \ldots).$$

We assume that a bi-nonsingular bijection $g_J : J \to X$ can be taken for each $J \in \mathscr{D}$. Then we call $T : Z \to Z$,

$$T(x, k) := \begin{cases} (x, k+1) & \text{for } x \in X_{k+1}, \\ (g_J(x), 0) & \text{for } x \in J \in \mathscr{D}_{k+1}, \end{cases}$$

a *tower map* on $(Z, \mathscr{A})$ induced by $\mathscr{G} := \{g_J : J \to X : J \in \mathscr{D}\}$. Remark that $J \times \{0\} \subset Z$ is injectively mapped onto $X_0 \times \{0\}$ by T^k if $J \in \mathscr{D}_k$. We denote by $k(J)$ the integer $k \geq 1$ such that $J \in \mathscr{D}_k$, i.e., $T^{k(J)}(J \times \{0\}) = X_0 \times \{0\}$.

In this paper we assume that a tower map $T : Z \to Z$ satisfies both of the admissibility and the bounded distortion conditions as below.

The admissibility condition. For any sequence $\{J_n\}_{n=0}^{\infty} \subset \mathscr{D}$ there is a unique point $z \in J_0 \times \{0\}$ such that $T^{k(J_0)+\cdots+k(J_{n-1})} z \in J_n \times \{0\}$ holds for all $n \geq 1$.

It follows from the admissibility condition that for any integer $l \geq 1$ the restriction of T^l to the set $\cap_{n=0}^{\infty} T^{-nl} (\sqcup_{A \in \mathscr{K}_l} A)$ is isomorphic to the full shift of $\sharp\mathscr{K}_l$-symbols if $\mathscr{K}_l$ is nonempty, where

$$\begin{aligned}\mathscr{K}_l := \{A \in \vee_{i=0}^{l-1} T^{-i}\mathscr{A} : A = \cap_{i=0}^{n-1} T^{-l_i}(J_i \times \{0\}),\\ 1 \le n \le l,\ \ J_0, J_1, \ldots, J_{n-1} \in \mathscr{D},\\ k(J_0) + k(J_1) + \cdots + k(J_{n-1}) = l,\ \ l_0 = 0,\\ l_i = k(J_0) + \cdots + k(J_{i-1})\ \ (i = 1, \ldots, n-1)\},\end{aligned}$$

and $\sharp B$ denotes the number of elements of a set B.

The bounded distortion condition. There are a version $\mathrm{Jac}(T) > 0$ of the Radon-Nikodym derivative $\frac{dm_Z \circ T}{dm_Z}$, a constant $D_T \ge 1$ and a sequence $\{\varepsilon_k\}_{k=0}^{\infty}$ of positive numbers with $\lim_{k\to\infty} \varepsilon_k = 0$ such that for any integer $n \ge 1$ and $A \in \vee_{i=0}^{n-1} T^{-i}\mathscr{A}$,

$$\frac{\mathrm{Jac}(T)(z)}{\mathrm{Jac}(T)(w)} \le e^{\varepsilon_l} \quad \text{and} \quad \frac{\prod_{i=0}^{n-1} \mathrm{Jac}(T)(T^i(z))}{\prod_{i=0}^{n-1} \mathrm{Jac}(T)(T^i(w))} \le D_T$$

hold whenever $z, w \in A$, where $l = \sharp\{1 \le j \le n : T^j(A) \subset X_0 \times \{0\}\}$.

We remark that $\mathrm{Jac}(T)(x, k) = 1$ holds if $x \notin J$ for all $J \in \mathscr{D}_k$. From the bounded distortion condition it follows that

$$D_T{}^{-1} \le \frac{m_Z(A) \prod_{i=0}^{n-1} \mathrm{Jac}(T)(T^i(z))}{m_Z(T^n A)} \le D_T$$

holds for any $A \in \vee_{i=0}^{n-1} T^{-i}\mathscr{A}$ and $z \in A$.

We take

$$\mathscr{F} := \left\{ \psi : Z \to \boldsymbol{R} : \text{a bounded function such that } \lim_{n\to\infty} \mathrm{var}_n(\psi) = 0 \right\}$$

as a class of observable functions on Z, where

$$\mathrm{var}_n(\psi) := \sup\{|\psi(z) - \psi(w)| : z, w \in A \text{ for some } A \in \vee_{i=0}^{n-1} T^{-i}\mathscr{A}\}$$

for a function $\psi : Z \to \boldsymbol{R}$ and $n \ge 1$. Notice that for any $\psi \in \mathscr{F}$ we have $\lim_{n\to\infty} \mathrm{var}_1(S_n\psi)/n = 0$, where

$$S_n\psi(z) := \sum_{i=0}^{n-1} \psi(T^i(z))$$

for each integer $n \geq 1$. In fact, for any $\epsilon > 0$ taking $N \geq 1$ so that $\mathrm{var}_N(\psi) \leq \varepsilon/2$ we have

$$\begin{aligned} |S_n\psi(z) - S_n\psi(w)| &\leq \left(\sum_{i=0}^{n-N-1} + \sum_{i=n-N}^{n-1}\right) |\psi(T^i(z)) - \psi(T^i(w))| \\ &\leq (n-N-1)\mathrm{var}_N(\psi) + N\mathrm{var}_1(\psi) \\ &\leq n\epsilon/2 + n\varepsilon/2 \quad = n\varepsilon \end{aligned}$$

whenever $z, w \in A$ for some $A \in \vee_{i=0}^{n-1} T^{-i}\mathscr{A}$ and $n \geq 1$ is large enough.

To give large deviations estimates for a tower map we define the following notion on the shape of a tower. We say that a tower $(Z, \mathscr{A})$ is *nonsteep*, or it satisfies *the nonsteep condition*, if there are sequences $\{l_k\}_{k=0}^{\infty} \subset \boldsymbol{N}$ with $\lim_{k\to\infty} l_k/k = 0$ and $\{\gamma_k\}_{k=0}^{\infty} \subset (0,1)$ with $\lim_{k\to\infty}(\log\gamma_k)/k = 0$ such that

$$m(J \setminus X_{k+l_k}) \geq \gamma_k m(J)$$

holds for all $k \geq 0$ and $J \in \mathscr{I}_k$. We always assume that the sequences above are monotone, i.e., $\{l_k\}_{k=0}^{\infty}$ is nondecreasing and $\{\gamma_k\}_{k=0}^{\infty}$ nonincreasing respectively without loss of generality. Moreover, we say that $(Z, \mathscr{A})$ has *bounded slope* if the sequences $\{l_k\}_{k=0}^{\infty}$ and $\{\gamma_k\}_{k=0}^{\infty}$ above can be taken as constants respectively. It is obvious that if the tower has bounded slope, then the tail decays exponentially fast, i.e., $\limsup_{k\to\infty}(\log m(X_k))/k < 0$, and then on which a tower map has the exponential decay of correlation for a function $\psi \in \mathscr{F}$ such that $\mathrm{var}_n(\psi)$ converges to zero sufficiently fast [29].

The main result of this paper is the following:

Theorem 1. *Let $(Z, \mathscr{A})$ be a nonsteep tower and $T : Z \to Z$ a tower map satisfying both of the admissibility and the bounded distortion conditions. Then for any $\psi \in \mathscr{F}$ there exists an upper semicontinuous concave function $q_\psi : \boldsymbol{R} \to [-\infty, 0]$ satisfying*

$$\liminf_{n\to\infty} \frac{1}{n} \log m\Big(\Big\{x \in X : \frac{1}{n} S_n\psi(x,0) > a\Big\}\Big) \geq \sup_{t>a} q_\psi(t)$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log m\Big(\Big\{x \in X : \frac{1}{n} S_n\psi(x,0) \geq a\Big\}\Big) \leq \max_{t\geq a} q_\psi(t)$$

for all $a \in \boldsymbol{R}$. Moreover, the function q_ψ above can be represented by

$$\begin{aligned} q_\psi(t) = \lim_{\varepsilon\to 0+} \sup\Big\{ h_\nu(T) - \int \log \mathrm{Jac}(T) d\nu : \nu \in \mathscr{M}_T \\ \textit{with } \nu(\sqcup_{k=0}^{K-1} X_k \times \{k\}) = 1 \textit{ for some } K \geq 1 \\ \textit{such that } |\int \psi d\nu - t| < \varepsilon \Big\} \end{aligned}$$

where $\mathscr{M}_T$ denotes the set of all T-invariant probability measures on Z and $h_\nu(T)$ the metric entropy of $\nu \in \mathscr{M}_T$ for T.

The theorem above is applicable to large deviations problems for nonuniformly hyperbolic dynamical systems. In fact, we obtain a criterion to satisfy the large deviation principle for smooth interval maps modelled by tower dynamical systems.

Let I be a compact interval of the real line and m denotes Lebesgue measure on I as a reference measure. We say that a map $f : I \to I$ is *topologically mixing* if for any nontrivial interval $L \subset I$ there is an integer $K \geq 1$ such that $f^K L = I$. Let $f : I \to I$ be a C^2 map of topologically mixing and assume that there are a closed subinterval J of I, a return time function $R : J \to \boldsymbol{N} \cup \{\infty\}$, i.e. $f^{R(x)}(x) \in J$ whenever $R(x) < \infty$, constants $\lambda > 1$, $D \geq 1$, sequences $\{\varepsilon_k\}_{k=0}^{\infty}$ of positive numbers with $\lim_{k\to\infty} \varepsilon_k = 0$, $\{l_k\}_{k=0}^{\infty} \subset \boldsymbol{N}$ with $\lim_{k\to\infty} l_k/k = 0$ and $\{\gamma_k\}_{k=0}^{\infty} \subset (0,1)$ with $\lim_{k\to\infty} (\log \gamma_k)/k = 0$ satisfying the following properties:

(1) if $k \geq 1$ and V is a connected component of $\{x \in J : R(x) = k\}$, then

$$f^k V = J \quad \text{and} \quad |(f^k)'(x)| \geq \lambda \ \ (x \in V);$$

(2) if $n = k_0 + \cdots + k_l \geq 1$ and U_n is a connected component of

$$\{x \in J : R(x) = k_0, R(f^{k_0}(x)) = k_1, \ldots, R(f^{k_0+\cdots+k_{l-2}}(x)) = k_{l-1} \\ \text{and } R(f^{k_0+\cdots+k_{l-1}}(x)) \geq k_l\},$$

then $m(f^j(U_n)) \leq \varepsilon_{n-j}$ for all $0 \leq j \leq n-1$.

(3) if $n = k_0 + \cdots + k_l \geq 1$ and V_n is a connected component of

$$\{x \in J : R(x) = k_0, R(f^{k_0}(x)) = k_1, \ldots, R(f^{k_0+\cdots+k_{l-1}}(x)) = k_l\},$$

then

$$\frac{|(f^{k_0})'(y)|}{|(f^{k_0})'(z)|} \leq e^{\varepsilon_l} \quad \text{and} \quad \frac{|(f^n)'(y)|}{|(f^n)'(z)|} \leq D$$

hold whenever $y, z \in V_n$;

(4) if $k \geq 1$ and U is a connected component of $\{x \in J : R(x) > k\}$, then

$$m\left(\{x \in U : R(x) \leq k + l_k\}\right) \geq \gamma_k m(U).$$

We remark that the function R does not necessarily correspond to a first return time on J.

We say that $\mu \in \mathscr{M}_f$ is *hyperbolic* if the Lyapunov exponent $\lambda(x) := \limsup_{n\to\infty} (\log |(f^n)'(x)|)/n$ is positive for μ-almost every $x \in I$. It follows from the assumptions for the map f that hyperbolic measures are dense in $\mathscr{M}_f$. If $\mu \in \mathscr{M}_f$ is ergodic then the Lyapunov exponents coincide with the constant

$\lambda_\mu(f) := \int \log|f'|d\mu$ μ-almost everywhere. A theory for hyperbolic measures [6, 10, 11] asserts the following:

Proposition 2. *Let $\mu \in \mathscr{M}_f$ be ergodic and hyperbolic. Then, for any continuous function $\varphi : I \to \boldsymbol{R}$ and $\varepsilon > 0$ there are integers $k, l \geq 1$ with $(\log l)/k \geq h_\mu(f) - \varepsilon$ and pairwise disjoint compact intervals $L_1, L_2, \ldots, L_l$ with $L \subset I$ such that $L_i \subset L, f^k(L_i) = L$ and L_i is injectively mapped to L by f^k for each $i = 1, 2, \ldots, l$. Moreover,*

$$|\frac{1}{k}\log|(f^k)'(x)| - \lambda_\mu(f)| \leq \varepsilon \quad \text{ and } \quad |\frac{1}{k}S_k\varphi(x) - \int \varphi d\mu| \leq \varepsilon$$

hold whenever $x \in \sqcup_{i=1}^{l} L_i$.

Now we define *the free energy function $F : \mathscr{M} \to \boldsymbol{R} \cup \{-\infty\}$* by

$$F(\mu) := \begin{cases} h_\mu(f) - \int \log|f'|d\mu & \text{for } \mu \in \mathscr{M}_f \text{ hyperbolic,} \\ -\infty & \text{otherwise.} \end{cases}$$

Then combining Proposition 2 with Theorem 1 we obtain the following:

Theorem 3 (The large deviation principle)**.** *Let $f : I \to I$ be a map satisfying the assumptions above. Then f satisfies the large deviation principle, and the rate function q coincides with the upper regularization of F, i.e.,*

$$q(\mu) = \inf\{Q(\mathscr{G}) : \mathscr{G} \text{ is a neighborhood of } \mu \text{ in } \mathscr{M}\}$$

where

$$Q(\mathscr{G}) := \sup\{F(\nu) : \nu \in \mathscr{G}\}.$$

Corollary 4 (The Ruelle inequality [22])**.** *For any $\mu \in \mathscr{M}_f$, $F(\mu) \leq 0$ holds.*

It should be noticed that we need the upper regularization for F to get the rate function without assuming uniform hyperbolicity of the map f, because the free energy function itself may not be upper semiconitinuous for a smooth interval map modelled by a tower dynamical system, see [4].

If $f : I \to I$ is a nonrenormalizable Collet-Eckmann unimodal map $f : I \to I$, then a subinterval J can be taken with a return time function R satisfying the assumptions above so that the sequences $\{l_k\}_{k=0}^{\infty}$ and $\{\gamma_k\}_{k=0}^{\infty}$ are constants respectively [29], see also [2]. Then the tower induced from the suspension by the return time function has bounded slope, and then the tail decays exponentially fast. It is known that the map f has an absolutely continuous invariant probability measure μ_0 and the correlation decays exponentially fast for any continuous function of bounded variation [13, 17, 28]. It is also known that all of the invariant Borel probability measures are hyperbolic for Collet-Eckmann unimodal maps [4, 17].

Let $f : I \to I$ be as in Theorem 3 and $\varphi : I \to \boldsymbol{R}$ a continuous function. Here φ is not assumed to be of bounded variation. Put

$$c_\varphi := \inf_{x\in I} \liminf_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \min_{\mu\in\mathscr{M}_f} \int \varphi\, d\mu$$

and

$$d_\varphi := \sup_{x\in I} \limsup_{n\to\infty} \frac{1}{n} S_n\varphi(x) = \max_{\mu\in\mathscr{M}_f} \int \varphi\, d\mu,$$

respectively. Then the function $F_\varphi : \boldsymbol{R} \to [-\infty, 0]$ defined by

$$F_\varphi(t) := \sup\left\{ F(\mu) : \int \varphi d\mu = t \right\}$$

is bounded and concave on the interval $[c_\varphi, d_\varphi]$. Thus it follows immediately from the theorem that:

Corollary 5 (The contraction principle)**.**

$$\lim_{n\to\infty} \frac{1}{n} \log m\left(\left\{ x \in I : a \le \frac{1}{n} S_n\varphi(x) \le b \right\} \right) = \max_{a\le t\le b} F_\varphi(t)$$

holds for any $a, b \in \boldsymbol{R}$ whenever $a \neq d_\varphi$ and $b \neq c_\varphi$.

As a consequence we obtain

$$\alpha(\varepsilon) = \sup\left\{ F(\mu) : \left| \int \varphi\, d\mu - \int \varphi\, d\mu_0 \right| \ge \varepsilon \right\}$$

for α in the large deviations theorem (1) for Collet-Eckmann unimodal maps. The above formula includes the large deviations theorem because $F(\mu) = 0$ holds if and only if μ is an absolutely continuous invariant probability measure, i.e., $\mu = \mu_0$ [15].

Another consequence of Theorem 3 follows from a general theory on large deviations in dynamical systems [24, 25]. It is the following:

Corollary 6 (The variational principle of Gibbs type)**.** *The limit*

$$P(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \int \exp S_n\varphi\, dm$$

exists for any continuous function $\varphi : I \to \boldsymbol{R}$. Moreover, the function $P : C(I) \to \boldsymbol{R}$, the pressure with respect to m, coincides with the Legendre transform of $-q$, i.e.,

$$P(\varphi) = \max_{\mu\in\mathscr{M}_f} \left\{ q(\mu) + \int \varphi\, d\mu \right\} \quad \textit{for all} \quad \varphi \in C(I),$$

and

$$q(\mu) = \min_{\varphi\in C(I)} \left\{ P(\varphi) - \int \varphi\, d\mu \right\} \quad \textit{for all} \quad \mu \in \mathscr{M}_f,$$

where $C(I)$ denotes the space of the continuous functions on I.

3. Examples of towers

In this section we give some of examples of towers. Throughout this section let $X := (0, 1]$ and the measure m on X is Lebesgue measure. For a sequence $\{a_k\}_{k=0}^{\infty}$ with $a_0 = 1 \geq a_1 \geq \cdots \geq a_k \geq \cdots > 0$ setting $X_k := (0, a_k]$ and $\mathscr{I}_k := \{X_{k+1}, X_k \setminus X_{k+1}\}$ we obtain a tower $(Z, \mathscr{A})$ as in the previous section.

For integers k with $a_{k+1} < a_k$ we define a linear bijection $g_k = g_{X_k \setminus X_{k+1}} : X_k \setminus X_{k+1} \to X$ by

$$g_k(x) = \frac{x - a_{k+1}}{a_k - a_{k+1}}.$$

Then a tower map $T : Z \to Z$ is also defined by

$$T(x, k) := \begin{cases} (x, k+1) & \text{if } x \in X_{k+1}, \\ (g_k(x), 0) & \text{if } x \in X_k \setminus X_{k+1}. \end{cases}$$

It has no distortion, that is, $D_T = 1$ holds in the bounded distortion condition.

Remark 1. *The tower map T defined as above gives a model for the countable piecewise linear map $f : [0, 1] \to [0, 1]$ with intermittency introduced originally by Takahashi* [23]*:*

$$f(x) := \begin{cases} (x - \beta_1)/(\beta_0 - \beta_1) & \textit{for } x \in (\beta_1, \beta_0], \\ \lambda_k(x - \beta_{k+1}) + \beta_k & \textit{for } x \in (\beta_{k+1}, \beta_k] \textit{ with } k \geq 1, \\ 0 & \textit{for } x = 0, \end{cases}$$

where $\{\beta_k\}_{k=0}^{\infty}$ is a decreasing sequence of positive numbers with $\beta_0 = 1$ and $\lambda_k := (\beta_{k-1} - \beta_k)/(\beta_k - \beta_{k+1})$ for each integer $k \geq 1$.

Remark 2. *By taking another family of functions $\mathscr{G} = \{g_k : a_{k+1} < a_k\}$ it also gives a model on a tower obtained from a sequence $\{a_k\}_{k=0}^{\infty}$ as above for a Manneville-Pomeau map, $f(x) = x + x^{1+s} (\mathrm{mod} 1)$ where $0 < s < 1$, on the interval $[0, 1]$. The sequence $\{a_k\}_{k=0}^{\infty}$ corresponds to the preimages of the discontinuity point of the map. An estimate on the upper bound is known for large deviations of this map* [20]*. Both of the lower and the upper bounds for large deviations are obtained from the result of this paper.*

The stochastic properties of the tower map $T : Z \to Z$ are completely determined by the sequence $\{a_k\}_{k=0}^{\infty}$. It is well-known that the absolutely continuous invariant probability measure exists for the map T if and only if the sequence is summable, i.e., $\sum_{k=0}^{\infty} a_k < \infty$. It is also known that the central limit theorem holds if $\sum_{k=n}^{\infty} a_k \leq C n^{-\alpha}$ $(n \geq 1)$ for some constants $C \geq 1$ and $\alpha > 1$. Moreover, if the sequence decays to zero exponentially fast, i.e., $\limsup_{n\to\infty} (\log a_n)/n < 0$, then so does the correlation function [29]. However, the large deviations estimates as in Theorem 1 do not follow from any conditions mentioned above. For example, the sequence $\{a_k\}_{k=0}^{\infty}$ given by $a_k = \exp\{-8^{l+1}\}$ $(8^l \le k < 8^{l+1}, l \ge 0)$ decays to zero exponentially fast, but the map T does not have a rate function q_ψ as in Theorem 1 for a locally constant function given by $\psi(x,k) := 1$ $(8^l \le k < 2 \cdot 8^l, l \ge 0); = 0$ (otherwise). In fact, since

$$\left\{x \in X : \frac{1}{8^l} S_{8^l}\psi(x,0) > \frac{7}{16}\right\} = \emptyset$$

for any integer $l \ge 1$, we have

$$\liminf_{n\to\infty} \frac{1}{n} \log m\left(\left\{x \in X : \frac{1}{n} S_n\psi(x,0) > \frac{7}{16}\right\}\right) = -\infty.$$

On the other hand, since

$$\left\{x \in X : \frac{1}{2\cdot 8^l} S_{2\cdot 8^l}\psi(x,0) \ge \frac{1}{2}\right\} \supset X_{2\cdot 8^l} = X_{8^l}$$

we have

$$\limsup_{n\to\infty} \frac{1}{n} \log m\left(\left\{x \in X : S_n\psi(x,0) \ge \frac{1}{2}\right\}\right) \ge -4.$$

Thus, we cannot take a function q_ψ to satisfy both of the lower and the upper estimates as in Theorem 1. The tower $(Z, \mathscr{A})$ obtained from the sequence $\{a_k\}_{k=0}^{\infty}$ as above is nonsteep if and only if

$$\lim_{k\to\infty} \frac{1}{k} \log \frac{a_k - a_{k+l_k}}{a_k} = 0$$

holds for some sequence $\{l_k\}_{k=0}^{\infty}$ of positive integers such that $\lim_{k\to\infty} l_k/k = 0$. Then the rate function q_ψ is given as in Theorem 1 for any $\psi \in \mathscr{F}$. It is given a typical example of the sequence $\{a_k\}_{k=0}^{\infty}$ for which the tower is nonsteep by $a_k = (1-p)^k$ $(k \ge 0)$ with $0 < p < 1$. Then for a function defined by $\psi(x,k) := 1$ $(k = 0); = 0$ $(k \ge 1)$ the rate function q_ψ satisfies

$$q_\psi(t) = \begin{cases} H(t, 1-t|p, 1-p) & \text{for} \quad 0 \le t \le 1, \\ -\infty & \text{otherwise,} \end{cases}$$

where $H(t, 1-t|p, 1-p)$ denotes the relative entropy, i.e.,

$$H(t, 1-t|p, 1-p) := -t\log t - (1-t)\log(1-t) + t\log p + (1-t)\log(1-p).$$

This is a classical result on large deviations obtained by Khinchin [14]. The tower obtained from this sequence is not only nonsteep but also having bounded slope. In general, the tower obtained from the sequence $\{a_k\}_{k=0}^{\infty}$ has bounded slope if and only if

$$\frac{a_k - a_{k+l}}{a_k} \ge c \quad (k = 0, 1, 2, \ldots)$$

holds for some $l \in \boldsymbol{N}$ and $c > 0$. Another example satisfying the nonsteep condition is given by $a_k = 1/k$ $(k \geq 1)$. Then the sequence $\{a_k\}_{k=0}^{\infty}$ is not summable and hence the map $T : Z \to Z$ has no absolutely continuous invariant probability measures, nevertheless the large deviations estimates hold by Theorem 1. It should be noticed that the nonsteep condition is not necessary for the large deviations estimates. In fact, the same estimates as in Theorem 1 still hold for the tower obtained from the sequence $\{a_k\}_{k=0}^{\infty}$ given by $a_k = \exp\{-8^{2(l+1)}\}$ $(8^l \leq k < 8^{l+1}, m \geq 0)$, although the tower fails the nonsteep condition. It can be checked that the large deviations estimates as in Theorem 1 valid without the nonsteep condition if the sequence decays super exponentially fast, i.e.,

$$\lim_{k\to\infty} \frac{1}{k} \log a_k = -\infty$$

holds, in general.

4. Proof of Theorem 1

Let $(Z, \mathscr{A})$ be a nonsteep tower, and $T : Z \to Z$ a tower map satisfying both of the admissibility and the bounded distortion conditions. We fix $\psi \in \mathscr{F}$ and $a \in \boldsymbol{R}$. Then the proof of Theorem 1 is devided into two estimates below:

(1) (The lower estimate)

$$\liminf_{n\to\infty} \frac{1}{n} \log m\Big(\Big\{x \in X : \frac{1}{n} S_n\psi(x,0) > a\Big\}\Big) \geq \sup_{t>a} q_\psi(t);$$

(2) (The upper estimate)

$$\limsup_{n\to\infty} \frac{1}{n} \log m\Big(\Big\{x \in X : \frac{1}{n} S_n\psi(x,0) \geq a\Big\}\Big) \leq \max_{t\geq a} q_\psi(t)$$

where

$$\begin{aligned} q_\psi(t) = \lim_{\varepsilon\to 0+} \sup\Big\{h_\nu(T) - \int \log \mathrm{Jac}(T) d\nu : \nu \in \mathscr{M}_T \\ \text{with } \nu(\sqcup_{k=0}^{K-1} X_k \times \{k\}) = 1 \text{ for some } K \geq 1 \\ \text{such that } |\int \psi d\nu - t| < \varepsilon\Big\}. \end{aligned}$$

The lower estimate. It is enough to show that for any $\nu \in \mathscr{M}_T$ with $\nu(\sqcup_{k=0}^{K-1} X_k \times \{k\}) = 1$ and $\varepsilon, \eta > 0$ the inequality

$$m\Big(\Big\{x \in X : |\frac{1}{n} S_n\psi(x,0) - \int \psi d\nu| \leq \varepsilon\Big\}\Big) \geq \exp\Big\{n\Big(h_\nu(T) - \int \log \mathrm{Jac}(T) d\nu - \eta\Big)\Big\} \tag{2}$$

holds for any sufficiently large integer $n \geq 1$. To prove the inequality above, first we assume that ν is ergodic. Let

$$\mathscr{A}_K := \sqcup_{k=0}^{K-1}\{J \times \{k\} : J \in \mathscr{I}_k\} \subset \mathscr{A}.$$

Then it is obvious that $\nu(\sqcup_{A \in \vee_{i=0}^{n-1} T^{-i}\mathscr{A}_K} A) = 1$ since $\nu \in \mathscr{M}_T$. Also, since $\psi \in \mathscr{F}$

$$|S_n\psi(z) - S_n\psi(w)| \leq n\varepsilon/8$$

holds whenever $z, w \in A$ for some $A \in \vee_{i=0}^{n-1} T^{-i}\mathscr{A}_K$ and $n \geq 1$ is large. Let

$$\begin{aligned}\mathscr{B}_n := \Big\{ A \in \vee_{i=0}^{n-1} T^{-i}\mathscr{A}_K : \nu(A) \leq \exp\{-n(h_\nu(T) - \eta/8)\}, \\ |\sum_{i=0}^{n-1} \log \mathrm{Jac}(T)(T^i(w_A)) - n\int \log \mathrm{Jac}(T) d\nu| \leq n\eta/8 \\ \text{and} \quad |S_n\psi(z_A) - n\int \psi d\nu| \leq n\varepsilon/8 \quad \text{for some } z_A, w_A \in A \Big\}\end{aligned}$$

for each integer $n \geq 1$. By the Birkhoff ergodic theorem and the Shannon-McMillan-Breimann theorem $\nu(\sqcup_{A \in \mathscr{B}_n} A) \geq 1/2$ holds, and hence the number of elements in $\mathscr{B}_n$ is not smaller than $e^{n(h_\nu(T) - \eta/8)}/2$ for large $n \geq 1$. Then we can choose integers k and l with $0 \leq k, l \leq K - 1$ so that the set

$$\mathscr{B}_{n,k,l} := \{A \in \mathscr{B}_n : A \subset X_k \times \{k\},\ T^{n+l}A = X_0 \times \{0\}\}$$

contains at least $e^{n(h_\nu(T) - \eta/4)}$ $(\leq e^{n(h_\nu(T) - \eta/8)}/(2K^2))$ elements. For large $n \geq 1$ and $A \in \mathscr{B}_n$

$$\begin{aligned} m_Z(A) &\geq {D_T}^{-1} m_Z(T^n A) \prod_{i=0}^{n-1} \mathrm{Jac}(T)(T^i(w_A))^{-1} \\ &\geq {D_T}^{-1} \min_{B \in \mathscr{A}_K} m_Z(B) \exp\left\{-n\left(\int \log \mathrm{Jac}(T) d\nu + \eta/8\right)\right\} \\ &\geq \exp\left\{-n\left(\int \log \mathrm{Jac}(T) d\nu + \eta/4\right)\right\} \end{aligned}$$

holds. Then we have

$$\begin{aligned} \sum_{A \in \mathscr{B}_{n,k,l}} m_Z(A) &\geq \sharp\mathscr{B}_{n,k,l} \cdot \min_{A \in \mathscr{B}_{n,k,l}} m_Z(A) \\ &\geq \exp\left\{n\Big(h_\nu(T) - \int \log \mathrm{Jac}(T) d\nu - \eta/2\Big)\right\}. \end{aligned}$$

Take $A^* \in \vee_{i=0}^{n+k-1} T^{-i}\mathscr{A}_K$ such that $T^k A^* = A$ for each $A \in \mathscr{B}_{n,k,l}$. Then it is obvious that $A^* \subset X_0 \times \{0\}$ and $T^{n+k+l}A^* = X_0 \times \{0\}$. Let

$$\mathscr{B}^*_{n,k,l} := \{A^* : A \in \mathscr{B}_{n,k,l}\}.$$

Then since $m_Z(A^*) = m_Z(A)$ for each $A \in \mathscr{B}_{n,k,l}$ we have

$$\sum_{A^* \in \mathscr{B}^*_{n,k,l}} m_Z(A^*) = \sum_{A \in \mathscr{B}_{n,k,l}} m_Z(A).$$

Moreover, for any $A \in \mathscr{B}_{n,k,l}$ and $z \in A$ there is $z^* \in A^*$ such that $T^k(z^*) = z$, and then

$$\begin{aligned}
&|S_n\psi(z^*) - n\int \psi d\nu| \\
&\leq |S_n\psi(z^*) - S_n\psi(z)| + |S_n\psi(z) - S_n\psi(z_A)| + |S_n\psi(z_A) - n\int \psi d\nu| \\
&\leq 2K \sup_{w\in Z} |\psi(w)| + n\varepsilon/8 + n\varepsilon/8 \qquad \leq n\varepsilon/2.
\end{aligned}$$

As a consequence we obtain

$$\begin{aligned}
&m\Big(\Big\{x \in X : |\frac{1}{n}S_n\psi(x,0) - \int \psi d\nu| \leq \varepsilon/2\Big\}\Big) \\
&\quad = m_Z\Big(\Big\{z^* \in X_0 \times \{0\} : |S_n\psi(z^*) - n\int \psi d\nu| \leq n\varepsilon/2\Big\}\Big) \\
&\quad \geq \sum_{A^* \in \mathscr{B}^*_{n,k,l}} m_Z(A^*) \qquad = \sum_{A \in \mathscr{B}_{n,k,l}} m_Z(A) \\
&\quad \geq \exp\Big\{n\Big(h_\nu(T) - \int \log \mathrm{Jac}(T) d\nu - \eta/2\Big)\Big\}.
\end{aligned}$$

The inequality (2) is proved for the case ν is ergodic. For $\nu \in \mathscr{M}_T$ not ergodic take a linear combination $\nu' = \alpha_1\nu_1 + \cdots + \alpha_p\nu_p$ of ergodic T-invariant probality measures $\nu_1, \dots, \nu_p$ supported on $\sqcup_{k=0}^{K-1} X_k \times \{k\}$ such that

$$|h_\nu(T) - h_{\nu'}(T)| \leq \eta/8, \quad |\int \log \mathrm{Jac}(T) d\nu - \int \log \mathrm{Jac}(T) d\nu| \leq \eta/8$$

and

$$|\int \psi d\nu - \int \psi d\nu'| \leq \varepsilon/4.$$

For large $n \geq 1$ and $q = 1, \dots, p$ put $n_q := [n\alpha_q]$ where $[\cdot]$ denotes the Gauss' symbol. Applying the above argument for ν_q we can take integers k_q, l_q with $0 \leq k_q, l_q \leq K - 1$ and $\mathscr{B}^*_n(q) \subset \vee_{i=0}^{n_q+k_q-1} T^{-i}\mathscr{A}_K$ which consists of at least $\exp\{n\alpha_q(h_{\nu_q}(T) - \eta/4)\}$ elements A such that:

(1) $A \subset X_0 \times \{0\}$ and $T^{n_q+k_q+l_q}A = X_0 \times \{0\}$;
(2) $m_Z(A) \geq \exp\left\{-n\alpha_q\left(\int \log \mathrm{Jac}(T) d\nu_q + \eta/8\right)\right\}$;
(3) $|S_{n_q}\psi(z) - n_q\int \psi d\nu_q| \leq n_q\varepsilon/2$ holds whenever $z \in A$.

Then for any $z \in A$ with $A \in \mathscr{B}_n^*(q)$ we have

$$\begin{aligned}\prod_{i=0}^{r_q(n)-1} \mathrm{Jac}(T)(T^i(z)) &\geq {D_T}^{-1} \frac{m_Z(X_0 \times \{0\})}{m_Z(A)} \\ &\geq {D_T}^{-1} m(X_0) \exp\left\{-n_q \left(\int \log \mathrm{Jac}(T) d\nu_q + \eta/8\right)\right\} \\ &\geq \exp\left\{-n\alpha_q \left(\int \log \mathrm{Jac}(T) d\nu_q + \eta/4\right)\right\}\end{aligned}$$

where $r_q(n) = n_q + k_q + l_q$. Let

$$\mathscr{B}_n^* := \{B = \cap_{q=1}^p T^{-s_{q-1}(n)} B_q : B_q \in \mathscr{B}_n^*(q) \text{ for all } q = 1, \dots, p\}$$

where $s_0(n) = 0$ and $s_q(n) = r_1(n) + \cdots + r_q(n)$ for $q = 1, \dots, p$. Then the number of elements of $\mathscr{B}_n^*$ is not smaller than $e^{n(h_{\nu'}(T) - \eta/4)}$. For each $B \in \mathscr{B}_n^*$ take $B_q \in \mathscr{B}_n^*(q)$ for $q = 1, \dots, p$ with $B = \cap_{q=1}^p T^{-s_{q-1}(n)} B_q$ and $z \in B$. Then we have $T^{s_{q-1}(n)}(z) \in B_q$ for each $q = 1, \dots, p$, and hence

$$\begin{aligned}m_Z(B) &\geq D_T^{-1} m_Z(X_0 \times \{0\}) \prod_{i=0}^{s_q(n)-1} \mathrm{Jac}(T)(T^i(z)) \\ &= D_T^{-1} m(X_0) \prod_{q=1}^{p} \prod_{i=0}^{r_q(n)-1} \mathrm{Jac}(T)(T^i(T^{s_{q-1}(n)}(z))) \\ &\geq D_T^{-1} m(X_0) \exp\left\{-n \sum_{q=1}^{p} \alpha_q \left(\int \log \mathrm{Jac}(T) d\nu_q + \eta/4\right)\right\} \\ &\geq \exp\left\{-n \left(\int \log \mathrm{Jac}(T) d\nu' + \eta/2\right)\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{B \in \mathscr{B}_n^*} m_Z(B) &\geq \sharp \mathscr{B}_n^* \cdot \min_{B \in \mathscr{B}_n^*} m_Z(B) \\ &\geq \exp\left\{n \left(h_{\nu'}(T) - \int \log \mathrm{Jac}(T) d\nu' - 3\eta/4\right)\right\} \\ &\geq \exp\left\{n \left(h_{\nu}(T) - \int \log \mathrm{Jac}(T) d\nu - \eta/4\right)\right\}.\end{aligned}$$

Moreover, for any $z \in B$ with $B \in \mathscr{B}_n^*$,

$$
\begin{aligned}
&|S_n\psi(z) - n\int \psi d\nu| \le |S_n\psi(z) - n\int \psi d\nu'| + |n\int \psi d\nu' - n\int \psi d\nu| \\
&\le \sum_{q=1}^{p}\Big(|S_{n_q}\psi(z) - n_q\int \psi d\nu_q| + 2(k_q + r_q)\sup_{w\in Z}|\psi(w)|\Big) \\
&+ n|\int \psi d\nu' - \int \psi d\nu_q| \\
&\le \sum_{q=1}^{p}(n_q\varepsilon/2 + 4K\sup_{w\in Z}|\psi(w)|) + n\varepsilon/4 \quad \le n\varepsilon.
\end{aligned}
$$

From the estimates above we conclude

$$
\begin{aligned}
m\Big(\Big\{x \in & X : |\frac{1}{n}S_n\psi(x,0) - \int \psi d\nu| \le \varepsilon\Big\}\Big) \\
&= m_Z\Big(\Big\{z \in X_0 \times \{0\} : |S_n\psi(z) - n\int \psi d\nu| \le n\varepsilon\Big\}\Big) \\
&\ge \sum_{B\in\mathscr{B}_n^*} m_Z(B) \\
&\ge \exp\left\{n(h_\nu(T) - n\left(\int \log \mathrm{Jac}(T)d\nu - \eta\right)\right\}.
\end{aligned}
$$

The inequality (2) is obtained for $\nu \in \mathscr{M}_T$ even if it is not ergodic. It finishes the proof of the lower estimate.

Remark 3. *The nonsteep condition is not needed for the lower estimate in the above proof.*

The upper estimate. To obtain the upper estimate we need a variational principle for dynamical systems of bounded distortion as below. Let $(Y, \mathscr{B}, m)$ be a finite measure space and $Y_1, \ldots, Y_l \in \mathscr{B}$ pairwise disjoint subsets of Y with positive measures. We consider a measurable map $g : \sqcup_{j=1}^{l} Y_j \to Y$ satisfying the following properties:

(1) for each $j = 1, \ldots, l$, $g_j := g|_{Y_j} : Y_j \to Y$ is a bi-nonsingular bijection;
(2) for any sequence $\{a_j\}_{j=0}^{\infty}$ with $a_j \in \{1, \ldots, l\}$ $(j = 0, 1, 2, \ldots)$ the set $\cap_{n=0}^{\infty} Y_{a_0\ldots a_{n-1}}$ consists of a single point, where $Y_{a_0\cdots a_{n-1}} := Y_{a_0} \cap g^{-1}Y_{a_1} \cap \cdots \cap g^{-(n-1)}Y_{a_{n-1}}$;
(3) there are a version $\mathrm{Jac}(g) > 0$ of the Radon-Nikodym derivative $\frac{dm\circ g}{dm}$ with $\lim_{n\to\infty}\sup_{J\in W_n}\sup_{x,y\in Y_J}|\log \mathrm{Jac}(g)(x) - \log \mathrm{Jac}(g)(y)| = 0$ and a distortion constant $C \ge 1$ such that for any integer $n \ge 1$ and $J \in W_n$

$$
\frac{\prod_{i=0}^{n-1}\mathrm{Jac}(g)(g^i(x))}{\prod_{i=0}^{n-1}\mathrm{Jac}(g)(g^i(y))} \le C
$$

holds whenever $x, y \in Y_J$, where W_n stands for the set of the words $J = (a_0 \cdots a_{n-1})$ of length n with $a_i \in \{1, 2, \ldots, l\}$ for each $i = 0, 1, \ldots, n-1$.

Then we call $g : \Lambda \to \Lambda$ a *finite Markov system* induced by $(Y, \{Y_i\}_{i=1}^{l}, g)$, where $\Lambda := \cap_{n=0}^{\infty} g^{-n}(\sqcup_{j=1}^{l} Y_j)$. It is isomorphic to a full shift of l-symbols, and the space of the probability measures supported on Λ is compact. The following lemma is obtained from a standard argument on the variatinal principle for pressure [26].

Lemma 7. *A finite Markov system $g : \Lambda \to \Lambda$ induced by $(Y, \{Y_j\}_{j=1}^{l}, g)$ with a distortion constant $C \geq 1$ has an invariant probability measure μ on Λ such that*

$$h_\mu(g) - \int \log \mathrm{Jac}(g) d\mu \geq \log \sum_{p=1}^{l} m(Y_p) - \log m(Y) - \log C.$$

Proof of Lemma 7. For each $J = (a_0 \ldots a_{n-1}) \in W_n$ and $p = 1, \ldots, l$ let $Y_{Jp} := Y_J \cap g^{-n} Y_p$. Then since

$$\frac{m(Y_{Jp})}{m(Y_J)} \geq \frac{\prod_{i=0}^{n-1} \inf_{x \in Y_{a_0 \cdots a_{n-1} p}} \mathrm{Jac}(g)(g^i(x))^{-1} m(Y_p)}{\prod_{i=0}^{n-1} \sup_{y \in Y_{a_0 \cdots a_{n-1}}} \mathrm{Jac}(g)(g^i(y))^{-1} m(Y)} \geq C^{-1} \frac{m(Y_p)}{m(Y)}$$

for each $p = 1, \ldots, l$, we have

$$\frac{\sum_{p=1}^{l} m(Y_{Jp})}{m(Y_J)} \geq C^{-1} \frac{\sum_{p=1}^{l} m(Y_p)}{m(Y)},$$

and hence

$$\begin{aligned}
\sum_{J \in W_n} m(Y_J) &= \sum_{a_0, \ldots, a_{n-1}=1}^{l} m(Y_{a_0 \ldots a_{n-1}}) \\
&\geq \left\{ C^{-1} \frac{\sum_{p=1}^{l} m(Y_p)}{m(Y)} \right\} \sum_{a_0, \ldots, a_{n-2}=1}^{l} m(Y_{a_0 \ldots a_{n-2}}) \\
\cdots &\geq \left\{ C^{-1} \frac{\sum_{p=1}^{l} m(Y_p)}{m(Y)} \right\}^{n-1} \sum_{a_0=1}^{l} m(Y_{a_0}) \\
&= \left\{ C^{-1} \frac{\sum_{p=1}^{l} m(Y_p)}{m(Y)} \right\}^{n} C m(Y).
\end{aligned}$$

Thus we obtain

$$\liminf_{n \to \infty} \frac{1}{n} \log \sum_{J \in W_n} m(Y_J) \geq \log \sum_{p=1}^{l} m(Y_p) - \log m(Y) - \log C. \tag{3}$$

On the other hand, taking $x_J \in Y_J \cap \Lambda$ for each $J \in W_n$ we obtain a sequence of probability measures supported on Λ by $\mu_n := \frac{1}{Z_n} \sum_{J \in W_n} m(Y_J) \delta_{x_J}^n$ for each $n \geq 1$, where $Z_n := \sum_{J \in W_n} m(Y_J)$ and $\delta_{x_J}^n := (\delta_{x_J} + \delta_{g(x_J)} + \cdots + \delta_{g^{n-1}(x_J)})/n$. Then an accumulation point μ of the sequence $\{\mu_n\}_{n=1}^{\infty}$ is a g-invariant probability measure supported on Λ. Since

$$\begin{aligned}
&\log \sum_{J\in W_n} m(Y_J) = \log Z_n \\
&= \sum_{J\in W_n} \left(\frac{m(Y_J)}{\sum_{J'\in W_n} m(Y_{J'})}\right)\left\{-\log\left(\frac{m(Y_J)}{\sum_{J'\in W_n} m(Y_{J'})}\right) + \log m(Y_J)\right\} \\
&= \sum_{J\in W_n} \mu_n(Y_J)\left\{-\log \mu_n(Y_J) + \log m(Y_J)\right\} \\
&\leq \sum_{J\in W_n} \mu_n(Y_J)\left\{-\log \mu_n(Y_J) + \log\left(Cm(Y)\prod_{i=0}^{n-1} \mathrm{Jac}(g)(g^i(x_J))^{-1}\right)\right\} \\
&= -\sum_{J\in W_n} \mu_n(Y_J)\log \mu_n(Y_J) - n\int \log \mathrm{Jac}(g)d\mu_n + \log\{Cm(Y)\},
\end{aligned}$$

we have

$$\limsup_{n\to\infty} \frac{1}{n}\log \sum_{J\in W_n} m(Y_J) \leq h_\mu(g) - \int \log \mathrm{Jac}(g)d\mu. \tag{4}$$

Combining (3) and (4) we obtain

$$h_\mu(g) - \int \log \mathrm{Jac}(g)d\mu \geq \sum_{j=1}^{l} \log m(Y_j) - \log m(Y) - \log C.$$

The lemma is proved.

Now we show the upper estimate. Take the monotone sequences $\{l_k\}_{k=0}^{\infty} \subset \boldsymbol{N}$ and $\{\gamma_k\}_{k=0}^{\infty} \subset (0,1)$ as in the definition of the nonsteep condition for T. Put

$$-\beta := \limsup_{n\to\infty} \frac{1}{n}\log m\Big(\Big\{x\in X : \frac{1}{n}S_n\psi(x,0) \geq a\Big\}\Big).$$

If $\beta = \infty$ then nothing has to be shown, and so we assume that $\beta < \infty$. We show that for any $\varepsilon, \eta > 0$ there is $\nu \in \mathscr{M}_T$ with $\nu(\sqcup_{k=0}^{K-1} X_k \times \{k\}) = 1$ for some $K \geq 1$ such that

$$\int \psi d\nu > a - \epsilon \quad \text{ and } \quad h_\nu(T) - \int \log \mathrm{Jac}(T)d\nu \geq -(\beta+\eta)$$

hold. Take an arbitrarily large integer $N \geq 1$ such that

$$e^{-N(\beta+\eta/4)} \leq m\Big(\Big\{x\in X : \frac{1}{N}S_N\psi(x,0) \geq a\Big\}$$

and

$$|S_N\psi(z) - S_N\psi(w)| \leq \epsilon N/8$$

whenever $z, w \in A$ for some $A \in \vee_{i=0}^{N-1} T^{-i}\mathscr{A}$. Let

$$\mathscr{B}_N := \Big\{ A \in \vee_{i=0}^{N-1} T^{-i}\mathscr{A} : A \subset X_0 \times \{0\},$$
$$\frac{1}{N} S_N \psi(z_A) \geq a \text{ for some } z_A \in A \Big\}.$$

Then it is obvious that

$$\sum_{A \in \mathscr{B}_N} m_Z(A) \geq m\Big(\Big\{ x \in X : \frac{1}{N} S_N \psi(x, 0) \geq a \Big\} \geq e^{-N(\beta+\eta/4)}.$$

For each $A \in \mathscr{B}_N$ take $0 \leq k_A \leq N-1$ and $J_A \in \mathscr{I}_{k_A}$ such that

$$T^{N-1} A = J_A \times \{k_A\},$$

and let

$$J_{A,j} := (J_A \cap X_{k_A+j-1}) \setminus X_{k_A+j} \quad (j = 1, \ldots l_N).$$

Then from the monotonicity of the sequences $\{l_k\}_{k=0}^{\infty}$ and $\{\gamma_k\}_{k=0}^{\infty}$ in the assumption on the nonsteepness we have

$$\begin{aligned} \sum_{j=1}^{l_N} m(J_{A,j}) = m(J_A \setminus X_{k_A+l_N}) &\geq m(J_A \setminus X_{k_A+l_{k_A}}) \\ \geq \gamma_{k_A} m(J_A) \qquad &\geq \gamma_N m(J_A). \end{aligned}$$

For $j = 1, 2, \ldots, l_N$ we set

$$\mathscr{B}_{N,j} := \Big\{ B \in \vee_{i=0}^{N+j-1} T^{-i}\mathscr{A} : B \subset A_j, A \in \mathscr{B}_N \Big\},$$

where $A_j := A \cap T^{-N+1}(J_{A,j} \times \{k_A\})$ for each $A \in \mathscr{B}_N$. Then since

$$\begin{aligned} \frac{\sum_{j=1}^{l_N} m_Z(A_j)}{m_Z(A)} &\geq {D_T}^{-1} \frac{\sum_{j=1}^{l_N} m_Z(J_{A,j} \times \{k_A\})}{m_Z(J_A \times \{k_A\})} \\ &= {D_T}^{-1} \frac{\sum_{j=1}^{l_N} m(J_{A,j})}{m(J_A)} \quad \geq {D_T}^{-1} \gamma_N \end{aligned}$$

for each $A \in \mathscr{B}_N$, we have

$$\begin{aligned} \sum_{j=1}^{l_N} \sum_{B \in \mathscr{B}_{N,j}} m_Z(B) &= \sum_{A \in \mathscr{B}_N} \sum_{j=1}^{l_N} m_Z(A_j) \\ &\geq {D_T}^{-1} \gamma_N \sum_{A \in \mathscr{B}_N} m_Z(A) \\ &\geq {D_T}^{-1} \gamma_N e^{-N(\beta+\eta/4)}, \end{aligned}$$

and hence,

$$\sum_{B \in \mathscr{B}_{N,j_N}} m_Z(B) \geq {D_T}^{-1} {l_N}^{-1} \gamma_N e^{-N(\beta+\eta/4)}$$

holds for some $1 \leq j_N \leq l_N$. Then $(X_0 \times \{0\}, \mathscr{B}_{N,j_N}, T^{N+j_N}|_{X_0\times\{0\}})$ induces a finite Markov system on $X_0 \times \{0\}$. Set $K := N + j_N$. Then we obtain a T^K-invariant probability measure μ on $\Lambda := \cap_{l=0}^{\infty} T^{-lK}(\sqcup_{B\in\mathscr{B}_{N,j_N}} B) \subset X_0 \times \{0\}$ such that

$$\begin{aligned} &h_\mu(T^K) - \int \log \mathrm{Jac}(T^K) d\mu \\ &\qquad \geq \log \sum_{B\in\mathscr{B}_{N,j_N}} m_Z(B) - \log m_Z(X_0 \times \{0\}) - \log D_T \end{aligned}$$

by Lemma 7. Then $\nu := \frac{1}{K}\sum_{i=0}^{K-1} \mu \circ T^{-i}$ is a T-invariant probability measure satisfying $\nu(\sqcup_{k=0}^{K-1}(X_k \times \{k\})) = 1$. Moreover,

$$\begin{aligned} h_\nu(T) - \int \log \mathrm{Jac}(T) d\nu &= \frac{1}{K}\Big\{ h_\mu(T^K) - \int \log \mathrm{Jac}(T^K) d\mu \Big\} \\ \geq &\frac{1}{K}\Big\{ \log \sum_{B\in\mathscr{B}_{N,j_N}} m_Z(B) - \log m_Z(X_0 \times \{0\}) - \log D_T \Big\} \\ \geq &\frac{1}{K} \log \sum_{B\in\mathscr{B}_{N,j_N}} m_Z(B) - \eta/4 \\ \geq &\frac{1}{K} \log\{ {D_T}^{-1} {l_N}^{-1} \gamma_N e^{-N(\beta+\eta/4)} \} - \eta/4 \\ \geq &-\frac{N}{N+j_N}(\beta + \eta/4) - \frac{1}{N}\log D_T - \frac{1}{N}\log l_N + \frac{1}{N}\log \gamma_N - \eta/4 \\ \geq &-(\beta + \eta) \end{aligned}$$

holds if N is large. Furthermore, for any $z \in B$ with $B \in \mathscr{B}_{N,j_N}$ we can take $A \in \mathscr{B}_N$ with $B \subset A$ and $z_A \in A$ such that $S_N\psi(z_A)/N \geq a$. Then for a large integer $N \geq 1$ we have

$$\begin{aligned} S_K\psi(z_A) &\geq S_N\psi(z_A) - j_N \sup_{w\in Z} |\psi(w)| \\ &\geq Na - l_N \sup_{w\in Z} |\psi(w)| \\ &\geq (N + j_N)(a - \varepsilon/4) \quad = K(a - \varepsilon/4), \end{aligned}$$

and

$$\begin{aligned} |S_K\psi(z) - S_K\psi(z_A)| &\leq |S_N\psi(z) - S_N\psi(z_A)| \\ &\quad + |S_{j_N}\psi(T^N(z)) - S_{j_N}\psi(T^N(z_A))| \\ &\leq N\varepsilon/8 + 2j_N \sup_{w\in Z} |\psi(w)| \\ &\leq N\varepsilon/8 + 2l_N \sup_{w\in Z} |\psi(w)| \\ &\leq N\varepsilon/4 \quad \leq K\varepsilon/4. \end{aligned}$$

Then,

$$\begin{aligned} S_K\psi(z) &= S_K\psi(z_A) + (S_K\psi(z) - S_K\psi(z_A)) \\ &\geq K(a - \varepsilon/4) - K\varepsilon/4 \quad = K(a - \epsilon/2). \end{aligned}$$

Since $\nu \in \mathscr{M}_T$ and it is supported on $\cap_{l=0}^{\infty} T^{-lK}(\sqcup_{B\in\mathscr{B}_{N,j_N}} \sqcup_{i=0}^{K-1} T^i B)$ we have

$$\int \psi d\nu \geq a - \epsilon/2 > a - \epsilon.$$

Thus we obtain the upper estimate. This completes the proof of Theorem 1.

Remark 4. *To obtain the large deviations estimates as in Theorem 1, we can relax the bounded distortion condition to weaker one by replacing the constant* $D_T \geq 1$ *with a sequence* $\{D_n\}_{n=1}^{\infty}$ *of positive numbers satisfying* $\lim_{n\to\infty}(\log D_n)/n = 0$ *such that*

$$\frac{\prod_{i=0}^{n-1} \mathrm{Jac}(T)(T^i(z))}{\prod_{i=0}^{n-1} \mathrm{Jac}(T)(T^i(w))} \leq D_n$$

holds whenever $z, w \in A$ *for some* $A \in \vee_{i=0}^{n-1} T^{-i}\mathscr{A}$ *and* $n \geq 1$. *A thermodynamic formalism for dynamical systems satisfying this weak bounded distortion condition has been studied by Yuri* [31].

5. Proof of Theorem 3

Throughout this section, I denotes a compact interval of the real line and m Lebesgue measure. Let $f : I \to I$ be a topological mixing C^2 map satisfying the assumptions stated in Section 2. We notice that the weak* topology on the space $\mathscr{M}$ of the probability measures is generated by open sets $\mathscr{G}$ of the form

$$\mathscr{G} := \Big\{\mu \in \mathscr{M} : \max_{i=1,\dots,l} \Big| \int \varphi_i d\mu - \alpha_i \Big| < \varepsilon\Big\}$$

for some $\varphi_1, \dots, \varphi_l \in C(I)$, $\alpha_1, \dots, \alpha_l \in \boldsymbol{R}$ and $\varepsilon > 0$, where $C(I)$ denotes the space of the continuous functions on I. Thus, the proof of Theorem 3 is reduced to the estimates on the level 1 large deviations for given $\varphi \in C(I)$ and $a \in \boldsymbol{R}$ as follows:

(1) (The lower estimate)

$$\liminf_{n\to\infty} \frac{1}{n} \log m\Big(\Big\{x \in I : \frac{1}{n} S_n\varphi(x) > a\Big\}\Big) \geq \sup\Big\{q(\mu) : \int \varphi d\mu > a\Big\};$$

(2) (The upper estimate)

$$\limsup_{n\to\infty} \frac{1}{n} \log m\Big(\Big\{x \in I : \frac{1}{n} S_n\varphi(x) \geq a\Big\}\Big) \leq \max\Big\{q(\mu) : \int \varphi d\mu \geq a\Big\},$$

where q is the upper regularization of F,

$$F(\mu) := \begin{cases} h_\mu(f) - \int \log |f'| d\mu & \text{for } \mu \in \mathscr{M}_f \text{ hyperbolic,} \\ -\infty & \text{otherwise.} \end{cases}$$

The lower estimate. To obtain the lower estimate of the rate function we show that

$$\liminf_{n\to\infty} \frac{1}{n} \log m \left(\left\{ x \in I : \frac{1}{n} S_n\varphi(x) > a \right\} \right) \geq h_\mu(f) - \int \log |f'| d\mu \tag{5}$$

holds for any $\mu \in \mathscr{M}_f$ hyperbolic with $\int \varphi d\mu > a$. First we assume that μ is ergodic. Taking $\varepsilon > 0$ small enough so that $\int \varphi d\mu > a + \varepsilon$. Then there are integers $k, l \geq 1$ with $(\log l)/k \geq h_\mu(f) - \varepsilon$ and pairwise disjoint compact intervals $L_1, L_2, \ldots, L_l$ with $L \subset I$ such that: $L_i \subset L, f^k(L_i) = L$ and $f^k \mid_{L_i}: L_i \to L$ is injective on L_i $(i = 1, 2, \ldots l)$;

$$|\frac{1}{k} \log |(f^k)'(x)| - \int \log |f'| d\mu| \leq \varepsilon \quad \text{ and } \quad \frac{1}{k} S_k\varphi(x) \geq \int \varphi d\mu - \epsilon > a$$

whenever $x \in \sqcup_{i=1}^{l} L_i$. Then

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{n} \log m \left(\left\{ x \in I : \frac{1}{n} S_n\varphi(x) > a \right\} \right) \\ &\qquad \geq \liminf_{n\to\infty} \frac{1}{kn} \log m \left(\cap_{j=0}^{n-1} f^{-kn} (\sqcup_{i=1}^{l} L_i) \right) \\ &\qquad \geq \liminf_{n\to\infty} \frac{1}{kn} \log \{ l^n \min_{x \in \sqcup_{i=1}^{l} L_i} |(f^k)'(x)|^{-n} m(J) \} \\ &\qquad \geq \frac{1}{k} \log l - \int \log |f'| d\mu - \varepsilon \\ &\qquad \geq h_\mu(f) - \int \log |f'| d\mu - 2\varepsilon \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain the inequlity (5) for the case that μ is ergodic. If μ is not ergodic, then take $\varepsilon > 0$ small enough so that $\displaystyle\int \varphi d\mu > \alpha + \varepsilon$ and a linear combination $\mu' = \alpha_1\mu_1 + \cdots + \alpha_p\mu_p$ of ergodic and hyperbolic measures $\mu_1, \ldots, \mu_p$ such that

$$|h_\mu(f) - h_{\mu'}(f)| \leq \varepsilon/4, \qquad |\int \log |f'| d\mu - \int \log |f'| d\mu'| \leq \varepsilon/4$$

and

$$|\int \varphi d\mu - \int \varphi d\mu'| \leq \varepsilon/2.$$

Applying the argument above for each μ_q, $q = 1, \ldots, p$, we can take integers $k_q, l_q \geq 1$ with $(\log l_q)/k_q \geq h_{\mu_q}(f) - \varepsilon/4$ and pairwise disjoint compact intervals $L_1^q, \ldots, L_{l_q}^q$

with $L^q \subset I$ such that: $L_i^q \subset L^q$, $f^{k_q}(L_i^q) = L^q$ and $f^{k_q}|_{L_i^q}: L_i^q \to L^q$ is injective on L_i^q $(i = 1, 2, \dots l_q)$;

$$|\frac{1}{k_q} \log |(f^{k_q})'(x)| - \int \log |f'| d\mu_q| \le \varepsilon/8 \quad \text{and} \quad |\frac{1}{k_q} S_{k_q}\varphi(x) - \int \varphi d\mu_q| \le \epsilon/4$$

whenever $x \in \sqcup_{i=1}^{l_q} L_i^q$. On the other hand, there are integers $K_1, \dots, K_p \ge 1$ such that $f^{K_q}(L^q) = I$, since f is topologically mixing. For any large integer $n \ge 1$ define $r_0(n) := 0$ and $r_q(n) := r_{q-1}(n) + [n\alpha_q/k_q]k_q + K_q$ inductively on $q = 1, \dots, p$, and set

$$B_n := \cap_{q=1}^{p} \cap_{j=0}^{[n\alpha_q/k_q]-1} f^{-r_{q-1}(n)-jk_q}(\sqcup_{i=1}^{l_q} L_i^q),$$

where $[\cdot]$ denotes the Gauss' symbol. Then since $n \ge 1$ is large, for any $x \in B_n$ we have

$$\begin{aligned}
|(f^{r_p(n)})'(x)| &\le (\max_{y \in I} |f'(y)|)^{K_1+\cdots+K_p} \cdot \prod_{q=1}^{p} |(f^{[n\alpha_q/k_q]k_q})'(f^{r_{q-1}(n)}(x))| \\
&\le (\max_{y \in I}|f'(y)|)^{K_1+\cdots+K_p} \cdot \sum_{q=1}^{p} \exp\left\{[n\alpha_q/k_q]k_q \left(\int \log |f'| d\mu_q + \varepsilon/8\right)\right\} \\
&\le \exp\left\{n\left(\int \log |f'| d\mu' + \varepsilon/4\right)\right\}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
m(B_n) &\ge {l_1}^{[n\alpha_1/k_1]} \cdots {l_p}^{[n\alpha_p/k_p]} \left(\max_{x \in B_n} |(f^{r_p(n)})'(x)|\right)^{-1} m(I) \\
&\ge {l_1}^{[n\alpha_1/k_1]} \cdots {l_p}^{[n\alpha_p/k_p]} \exp\left\{-n\left(\int \log |f'| d\mu' + \varepsilon/4\right)\right\} m(I).
\end{aligned}$$

Moreover, for any $x \in B_n$ we have

$$\begin{aligned}
&\left|S_n\varphi(x) - n\int \varphi d\mu'\right| \\
&\qquad \le \sum_{q=1}^{p} |S_{[n\alpha_q/k_q]k_q}\varphi(f^{r_{q-1}(n)}(x)) - \left[\frac{n\alpha_q}{k_q}\right] k_q \int \varphi d\mu_q| \\
&\qquad + \sum_{q=1}^{p} 2(k_q + K_q) \max_{y \in I} |\varphi(y)| \\
&\qquad \le n\varepsilon/4 + n\varepsilon/4 \qquad = n\varepsilon/2,
\end{aligned}$$

and then

$$\begin{aligned}
|\frac{1}{n} S_n\varphi(x) - \int \varphi d\mu| &\le |\frac{1}{n} S_n\varphi(x) - \int \varphi d\mu'| + |\int \varphi d\mu' - \int \varphi d\mu| \\
&\le \varepsilon/2 + \varepsilon/2 \qquad = \varepsilon.
\end{aligned}$$

Thus,

$$\frac{1}{n}S_n\varphi(x) > \int \varphi d\mu - \varepsilon > a.$$

As a conclusion we have

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{n}\log m\Big(\Big\{x\in I : \frac{1}{n}S_n\varphi(x) > a\Big\}\Big) &\geq \liminf_{n\to\infty}\frac{1}{n}\log m(B_n)\\ &\geq \sum_{q=1}^{p}\frac{\alpha_q}{k_q}\log l_q - \Big(\int \log|f'|d\mu' + \varepsilon/4\Big)\\ &\geq h_{\mu'}(f) - \int \log|f'|d\mu' - \varepsilon/2\\ &\geq h_{\mu}(f) - \int \log|f'|d\mu - \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain (5), and hence the lower estimate of the rate function.

The upper estimate. We show that for any $\varepsilon, \eta > 0$ there exists $\mu \in \mathscr{M}_f$ hyperbolic with $\int \varphi d\mu > a - \varepsilon$ such that

$$\limsup_{n\to\infty}\frac{1}{n}\log m\Big(\Big\{x\in I : \frac{1}{n}S_n\varphi(x) \geq a\Big\}\Big) \leq h_{\mu}(f) - \int \log|f'|d\mu + \eta \tag{6}$$

holds whenever the left hand side of the inequality (6) is not $-\infty$. Take a subinterval $J \subset I$ and a return time function $R : J \to \boldsymbol{N} \cup \{\infty\}$ as in the assumptions for f stated in Section 2. Then setting $X_k := \{x \in J : R(x) > k\}$ $(k = 0, 1, 2, \ldots)$ we obtain a tower $(Z, \mathscr{A})$ by

$$Z := \sqcup_{k=0}^{\infty} X_k \times \{k\}$$

and

$$\mathscr{A} := \{J \times \{k\} : J \in \mathscr{I}_k, k = 0, 1, 2, \ldots\}$$

where $\mathscr{I}_k$ is the partition of X_k which consists of the connected components of both $\{x \in J : R(x) = k+1\}$ and $\{x \in J : R(x) > k+1\}$. Then it follows that the tower $(Z, \mathscr{A})$ is nonsteep from the assumptions for the map. A tower map $T : Z \to Z$ defined by

$$T(x,k) := \begin{cases}(x, k+1) & \text{if } R(x) > k+1,\\ (f^{k+1}(x), 0) & \text{if } R(x) = k+1,\end{cases}$$

satisfies $\pi \circ T = f \circ \pi$ on Z where $\pi(x,k) = f^k(x)$. It also follows from the assumptions for f that the map $T : Z \to Z$ satisfies both of the admissibility and the bounded distortion conditions. Moreover, if $A \in \vee_{i=0}^{n-1} T^{-i}\mathscr{A}$, then $\pi(A)$ is an interval with length less than or equal to ε_n, and it implies that the function $\psi : Z \to \boldsymbol{R}$

defined by $\psi(x,k) = \varphi(f^k(x))$ is contained in the class $\mathscr{F}$. Then since there is an integer $l \geq 1$ such that $f^l J = I$, taking large $n \geq 1$ we have

$$\begin{aligned} m\Big(\Big\{x \in I : \frac{1}{n}S_n\varphi(x) \geq a\Big\}\Big) &= m\Big(\Big\{x \in f^l J : \frac{1}{n}S_n\varphi(x) \geq a\Big\}\Big) \\ &\leq m\Big(f^l\Big(\Big\{x \in J : \frac{1}{n}S_n\varphi(x) \geq a - \epsilon/2\Big\}\Big)\Big) \\ &\leq (\max_{y\in I}|(f)'(y)|)^l m\Big(\Big\{x \in J : \frac{1}{n}S_n\varphi(x) \geq a - \epsilon/2\Big\}\Big) \\ &= (\max_{y\in I}|(f)'(y)|)^l m\Big(\Big\{x \in X_0 : \frac{1}{n}S_n\psi(x,0) \geq a - \epsilon/2\Big\}\Big). \end{aligned}$$

Hence, there is $\nu \in \mathscr{M}_T$ with $\nu(\sqcup_{k=0}^{K-1} X_k \times \{k\}) = 1$ for some $K \geq 1$ such that $\displaystyle\int \psi d\nu > a - \varepsilon$ and

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n}\log m\Big(\Big\{x \in I : \frac{1}{n}S_n\varphi(x) \geq a\Big\}\Big) \\ &\qquad\qquad \leq \limsup_{n\to\infty} \frac{1}{n}\log m\Big(\Big\{x \in X_0 : \frac{1}{n}S_n\psi(x,0) \geq a - \epsilon/2\Big\}\Big) \\ &\qquad\qquad \leq h_\nu(T) - \int \log \mathrm{Jac}(T) d\nu + \eta \end{aligned}$$

by Theorem 1. Then the f-invariant probability measure $\mu := \nu \circ \pi^{-1}$ on I is hyperbolic because the Lyapunov exponents on the support of μ are not smaller than $(\log\lambda)/K$ uniformly. Also, μ satisfies

$$\int \varphi d\mu = \int \psi d\nu > a - \varepsilon$$

and

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n}\log m\Big(\Big\{x \in I : &\frac{1}{n}S_n\varphi(x) \geq a\Big\}\Big) \\ &\leq h_\nu(T) - \int \log \mathrm{Jac}(T) d\nu + \eta \\ &= h_\mu(f) - \int \log|f'| d\mu + \eta. \end{aligned}$$

This completes the proof of Theorem 3.

References

[1] V. Araújo and M.J. Pacifico *Large deviations for non-uniformly expanding maps*, J. Stat. Phys. **125** (2006), 411–453.

[2] V. Baladi, *Positive Transfer Operators and Decay of Correlations*, World Scientific, Singapore, 2000.

[3] M. Benedicks and L. Carleson, *On iterations of* $1-ax^2$ *on* $(-1,1)$, Ann. of Math. **122** (1985), 1–25.

[4] H. Bruin and G. Keller, *Equilibrium states for S-unimodal maps*, Ergod. Th. Dynam. Sys. **18** (1998), 765–789.

[5] H. Bruin, S. Luzzatto and S. van Strien, *Decay of correlations in one-dimensional dynamics*, Ann. Sci. École Norm. Sup. **36** (2003), 621–646.

[6] Y.M. Chung, *Shadowing property of non-invertible maps with hyperbolic measures*, Tokyo J. Math. **22** (1999), 145–166.

[7] Y.M. Chung, *Birkhoff spectra for one-dimensional maps with some hyperbolicity*, in preparation.

[8] P. Collet and J.-P. Eckmann, *Positive Lyapunov exponents and absolutely continuity for maps of the interval*, Ergod. Th. Dynam. Sys. **3** (1983), 13–46.

[9] R.S. Ellis, *Entropy, Large deviations and Statistical Mechanics*, Grundlehren der Mathematischen Wissenschaften **271**, Springer-Verlag, New York, 1985.

[10] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Inst. Hautes Etudes Sci. Publ. Math. **51** (1980), 137–173.

[11] A. Katok and L. Mendoza, *Dynamical systems with nonuniformly hyperbolic behavior*, supplement to "Introduction to the modern theory of dynamical systems" written by A. Katok and B. Hasselblatt, Cambridge Univ. Press, Cambridge, 1995, pp 659–700.

[12] G. Keller, *Equilibrium states in ergodic theory*, London Mathematical Society Student texts **42**, Cambridge Univ. Press, Cambridge, 1998.

[13] G. Keller and T. Nowicki, *Spectral theory, zeta functions and the distribution of periodic points for Collet-Eckmann maps*, Commun. Math. Phys. **149** (1992), 31–69.

[14] A.I. Khinchin, *Über einen neuen Grenzwertsats der Wahrscheinlichkeitsrechnung*, Math. Annalen **101**, (1929), 745–752.

[15] F. Ledrappier, *Some properties of absolutely continuous invariant measures of an interval*, Ergod. Th. Dynam. Sys. **1** (1981), 77–93.

[16] I. Melbourne and M. Nicol, *Large deviations for nonuniformly hyperbolic systems*, to appear in Trans. Amer. Math. Soc.

[17] T. Nowicki and D. Sands, *Non-uniform hyperbolicity and universal bounds for S-unimodal maps*, Invent. Math. **132** (1998), 633–680.

[18] S. Orey and S. Pelikan, *Deviation of trajectory averages and the defect in Pesin's formula for Anosov diffeomorphisms*, Trans. Amer. Math. Soc. **315** (1989), 741–753.

[19] Ya. Pesin, *Characteristic Lyapunov exponents and smooth ergodic theory*, Russian Math. Survays **32** (1977), 55–114.

[20] M. Pollicott, R. Sharp and M. Yuri, *Large deviations for maps with indifferent fixed points*, Nonlinearity **11** (1998), 1173–1184.

[21] Y. Pomeau and P. Manneville, *Intermittent transition to turbulence in dissipative dynamical systems*, Commun. Math. Phys. **74** (1980), 189–197.

[22] D. Ruelle, *An inequality for the entropy of differentiable maps*, Bol. Soc. Brasil. Math. **9** (1978), 83–87.

[23] Y. Takahashi, *Fredholm determinant of unimodal linear maps* Sci. Papers College Gen. Ed. Univ. Tokyo **31** (1981), 61–87. 58F13 (28A05)

[24] Y. Takahashi, *Entropy functional (free energy) for dynamical systems and their random perturbations*, in "Stochastic analysis (Katata/Kyoto, 1982)", North-Holland, Amsterdam, 1984, pp 437–467.

[25] Y. Takahashi, *Two aspects of large deviation theory for large time*, in "Probabilistic methods in mathematical physics (Katata/Kyoto, 1985)", Academic Press, Boston, 1987, pp 363–384.

[26] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics **79**, Springer-Verlag, New York, 1982.

[27] L.-S. Young, *Some large deviation results for dynamical systems*, Trans. Amer. Math. Soc. **318** (1990), 525–543.

[28] L.-S. Young, *Decay of correlations for certain quadratic maps*, Commun. Math. Phys. **146** (1992), 123–138.

[29] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity*, Ann. of Math. **147** (1998), 585–650.

[30] L.-S. Young, *Recurrence times and rates of mixing*, Israel J. Math. **110** (1999), 153–188.

[31] M. Yuri, *Thermodynamic formalism for certain nonhyperbolic maps*, Ergod. Th. Dynam. Sys. **19** (1999), 1365–1378.

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