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THE STUDY OF A FOURTH ORDER NONLINEAR ELLIPTIC EQUATION BY CRITICAL POINT THEORY

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Abstract. We investigate the existence of solutions of the fourth order nonlinear elliptic boundary value problem under Dirichlet boundary condition $\Delta^2 u + c\Delta u = g(u)$ in Ω, where Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary.

0. Introduction

We investigate the existence of solutions of the fourth order nonlinear elliptic boundary value problem

$$\begin{aligned} \Delta^2 u + c\Delta u = g(u) \quad \text{in} \quad \Omega, \\ u = 0, \quad \Delta u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{0.1}$$

where c is not an eigenvalue of $-\Delta$ under Dirichlet boundary condition. Here we assume that Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The operator Δ^2 denotes the biharmonic operator. We assume that b is not an eigenvalue of $\Delta^2 + c\Delta$ under Dirichlet boundary condition.

The nonlinear equation with jumping nonlinearity have been extensively studied by many authors [6,7,8]. They studied the existence of solutions of the nonlinear equation with jumping nonlinearity for the second order elliptic operator [6], for one dimensional wave operator [3,4], and for the other operators [7,8] when the source term is a multiple of the positive eigenfunction.

In [10], Tarantello considered the fourth order, nonlinear elliptic problem under the Dirichlet boundary condition

$$\begin{aligned} \Delta^2 u + c\Delta u = b[(u+1)^+ + 1] \quad \text{in} \quad \Omega, \\ u = 0, \quad \Delta u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{0.2}$$

She showed by degree theory that if $b \geq \lambda_1(\lambda_1 - c)$, then (0.2) has a solution u such that $u(x) < 0$ in Ω.

In this paper we investigate the existence of solutions of the fourth order nonlinear equation (0.1) when the nonlinearity bu^+ crosses eigenvalues of $\Delta^2 + c\Delta$ under Dirichlet boundary condition.

In section 1, we introduce the Banach space spanned by eigenfunctions of $\Delta^2 + c\Delta$ and investigate the existence of solutions of (0.1) when the nonlinearity $g(u) = b^+ + s$ satisfies $\lambda_1 < c$, $b < \lambda_1(\lambda_1 - c)$ and when it satisfies $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b$. We

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investigate the multiplicity of solutions of (0.1) under the following two conditions.
Condition(1) : $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $f = s > 0$.
Condition(2) : $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$ $(k = 1, 2, \cdots)$ and $s < 0$.

In the section 2, we recall a variation of linking theorem (see Theorem 3.4 of [3] and [5]) of the existence of critical levels for a functional corresponding to equation (0.1).

1. The Jumping Nonlinearity

In this section we introduce the Banach space spanned by eigenfunctions of the operator $\Delta^2 + c\Delta$ and we investigate the existence of solutions of the boundary value problem

$$\Delta^2 u + c\Delta u = bu^+ + s \quad \text{in} \quad \Omega, \tag{1.1}$$
$$u = 0, \quad \Delta u = 0 \quad \text{on} \quad \partial\Omega.$$

Here s is real, c is not an eigenvalue of $-\Delta$ under Dirichlet boundary condition and the nonlinearity bu^+ satisfies $\lambda_1 < c$, $b < \lambda_1(\lambda_1 - c)$ or $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b$.

Let $\lambda_k (k = 1, 2, \cdots)$ denote the eigenvalues and $\phi_k (k = 1, 2 \cdots)$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem $\Delta u + \lambda u = 0$ in Ω, under Dirichlet boundary condition, where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, $\lambda_i \to +\infty$ and that $\phi_1(x) > 0$ for $x \in \Omega$. The eigenvalue problem $\Delta^2 u + c\Delta u = \mu u$ in Ω, with Dirichlet boundary condition, has infinitely many eigenvalues $\mu_k = \lambda_k(\lambda_k - c)$, $(k = 1, 2, \cdots)$ and corresponding eigenfunctions $\phi_k(x)$. The set of functions $\{\phi_k\}$ is an orthogonal base for $W_0^{1,2}(\Omega)$. Let us denote an element u of $W_0^{1,2}(\Omega)$ as

$$u = \sum h_k \phi_k, \qquad \sum h_k^2 < \infty.$$

Let c be not an eigenvalue of $-\Delta$ and define a subspace H of $W_0^{1,2}(\Omega)$ as follows

$$H = \{u \in W_0^{1,2}(\Omega) : \sum |\lambda_k(\lambda_k - c)| h_k^2 < \infty\}.$$

Then this is a complete normed space with a norm $|||u||| = [\sum |\lambda_k(\lambda_k - c)| h_k^2]^{1/2}$.

Lemma 1.1. *Let d be not an eigenvalue of $\Delta^2 + c\Delta$ and $u \in L^2(\Omega)$. Then $(\Delta^2 + c\Delta + d)^{-1} u \in H$.*

Lemma 1.2. *Let $f \in L^2(\Omega)$. Let b be not an eigenvalue of $\Delta^2 + c\Delta$. Then all solutions in $W_0^{1,2}(\Omega)$ of*

$$\Delta^2 u + c\Delta u = bu^+ + f(x)$$

belong to H.

With the aid of Lemma 1.2, it is enough to investigate the existence of solutions of (1.1) in the subspace H of $W_0^{1,2}(\Omega)$, namely,

$$\Delta^2 u + c\Delta u = bu^+ + s \quad \text{in} \quad H. \tag{1.2}$$

Let $\lambda_k < c < \lambda_{k+1}$ and $\lambda_k(\lambda_k - c)$, $\lambda_{k+1}(\lambda_{k+1} - c)$ be successive eigenvalues of $\Delta^2 + c\Delta$ such that there is no eigenvalue between $\lambda_k(\lambda_k - c)$ and $\lambda_{k+1}(\lambda_{k+1} - c)$. Then $\lambda_k(\lambda_k - c) < 0 < \lambda_{k+1}(\lambda_{k+1} - c)$ and we have the uniqueness theorem.

Theorem 1.1. *Suppose $\lambda_k < c < \lambda_{k+1}$ and $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$. Then equation (1.2) has exactly one solution in $L^2(\Omega)$ for all real s. Furthermore equation (1.2) has a unique solution in H.*

We now examine equation (1.2) when $\lambda_1 < c$ and $b < \lambda_1(\lambda_1 - c) < 0$.

Theorem 1.2. *Assume that $\lambda_1 < c$ and $b < \lambda_1(\lambda_1 - c) < 0$. Then we have :*
(i) If $s < 0$, then equation (1.2) has no solution.
(ii) If $s = 0$, then equation (1.2) has only the trivial solution.

For the case $s > 0$ in Theorem 1.2, we shall investigate the existence of solutions of (1.2) in the next section.

If $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b$ and $s > 0$, then the left hand side of (1.5) is larger than or equal to zero and the right hand side of it is negative.

Therefore we have the following theorem.

Theorem 1.3. *Assume that $c < \lambda_1$ and $0 < \lambda_1(\lambda_1 - c) < b$, $b \neq \lambda_k(\lambda_k - c)$, $k = 2, 3, \cdots$. Then we have :*
(i) If $s > 0$, then equation (1.2) has no solution.
(ii) If $s = 0$, then equation (1.2) has only the trivial solution.

Proof. Assume $s \geq 0$. We rewrite (1.2) as

$$\{\Delta^2 + c\Delta - \lambda_1(\lambda_1 - c)\}u + [\lambda_1(\lambda_1 - c) - b]u^+ - \lambda_1(\lambda_1 - c)u^- = s.$$

Multiply across by ϕ_1 and integrate over Ω. Since $(\{\Delta^2 + c\Delta - \lambda_1(\lambda_1 - c)\}u, \phi_1) = 0$, we have

$$\int_\Omega \{[\lambda_1(\lambda_1 - c) - b]u^+ - \lambda_1(\lambda_1 - c)u^-\}\phi_1 = s \int \phi_1. \tag{1.6}$$

But $[\lambda_1(\lambda_1 - c) - b]u^+ - \lambda_1(\lambda_1 - c)u^- \leq 0$ for any real valued function u. Also $\phi_1(x) > 0$ in Ω. Therefore, if $s > 0$, then equation (1.2) has no solution and if $s = 0$, then the only possibility is that $u = 0$. □

We investigate the multiplicity of solutions of (1.2) under the following two conditions.

Condition(1) : $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$.

Condition(2) : $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$ $(k = 1, 2, \cdots)$ and $s < 0$.

First we investigate the multiplicity of solutions of (1.2) under the Condition(1).

Theorem 1.4. *Assume that $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Then the problem (1.2) has at least two solutions.*

One solution is positive and we show the existence of the other solution of (1.2) by showing the existence of the critical point of the functional

$$F_b(u,s) = \int_\Omega \left[\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \frac{b}{2}|u^+|^2 - su\right] dx$$

Next, we investigate the multiplicity of solutions of (1.2) under the Condition (2).

Theorem 1.5. *Assume that $c < \lambda_1$, $0 < \lambda_1(\lambda_1 - c)$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$, $(k \geq 0)$ and $s < 0$. Then the problem (1.2) has at least two solutions.*

One solution is a negative solution and the existence of another solution be shown by showing the existence of the critical point of the functional $F_b(u,s)$.

2. A Variation of Lingking Theorem

Let $g : R \to R$ be a differentiable function such that $g(0) = 0$, and

$$g'(\infty) = \lim_{|u|\to\infty} \frac{g(u)}{u} \in R.$$

Let Ω be a smooth bounded region in R^n with smooth boundary $\partial\Omega$. Let $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$ be the eigenvalues of $-\Delta$ with Dirichlet boundary condition in Ω. In this paper we are concerned with the multiplicity of the solutions of the nonlinear biharmonic equation with Dirichlet boundary condition

$$\Delta^2 u + c\Delta u = g(u) \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega,$$

where $c \in R$ and Δ^2 denotes the biharmonic operator.

We recall a variation of linking theorem (see Theorem 3.4 of [3] and [5]) of the existence of critical levels for a functional, which will be used for the proof of the main theorem.

Let X be an Hilbert space, $Y \subset X$, $\rho > 0$ and $e \in X\backslash Y$, $e \neq 0$. Set:

$$B_\rho(Y) = \{x \in Y : \|x\|_X \leq \rho\},$$

$$S_\rho(Y) = \{x \in Y : \|x\|_X = \rho\},$$

$$\Delta_\rho(e,Y) = \{\sigma e + v : \sigma \geq 0, v \in Y, \|\sigma e + v\|_X \leq \rho\},$$

$$\Sigma_\rho(e,Y) = \{\sigma e + v : \sigma \geq 0, v \in Y, \|\sigma e + v\|_X = \rho\} \cup \{v : v \in Y, \|v\|_X \leq \rho\}.$$

Theorem 2.1. *("A Variation of Linking") Let X be an Hilbert space, which is topological direct sum of the subspaces X_1 and X_2. Let $F \in C^1(X,R)$. Moreover assume:*

(a) $\dim X_1 < +\infty$;

(b) there exist $\rho > 0$, $R > 0$ and $e \in X_1$, $e \neq 0$ such that $\rho < R$ and

$$\sup_{S_\rho(X_1)} F < \inf_{\Sigma_R(e,X_2)} F;$$

(c) $-\infty < a = \inf_{\Delta_R(e,X_2)} F$;

(d) $(P.S.)_c$ holds for any $c \in [a,b]$, where $b = \sup_{B_\rho(X_1)} F$.

Then there exist at least two critical levels c_1 and c_2 for the functional F such that :

$$\inf_{\Delta_R(e,X_2)} F \leq c_1 \leq \sup_{S_\rho(X_1)} F < \inf_{\Sigma_R(e,X_2)} F \leq c_2 \leq \sup_{B_\rho(X_1)} F.$$

By the Theorem 2.1 we show the following.

THEOREM 2.2. *Assume that $\lambda_i < c < \lambda_{i+1}$, $\lambda_{i+1}(\lambda_{i+1} - c) < \lambda_k(\lambda_k - c) < g'(\infty) < \lambda_{k+1}(\lambda_{k+1} - c)$, $\lambda_{k+m}(\lambda_{k+m} - c) < g'(0) < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$ for $m \geq 1$, and $g'(t) \leq \gamma < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$ where $k > i+1$. Then problem (1.1) has at least two nontrivial solutions.*

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