

DENSITY OF CLOSED GEODESICS IN TWO STEP NILMANIFOLDS

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ABSTRACT. Let M be a compact Riemannian manifold. It is very interesting to know how many closed geodesics there are in the manifold M . The manifold M is said to have the density of closed geodesics if the set of closed geodesics in M is dense. In this survey, we will give some progress in the problem related to the density of closed geodesics in two step nilmanifolds. The most results introduced in this paper are in [?], [2], [7], [1],[6] and [?].

1. CLOSED GEODESICS AND DENSITY OF CLOSED GEODESICS

Let M be a compact Riemannian manifold. A geodesic $\gamma(t)$ in M is called the closed geodesic if for some $\omega > 0$, $\gamma(t + \omega) = \gamma(t)$, $t \in \mathbb{R}$. A compact Riemannian manifold M is said to have the density of closed geodesics if the set of closed geodesics is dense in M . For example, consider 2-sphere S^2 . The great circles in S^2 are closed geodesics and the set of closed geodesics is equal to S^2 . So, 2-sphere S^2 has the density of closed geodesics.

Another example is the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$. Let $(\mathbb{R}^2, +)$ be the abelian group with the Euclidean inner product and \mathbb{Z}^2 its integral lattice. Then, the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ has the density of closed geodesics.

Let \mathcal{N} be a n -dimensional Lie algebra over the real numbers \mathbb{R} . It is well-known, for example [9], that a Lie algebra \mathcal{N} determines unique simply-connected Lie group N with its Lie algebra \mathcal{N} .

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The Lie algebra \mathcal{N} is called abelian if $[\mathcal{N}, \mathcal{N}] = 0$. So, $(R^n, +)$ is unique simply-connected Lie group determined by abelian Lie algebra \mathcal{N} . As in the 2-torus R^2/Z^2 , n -torus R^n/Z^n has also the density of closed geodesics.

A Lie algebra \mathcal{N} is called two step nilpotent if $[[\mathcal{N}, \mathcal{N}], \mathcal{N}] = 0$ and \mathcal{N} is non-abelian. A Lie group N is said to be two step nilpotent if its Lie algebra is two step nilpotent.

Let N be a two step nilpotent group with a left-invariant metric, Γ a cocompact lattice in N and $\Gamma \backslash N$ the two step nilmanifold with the metric induced naturally by the left-invariant metric.

Problem : What conditions are necessary and/or sufficient for $\Gamma \backslash N$ having the density of closed geodesics?

2. TWO STEP NILPOTENT METRIC GROUPS

Let \mathcal{N} be a two step nilpotent Lie algebra with an inner product \langle, \rangle and N its unique simply connected two step nilpotent Lie group with the left invariant metric induced by \langle, \rangle on \mathcal{N} .

Denote the center of \mathcal{N} by \mathcal{Z} and let \mathcal{Z}^\perp the orthogonal complement of the center. Then, \mathcal{N} is the orthogonal direct sum of \mathcal{Z} and \mathcal{Z}^\perp , that is

$$\mathcal{N} = \mathcal{Z} \oplus \mathcal{Z}^\perp.$$

For each $Z \in \mathcal{Z}$, a skew symmetric linear transformation

$$j(Z) : \mathcal{Z}^\perp \rightarrow \mathcal{Z}^\perp$$

is defined by

$$j(Z)X = (adX)^*Z$$

for $X \in \mathcal{Z}^\perp$. Or, equivalently,

$$\langle j(Z)X, Y \rangle = \langle [X, Y], Z \rangle$$

for all $X, Y \in \mathcal{Z}^\perp$.

Definition 2.1. A two step nilpotent Lie algebra \mathcal{N} is called the algebra of Heisenberg type if

$$j(Z)^2 = -|Z|^2 id$$

for all $Z \in \mathcal{Z}$. And a Lie group N is called the group of Heisenberg type if its Lie algebra \mathcal{N} is the algebra of Heisenberg type.

The classical Heisenberg groups are examples of Heisenberg type. That is, let $n \geq 1$ be any integer and let $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ be any basis of $R^{2n} = \mathcal{V}$. Let \mathcal{Z} be an 1-dimensional vector space spanned by $\{Z\}$. Define

$$[X_i, Y_i] = -[Y_i, X_i] = Z$$

for any $i = 1, 2, \dots, n$ with all other brackets are zero. Give on $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ the inner product such that the vectors $\{X_i, Y_i, Z | i = 1, 2, \dots, n\}$ form an orthonormal basis. Then, $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ is an algebra of Heisenberg type. The simply-connected group of Heisenberg type, N which is determined by \mathcal{N} and equipped with a left-invariant metric induced by the inner product in \mathcal{N} is called the $(2n+1)$ -dimensional Heisenberg group.

Another example is the quaternionic Heisenberg group of dimension $4n+3$. Let $\mathcal{N} = \mathcal{Z}^\perp \oplus \mathcal{Z}$ be a $(4n+3)$ -dimensional real vector space with basis $\{X_i, Y_i, V_i, W_i | 1 \leq i \leq n\}$ of \mathcal{Z}^\perp and $\{\xi_1, \xi_2, \xi_3\}$ of \mathcal{Z} . Define a Lie bracket on \mathcal{N} as follows:

$$[X_i, Y_i] = \xi_1 = [V_i, W_i], \quad [Y_i, X_i] = -\xi_1 = [W_i, V_i],$$

$$[X_i, V_i] = \xi_2 = [W_i, Y_i], \quad [V_i, X_i] = -\xi_2 = [Y_i, W_i],$$

$$[X_i, W_i] = \xi_3 = [Y_i, V_i], \quad [W_i, X_i] = -\xi_3 = [V_i, Y_i],$$

and all other brackets are zero. Define on \mathcal{N} the inner product by giving that

$$\{X_i, Y_i, V_i, W_i, \xi_1, \xi_2, \xi_3 | 1 \leq i \leq n\}$$

forms an orthonormal basis. The simply connected group of Heisenberg type, N which is determined by \mathcal{N} and equipped with a left-invariant metric induced by the inner product in \mathcal{N} is called the quaternionic Heisenberg group.

3. GEODESIC EQUATIONS IN THE TWO STEP NILPOTENT METRIC GROUP

Recall that, for nonzero Z_0 in \mathcal{Z} , a skew-symmetric linear transformation $j(Z) : \mathcal{Z}^\perp \rightarrow \mathcal{Z}^\perp$ is defined by

$$\langle j(Z_0)X, Y \rangle = \langle [X, Y], Z_0 \rangle$$

for $X, Y \in \mathcal{Z}^\perp$.

Let $\{\pm\theta_1 i, \pm\theta_2 i, \dots, \pm\theta_p i\}$ be the distinct eigenvalues of $j(Z_0)$ with $\theta_k > 0$ for any $k = 1, 2, \dots, p$, and let $\{W_1, W_2, \dots, W_p\}$ be the invariant subspaces of $j(Z_0)$ such that $j(Z_0)^2 = -\theta_k^2 \text{id}$ on W_k for every $k = 1, 2, \dots, p$. Then, \mathcal{Z}^\perp is expressed as an orthogonal direct sum,

$$\begin{aligned}\mathcal{Z}^\perp &= \text{Ker}j(Z_0) \oplus \text{Ker}j(Z_0)^\perp \\ \text{Ker}j(Z_0)^\perp &= \bigoplus_{k=1}^p W_k.\end{aligned}$$

Let $\gamma(t)$ be a curve in N such that $\gamma(0) = e$ (identity element in N) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^\perp$ and $Z_0 \in \mathcal{Z}$. Since $\exp: \mathcal{N} \rightarrow N$ is a diffeomorphism([8]), the curve $\gamma(t)$ can be expressed uniquely by $\gamma(t) = \exp(X(t) + Z(t))$ with

$$\begin{aligned}X(t) &\in \mathcal{Z}^\perp, & X'(0) &= X_0, & X(0) &= 0 \\ Z(t) &\in \mathcal{Z}, & Z'(0) &= Z_0, & Z(0) &= 0.\end{aligned}$$

A. Kaplan [4, 5] shows that the curve $\gamma(t)$ is a geodesic in N if and only if

$$\begin{aligned}X''(t) &= j(Z_0)X'(t), \\ Z'(t) + \frac{1}{2}[X'(t), X(t)] &\equiv Z_0.\end{aligned}$$

Note that $j(Z_0)$ is nonsingular on $\text{Ker}j(Z_0)^\perp$.

Proposition 3.1. *The solution to this geodesic equation is following :*

$$\begin{aligned}X(t) &= tX_1 + (e^{tJ} - \text{id})(J^{-1}X_2) \\ Z(t) &= tZ_1(t) + Z_2(t),\end{aligned}$$

where

$$\begin{aligned}Z_1 &= Z_0 + \frac{1}{2} \sum_{k=1}^p [J^{-1}\xi_k, \xi_k] \\ Z_2(t) &= \frac{1}{2}[e^{tJ}J^{-1}X_0, J^{-1}X_0] - \frac{1}{2} \sum_{i \neq k=1}^p \frac{1}{\theta_k^2 - \theta_i^2} \{ [e^{tJ}J\xi_i, e^{tJ}J^{-1}\xi_k] \\ &\quad - [e^{tJ}\xi_i, e^{tJ}\xi_k] \} + \frac{1}{2} \sum_{i \neq k=1}^p \frac{1}{\theta_k^2 - \theta_i^2} \{ [J\xi_i, J^{-1}\xi_k] - [\xi_i, \xi_k] \}\end{aligned}$$

with $J = j(Z_0)$, $X_0 = X_1 + X_2 \in \text{ker}j(Z_0) \oplus \text{ker}j(Z_0)^\perp$ and $X_2 = \sum_{k=1}^p \xi_k$, $\xi_k \in W_k(Z_0)$ ([2]).

P. Eberlein defined the following which is related to problem, the density of closed geodesics in the two step nilmanifolds.

Definition 3.2. For $Z \in \mathcal{Z} - \{0\}$, $j(Z)$ is *in resonance* if every ratio of non-zero eigenvalues of $j(Z)$ is rational([2]).

Proposition 3.3. $j(Z)$ is *in resonance* if and only if $e^{\omega j(Z)} = I$ for some $\omega > 0$, where e is the matrix exponential map, and ω depends only on the eigenvalues of $j(Z)$ ([2]).

If $j(Z_0)$ is nonsingular and in resonance, then $X_1 = 0$ and by Proposition 3.1, $e^{\omega j(Z)} = I$ for some $\omega > 0$. So, in the geodesic equations of Proposition 3.1, we see that $X(\omega) + Z(\omega)$ is not complicated, that is:

$$X(\omega) + Z(\omega) = \omega(Z_0 + \frac{1}{2} \sum_{k=1}^p [j(Z_0)^{-1} \xi_k, \xi_k])$$

and $X(\omega) + Z(\omega)$ is a point where $\log \gamma(t)$ hits \mathcal{Z} , where $\log : N \rightarrow \mathcal{N}$ is the inverse map of \exp . In fact, the curve $\log \gamma(t)$ hits \mathcal{Z} periodically with period ω .

For the unique simply connected Lie group N corresponding to a given two step nilpotent Lie algebra \mathcal{N} , its Campbell-Baker-Hausdorff formula has a simple form

$$\exp(X) \cdot \exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y]) \quad \text{for any } X, Y \in \mathcal{N}.$$

From this formula, it is easy to see that $\log(Z(N)) = \mathcal{Z}$. Moreover, it follows that, if Γ is a lattice in N , $Z(\Gamma) = \Gamma \cap Z(N)$ is a lattice in $Z(N)$ and $\log(Z(\Gamma)) = \log \Gamma \cap \mathcal{Z}$ is a lattice in \mathcal{Z} (see [2]).

4. DENSITY OF CLOSED GEODESIC IN TWO STEP NILMANIFOLD

The concrete definition of having the density of closed geodesics is given as follow:

Definition 4.1. A compact manifold M is said to have the density of closed geodesics if the unit vectors tangent to smoothly closed geodesic are dense in the unit tangent bundle.

In 1994, P. Eberlein first started to study the density of closed geodesic property for two step nilmanifolds. He proved the following Theorem for the group of Heisenberg type..

Theorem 4.2. *Let N be a group of Heisenberg type, Then $\Gamma \backslash N$ has the density of closed geodesics for any lattice Γ in N ([2]).*

Note that if N is group of Heisenberg type, then $j(Z_0)$ is nonsingular and in resonance for any nonzero $Z_0 \in \mathcal{Z}$. And he proved the following Theorem for the two step nilpotent groups with one dimensional center.

Theorem 4.3. *Let N be a two step nilpotent metric group with one dimensional center. Then $\Gamma \backslash N$ has the density of closed geodesics for some (hence any) lattice Γ in N if and only if $j(Z)$ is in resonance for all non-zero Z in $\mathcal{Z}([2])$.*

Note that if N is a two step nilpotent metric group with one dimensional center, then $j(Z_0)$ is nonsingular for any nonzero $Z_0 \in \mathcal{Z}$.

Definition 4.4.

- (1) \mathcal{N} is nonsingular if $j(Z)$ is nonsingular for any $Z \neq 0$.
- (2) \mathcal{N} is almost nonsingular if $j(Z)$ is nonsingular for all Z in some open dense subset of \mathcal{Z} .
- (3) \mathcal{N} is singular if $j(Z)$ is singular for any $Z \neq 0$.

So, the groups of Heisenberg type and the two step nilpotent groups with one dimensional center are nonsingular. M. Mast proved the following Theorems.

Theorem 4.5. *Let N be non-singular. If $\Gamma \backslash N$ has the density of closed geodesics for some lattice Γ in N , then $j(Z)$ is in resonance for all nonzero Z in a dense subset R of $\mathcal{Z}([7])$.*

Theorem 4.6. *Let N be non-singular. If $j(Z)$ is in resonance for all nonzero Z in \mathcal{Z} , then $\Gamma \backslash N$ has the density of closed geodesics for any lattice Γ in N ([7]).*

K. B. Lee and K. Park solved completely the problem, the density of closed geodesics in case of the almost nonsingular(so, nonsingular) two step nilmanifolds as follows:

Theorem 4.7. *Let N be a two step nilpotent metric group. Assume that \mathcal{N} is almost nonsingular. Then, $\Gamma \backslash N$ has the density of closed geodesics if and only if there exists a dense subset R of \mathcal{Z} such that $j(Z)$ is in resonance for every $Z \in R$ ([6]).*

L. DeMeyer showed that the density of closed geodesics property holds in a certain singular two step nilpotent metric group which is very interesting. See [1] more details. Let $\rho : SU(2) \rightarrow Aut(R^n)$ be an irreducible representation with $n(\geq 5)$ is an odd integer, R^n be given an inner product \langle, \rangle_1 and $su(2)$ be given an inner product \langle, \rangle_2 defined by

$$\langle Z, Z^* \rangle_2 = -tr(j(Z)j(Z^*)) \text{ for } Z, Z^* \in su(2)$$

where $j = d\rho$. Let $\mathcal{N} = R^n \oplus su(2)$ and define the Lie bracket on \mathcal{N} by $[\mathcal{N}, su(2)] = 0$ and

$$\langle [X, Y], Z \rangle_2 = \langle j(Z)X, Y \rangle_1 \text{ for any } X, Y \in R^n, Z \in su(2).$$

Then, $\mathcal{N} = R^n \oplus su(2)$ is a two step nilpotent Lie algebra with center $su(2)$.

Theorem 4.8. *If N be unique simply-connected two step metric group determined by $\mathcal{N} = R^n \oplus su(2)$, then $\Gamma \backslash N$ has the density of closed geodesics for any lattice Γ in $N([1])$.*

Another class of two step nilpotent Lie groups is the generalized Heisenberg groups $H(n, n)$. Let $\mathcal{N} = \mathcal{Z}^\perp \oplus \mathcal{Z}$ be a $n^2 + 2n$ -dimensional real vector space with basis $\{X_i, Y_i, |1 \leq i \leq n\}$ of \mathcal{Z}^\perp and $\{Z_{ij} | 1 \leq i, j \leq n\}$ of \mathcal{Z} . Define a Lie bracket on \mathcal{N} as follows:

$$[X_i, Y_j] = Z_{ij}, [Y_j, X_i] = -Z_{ij}$$

and all other brackets are zero. Define on \mathcal{N} the inner product by giving that

$$\{X_i, Y_i, Z_{ij} | 1 \leq i, j \leq n\}$$

forms an orthonormal basis. The simply connected two step nilpotent group N which is determined by \mathcal{N} and equipped with a left-invariant metric induced by the inner product in \mathcal{N} is called the generalized Heisenberg group and denoted by $H(n, n)$. Clearly $H(n, n)$ is the group of Heisenberg type if and only if $n = 1$. And \mathcal{N} is almost nonsingular, but not singular. So, Theorem 4.7 is applied.

Theorem 4.9. *If $N = H(n, n)$, then $\Gamma \backslash N$ has the density of closed geodesics for any lattice Γ in $N([?])$.*

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