

## MOMENT-ANGLE MANIFOLDS AND COMPLEXES

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ABSTRACT. These are notes of the lectures given during the Toric Topology Workshop at the Korea Advanced Institute of Science and Technology in February 2010. We describe several approaches to moment-angle manifolds and complexes, including the intersections of quadrics, complements of subspace arrangements and level sets of moment maps. We overview the known results on the topology of moment-angle complexes, including the description of their cohomology rings, as well as the homotopy and diffeomorphism types in some particular cases. We also discuss complex-analytic structures on moment-angle manifolds and methods for calculating invariants of these structures.

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## Preface

Moment-angle manifolds and complexes are one of the key players in *toric topology*, a new and actively developing field on the borders of equivariant topology, symplectic and algebraic geometry, and combinatorics. As it was remarkably observed by Bosio and Meersseman in [BM], polytopal moment-angle manifolds admit complex-analytic structures as LVM manifolds, therefore creating a new link from toric topology to complex geometry. The general idea behind these lectures was to broaden this link and whenever possible unify the complex-analytic approach to moment-angle manifolds via the LVM-manifolds and the combinatorial topological approach of [BP2] via the moment-angle complexes and coordinate subspace arrangements.

In Lecture I we review the convex-geometrical construction of the moment-angle manifold  $\mathcal{Z}_P$  corresponding to a simple polytope  $P$ , following [BP2] and [BPR]. This manifold embeds into  $\mathbb{C}^m$  with trivial normal bundle, and an explicit framing is specified by writing the embedded submanifold as a nondegenerate intersection of quadratic surfaces. This chimes with the approach of [BM], who considered nondegenerate (or transverse) intersections of several homogeneous real quadrics with a unit sphere in  $\mathbb{C}^m$ ; these manifolds were referred to as *links*. A result of [BM] shows that the class of links coincides with the class of polytopal moment-angle manifolds  $\mathcal{Z}_P$  if we allow redundant inequalities in the presentation of the polytope  $P$ . The two approaches to define the same manifolds are connected by a sort of duality related to the Gale duality in convex geometry. This duality was already pretty well understood and described in [BM] from the point of view of links of quadrics (‘from quadrics to polytopes’). We take time to go the other way (‘from polytopes to quadrics’) thoroughly, by starting from a moment-angle manifold  $\mathcal{Z}_P$  and eventually coming to different presentations of it by quadrics, and understanding the underlying convex geometry on the way.

In Lecture II we review several other approaches to moment-angle manifolds: the original approach of Davis–Januszkiewicz [DJ] where  $\mathcal{Z}_P$  was defined as a certain identification space; the approach of [BP2] via the *moment-angle complexes*  $\mathcal{Z}_{\mathcal{K}}$  corresponding to simplicial complexes  $\mathcal{K}$  (which also leads to a much wider class of *non polytopal* moment-angle manifolds corresponding to sphere triangulations  $\mathcal{K}$ ); coordinate subspace arrangements and their complements; and the symplectic approach where  $\mathcal{Z}_P$  appears as the level set for the moment map used in the construction of *Hamiltonian toric manifolds* via symplectic reduction. We omit the homotopy theoretic approach to moment-angle complexes because of the geometric nature of these lectures; the reader is referred to [BP2, Ch. 6] for the basics (including the homotopy fibre interpretation of  $\mathcal{Z}_{\mathcal{K}}$ ), and [GT], [BBCG] for more recent developments.

Lecture III is dedicated to the topology of moment-angle manifolds and complexes. The cohomology ring of a moment-angle complex  $\mathcal{Z}_{\mathcal{K}}$  is isomorphic to the Tor-algebra of the *Stanley–Reisner face ring* of  $\mathcal{K}$  and may be described as the cohomology of a Koszul-like dga with an explicit differential. This result was first proved in [BP1]; here we give a sketch of a more recent proof from [BBP]. We also derive several corollaries of the calculation of  $H^*(\mathcal{Z}_{\mathcal{K}})$ , including a description in terms of full subcomplexes of  $\mathcal{K}$ ; the corresponding formula for the Tor-groups of the face ring is known in the combinatorial commutative algebra as the *Hochster theorem*. It is quite clear from any of these descriptions of  $H^*(\mathcal{Z}_{\mathcal{K}})$  that there is no

hope for a reasonable topological classification of moment-angle manifolds. However, there are two methods of quite different origin and flavour which may be used to describe the topology of  $\mathcal{Z}_P$  and  $\mathcal{Z}_\mathcal{K}$  explicitly for some particular polytopes  $P$  and complexes  $\mathcal{K}$ . The first method is geometric and may be applied to polytopal moment-angle manifolds: it is based on the interpretation of  $\mathcal{Z}_P$  as an intersection of quadrics and uses the surgery theory. The topology of intersections of quadrics is well understood when the number of quadrics is small (this corresponds to polytopes  $P$  with few facets; the case of three quadrics was done in [Lo]), or when the polytope  $P$  is of special type, e.g. obtained by iteratively cutting vertices off a simplex [BM], [GL]. The second method is homotopical; different homotopy-theoretical interpretations of the moment-angle complex  $\mathcal{Z}_\mathcal{K}$  from [BP2] may be used to identify its homotopy type as that of a wedge of spheres for some special series of simplicial complexes  $\mathcal{K}$  [GT].

Complex-analytic aspects of the theory of moment-angle complexes are the subject of Lecture IV. Nondegenerate intersections of quadrics were studied in holomorphic dynamics as the transverse sets to certain complex foliations. This study led to a discovery of a new class of compact non-Kähler complex-analytic manifolds in the work of Lopez de Medrano and Verjovsky [LV] and Meersseman [Me], now known as the *LVM-manifolds*. As we already mentioned above, Bosio and Meersseman [BM] were first to observe that the smooth manifolds underlying a large class of LVM-manifolds are exactly the polytopal moment-angle manifolds. It therefore became clear that the moment-angle manifolds  $\mathcal{Z}_P$  admit non-Kähler complex structures generalising the families of Hopf and Calabi–Eckmann manifolds. We review the construction of LVM-manifolds and give an argument of [BM] providing  $\mathcal{Z}_P$  with a complex structure of an LVM manifold. In the final section we apply the spectral sequence of Borel to analyse the Dolbeault cohomology and Hodge numbers of these complex structures.

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## Lecture I. Moment-angle manifolds from polytopes

### I.1. FROM POLYTOPES TO QUADRICS

Let  $\mathbb{R}^n$  be a Euclidean space with scalar product  $\langle \cdot, \cdot \rangle$ . We consider convex polyhedrons defined as intersections of  $m$  halfspaces:

$$(I.1) \quad P = \{ \mathbf{x} \in \mathbb{R}^n : \langle \mathbf{a}_i, \mathbf{x} \rangle + b_i \geq 0 \text{ for } i = 1, \dots, m \},$$

where  $\mathbf{a}_i \in \mathbb{R}^n$  and  $b_i \in \mathbb{R}$ . Assume that the hyperplanes defined by the equations  $\langle \mathbf{a}_i, \mathbf{x} \rangle + b_i = 0$  are in general position, i.e. at most  $n$  of them meet at a single point. Assume further that  $\dim P = n$  and  $P$  is bounded (which implies that  $m > n$ ). Then  $P$  is an  $n$ -dimensional *simple polytope*. Set

$$F_i = \{ \mathbf{x} \in P : \langle \mathbf{a}_i, \mathbf{x} \rangle + b_i = 0 \}.$$

Since the hyperplanes are in general position, each  $F_i$  is either empty or a *facet* (an  $(n-1)$ -dimensional face) of  $P$ . If  $F_i$  is empty, then the  $i$ th inequality in (I.1) is *redundant*; removing it does not change the set  $P$ .

Let  $A_P$  be the  $m \times n$  matrix of row vectors  $\mathbf{a}_i$ , and  $\mathbf{b}_P$  be the column vector of scalars  $b_i \in \mathbb{R}^n$ . Then we can write (I.1) as

$$P = \{ \mathbf{x} \in \mathbb{R}^n : A_P \mathbf{x} + \mathbf{b}_P \geq \mathbf{0} \},$$

and consider the affine map

$$i_P : \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad i_P(\mathbf{x}) = A_P \mathbf{x} + \mathbf{b}_P.$$

It embeds  $P$  into

$$\mathbb{R}_{\geq}^m = \{ \mathbf{y} \in \mathbb{R}^m : y_i \geq 0 \text{ for } 1 \leq i \leq m \}.$$

We identify  $\mathbb{C}^m$  (as a real vector space) with  $\mathbb{R}^{2m}$  using the map

$$(I.2) \quad \mathbf{z} = (z_1, \dots, z_m) \mapsto (x_1, y_1, \dots, x_m, y_m),$$

where  $z_k = x_k + iy_k$  for  $k = 1, \dots, m$ .

We define the space  $\mathcal{Z}_P$  from the commutative diagram

$$(I.3) \quad \begin{array}{ccc} \mathcal{Z}_P & \xrightarrow{i_Z} & \mathbb{C}^m \\ \downarrow & & \downarrow \mu \\ P & \xrightarrow{i_P} & \mathbb{R}_{\geq}^m \end{array}$$

where  $\mu(z_1, \dots, z_m) = (|z_1|^2, \dots, |z_m|^2)$ . The latter map may be thought of as the quotient map for the coordinatewise action of the standard torus

$$\mathbb{T}^m = \{ \mathbf{z} \in \mathbb{C}^m : |z_i| = 1 \text{ for } 1 \leq i \leq m \}$$

on  $\mathbb{C}^m$ . Therefore,  $\mathbb{T}^m$  acts on  $\mathcal{Z}_P$  with quotient  $P$ , and  $i_Z$  is a  $\mathbb{T}^m$ -equivariant embedding.

The image of  $\mathbb{R}^n$  under  $i_P$  is an  $n$ -dimensional affine plane in  $\mathbb{R}^m$ , which can be written as

$$(I.4) \quad \begin{aligned} i_P(\mathbb{R}^n) &= \{ \mathbf{y} \in \mathbb{R}^m : \mathbf{y} = A_P \mathbf{x} + \mathbf{b}_P \text{ for some } \mathbf{x} \in \mathbb{R}^n \} \\ &= \{ \mathbf{y} \in \mathbb{R}^m : \Gamma \mathbf{y} = \Gamma \mathbf{b}_P \}, \end{aligned}$$

where  $\Gamma = (\gamma_{jk})$  is an  $(m-n) \times m$  matrix whose rows form a basis of linear relations between the vectors  $\mathbf{a}_i$ . That is,  $\Gamma$  is of full rank and satisfies the identity  $\Gamma A_P = 0$ .

Then we obtain from (I.3) that  $\mathcal{Z}_P$  embeds into  $\mathbb{C}^m$  as the set of common zeros of  $m - n$  real quadratic equations:

$$(I.5) \quad i_Z(\mathcal{Z}_P) = \left\{ \mathbf{z} \in \mathbb{C}^m : \sum_{k=1}^m \gamma_{jk} |z_k|^2 = \sum_{k=1}^m \gamma_{jk} b_k, \text{ for } 1 \leq j \leq m - n \right\}.$$

The following properties of  $\mathcal{Z}_P$  easily follow from its construction.

**Proposition I.1.** (a) *Given a point  $z \in \mathcal{Z}_P$ , the  $i$ th coordinate of  $i_Z(z) \in \mathbb{C}^m$  vanishes if and only if  $z$  projects onto a point  $\mathbf{x} \in P$  such that  $\mathbf{x} \in F_i$ .*

(b) *Adding a redundant inequality to (I.1) results in multiplying  $\mathcal{Z}_P$  by a circle.*

**Theorem I.2** ([BP2, Cor. 3.9]).  *$\mathcal{Z}_P$  is a smooth manifold of dimension  $m + n$ . Moreover, the embedding  $i_Z: \mathcal{Z}_P \rightarrow \mathbb{C}^m$  has  $\mathbb{T}^m$ -equivariantly trivial normal bundle; a  $\mathbb{T}^m$ -framing is determined by any choice of matrix  $\Gamma$  in (I.4).*

*Proof.* All assertions will follow from the fact that the intersection of quadrics (I.5) defining  $i_Z(\mathcal{Z}_P)$  is nondegenerate (transverse). For simplicity we identify  $\mathcal{Z}_P$  with its image  $i_Z(\mathcal{Z}_P) \subset \mathbb{C}^m$ . The gradients of the  $m - n$  quadratic forms in (I.5) at a point  $\mathbf{z} = (x_1, y_1, \dots, x_m, y_m) \in \mathcal{Z}_P$  are

$$(I.6) \quad 2(\gamma_{j1}x_1, \gamma_{j1}y_1, \dots, \gamma_{jm}x_m, \gamma_{jm}y_m), \quad 1 \leq j \leq m - n.$$

These gradients form the rows of the  $(m - n) \times 2m$  matrix  $2\Gamma\Delta$ , where

$$\Delta = \begin{pmatrix} x_1 & y_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & x_m & y_m \end{pmatrix}.$$

Assume that for the chosen point  $\mathbf{z} \in \mathcal{Z}_P$  we have  $z_{j_1} = \dots = z_{j_k} = 0$ , and the other complex coordinates are nonzero. Then the rank of the gradient matrix  $2\Gamma\Delta$  at  $\mathbf{z}$  is equal to the rank of the  $(m - n) \times (m - k)$  matrix  $\Gamma'$  obtained by deleting the columns  $j_1, \dots, j_k$  from  $\Gamma$ .

By Proposition I.1 (a), the point  $\mathbf{z}$  projects onto  $\mathbf{x} \in F_{j_1} \cap \dots \cap F_{j_k} \neq \emptyset$ . Let  $\iota: \mathbb{R}^{m-k} \rightarrow \mathbb{R}^m$  be the inclusion of the coordinate subspace  $\{\mathbf{y}: y_{j_1} = \dots = y_{j_k} = 0\}$ , and  $\kappa: \mathbb{R}^m \rightarrow \mathbb{R}^k$  the projection onto the quotient space. Then  $\Gamma' = \Gamma \cdot \iota$ , and  $A' = \kappa \cdot A_P$  is the  $k \times n$  matrix formed by the row vectors  $\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_k}$ . These vectors are linearly independent by the assumption, and therefore  $A'$  is of rank  $k$  and  $\kappa \cdot A_P$  is epic. We claim that  $\Gamma' = \Gamma \cdot \iota$  is also epic. Indeed, consider the diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & \mathbb{R}^n & & \\ & & & & \downarrow A_P & & \\ 0 & \longrightarrow & \mathbb{R}^{m-k} & \xrightarrow{\iota} & \mathbb{R}^m & \xrightarrow{\kappa} & \mathbb{R}^k \longrightarrow 0 \\ & & & & \downarrow \Gamma & & \\ & & & & \mathbb{R}^{m-n} & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

Take  $\alpha \in \mathbb{R}^{m-n}$ . There is  $\beta \in \mathbb{R}^m$  such that  $\Gamma\beta = \alpha$ . Assume  $\beta \notin \iota(\mathbb{R}^{m-k})$ , then  $\gamma := \kappa(\beta) \neq 0$ . Choose  $\delta \in \mathbb{R}^n$  such that  $\kappa \cdot A_P(\delta) = \gamma$ . Set  $\eta := A_P(\delta) \neq 0$ .

Hence,  $\kappa(\eta) = \kappa(\beta) (= \gamma)$  and there is  $\xi \in \mathbb{R}^{m-k}$  such that  $\iota(\xi) = \beta - \eta$ . Then  $\Gamma \cdot \iota(\xi) = \Gamma(\beta - \eta) = \alpha - \Gamma \cdot A_P \delta = \alpha$ . Thus,  $\Gamma \cdot \iota$  is epic, and  $\Gamma'$  has rank  $m - n$ .  $\square$

We refer to  $\mathcal{Z}_P$  as the *moment-angle manifold corresponding to  $P$* , or simply a *polytopal moment-angle manifold*.

There is an indeterminacy in the choice of  $\Gamma$ . How to make this choice more canonical?

**Construction I.3** (1st method). By reordering the facets and changing  $\mathbb{R}^n$  by a linear isomorphism we may achieve that the first  $n$  facets of  $P$  meet at a vertex and  $\mathbf{a}_1, \dots, \mathbf{a}_n$  form the standard basis of  $\mathbb{R}^n$ . Then  $A_P = \begin{pmatrix} I_n \\ A_P^* \end{pmatrix}$ , where  $I_n$  is a unit  $n \times n$  matrix, and  $A_P^*$  is an  $(m - n) \times n$  matrix. We may take  $\Gamma = \begin{pmatrix} -A_P^* & I_{m-n} \end{pmatrix}$ . This choice of  $\Gamma$  and the corresponding framing of  $\mathcal{Z}_P$  was used in [BPR].

**Construction I.4** (2nd method). Assume for simplicity that all redundant inequalities in (I.5) are of the form  $1 \geq 0$  (that is,  $\mathbf{a}_i = \mathbf{0}$  and  $b_i = 1$ ). The rows of the coefficient matrix  $\Gamma = (\gamma_{jk})$  of (I.5) form a basis in the space of linear relations between the vectors  $\mathbf{a}_1, \dots, \mathbf{a}_m$ . Since these vectors are determined only up to a positive multiple, we may assume that  $|\mathbf{a}_i| = 1$  if the  $i$ th inequality is irredundant. Since  $P$  is a convex polytope, one of the relations between the  $\mathbf{a}_i$ 's is  $\sum_{i=1}^m (\text{vol } F_i) \mathbf{a}_i = \mathbf{0}$ , where  $\text{vol } F_i \geq 0$  is the volume of the facet  $F_i$ . By rescaling the vectors  $\mathbf{a}_i$  again we may achieve that  $\sum_{i=1}^m \mathbf{a}_i = \mathbf{0}$  (this still leaves one scaling parameter free), and choose the coefficients of this relation as the last row of  $\Gamma$ , that is,  $\gamma_{m-n,k} = 1$  for  $1 \leq k \leq m$ . Then the last of the quadrics in (I.5) takes the form  $\sum_{k=1}^m |z_k|^2 = \sum_{k=1}^m b_k$ . Summing up all  $m$  inequalities of (I.1), using the fact that  $\sum_{i=1}^m \mathbf{a}_i = \mathbf{0}$  and noting that at least one of the inequalities is strict for some  $\mathbf{x} \in \mathbb{R}^n$ , we obtain  $\sum_{k=1}^m b_k > 0$ . Using up the last free scaling parameter we achieve that  $\sum_{k=1}^m b_k = 1$ . Then the last quadric in (I.5) becomes  $\sum_{k=1}^m |z_k|^2 = 1$ . Subtracting this equation with an appropriate coefficient from the other equations in (I.5) we finally transform (I.5) to

$$(I.7) \quad \left\{ \begin{array}{l} \mathbf{z} \in \mathbb{C}^m: \quad \sum_{k=1}^m \gamma_{jk} |z_k|^2 = 0, \quad \text{for } 1 \leq j \leq m - n - 1, \\ \sum_{k=1}^m |z_k|^2 = 1. \end{array} \right\}$$

We therefore have written  $\mathcal{Z}_P$  as a transverse intersection of  $m - n - 1$  *homogeneous* quadrics with a unit sphere in  $\mathbb{C}^m$ . Such intersections were called *links* in [BM]. We denote by  $\Gamma^*$  the submatrix of  $\Gamma$  formed by  $\gamma_{jk}$  with  $j \leq m - n - 1$ , and denote the columns of  $\Gamma^*$  by  $\mathbf{g}_1, \dots, \mathbf{g}_m$ , so that  $\Gamma = \begin{pmatrix} \mathbf{g}_1 & \cdots & \mathbf{g}_m \\ 1 & \cdots & 1 \end{pmatrix}$ .

The identity  $\Gamma A_P = 0$  also means that the columns of  $A_P$  form a basis in the space of linear relations between the columns of  $\Gamma$ , that is, a basis of solutions of the homogeneous system

$$(I.8) \quad \sum_{k=1}^m \mathbf{g}_k y_k = \mathbf{0}, \quad \sum_{k=1}^m y_k = 0.$$

The passage from the vectors  $\mathbf{g}_1, \dots, \mathbf{g}_m \in \mathbb{R}^{m-n-1}$  to the vectors  $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{R}^n$  forming the transpose to a basis of solutions of (I.8) is known in convex geometry as the *Gale transform*.

**Example I.5.** Let  $P$  be the standard  $n$ -simplex given by the equations  $x_i \geq 0$  for  $i = 1, \dots, n$  and  $-x_1 - \dots - x_n + 1 \geq 0$  in  $\mathbb{R}^n$ . We therefore have  $m = n + 1$ ,  $\mathbf{a}_i = \mathbf{e}_i$  (the  $i$ th standard basis vector) for  $i = 1, \dots, n$  and  $\mathbf{a}_{n+1} = -\mathbf{e}_1 - \dots - \mathbf{e}_n$ ,  $A_P = \begin{pmatrix} I_n \\ -1 \dots -1 \end{pmatrix}$ ,  $\Gamma = (1 \dots 1)$ ,  $\Gamma^* = \emptyset$ , and  $\mathcal{Z}_P$  is a unit sphere in  $\mathbb{C}^{n+1}$ .

The combinatorial structure of  $P$  can be read from the configuration of vectors  $\mathbf{g}_1, \dots, \mathbf{g}_m$  in  $\mathbb{R}^{m-n-1}$  using the following lemma.

**Lemma I.6.** *Let  $P$  be a simple polytope (I.1). The following conditions are equivalent:*

- (a)  $F_{i_1} \cap \dots \cap F_{i_k} \neq \emptyset$ ,
- (b)  $\mathbf{0} \in \text{conv}(\mathbf{g}_j : j \notin \{i_1, \dots, i_k\})$ .

Here  $\text{conv}(\cdot)$  denotes the convex hull of a set of points.

*Proof.* Given  $\mathbf{x} \in P$ , set  $\mathbf{y} = i_P(\mathbf{x})$  and  $y_i = \langle \mathbf{a}_i, \mathbf{x} \rangle + b_i$  for  $1 \leq i \leq m$ . Assume (a) is satisfied. Choose  $\mathbf{x} \in F_{i_1} \cap \dots \cap F_{i_k}$ ; then  $y_j = 0$  for  $j \in \{i_1, \dots, i_k\}$  and  $y_j \geq 0$  for  $j \notin \{i_1, \dots, i_k\}$ . Now  $\Gamma^* \mathbf{y} = \mathbf{0}$  implies

$$\mathbf{g}_1 y_1 + \dots + \mathbf{g}_m y_m = \sum_{j \notin \{i_1, \dots, i_k\}} \mathbf{g}_j y_j = \mathbf{0},$$

which is equivalent to condition (b). Conversely, if (b) is satisfied, then there is  $\mathbf{y} \in \mathbb{R}^m$  with  $y_j = 0$  for  $j \in \{i_1, \dots, i_k\}$  and  $y_j \geq 0$  for  $j \notin \{i_1, \dots, i_k\}$  such that  $\Gamma \mathbf{y} = \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$ . Therefore,  $\mathbf{y} = i_P(\mathbf{x})$  for some  $\mathbf{x} \in P$ . For this  $\mathbf{x}$  we have  $\langle \mathbf{a}_j, \mathbf{x} \rangle + b_j = 0$  if  $j \in \{i_1, \dots, i_k\}$ , which implies that  $\mathbf{x} \in F_{i_1} \cap \dots \cap F_{i_k}$ .  $\square$

The *polar set* of a polyhedron given by (I.1) is defined as

$$P^* := \{ \mathbf{u} \in \mathbb{R}^n : \langle \mathbf{u}, \mathbf{x} \rangle \geq -1 \text{ for all } \mathbf{x} \in P \}.$$

If  $P$  is a convex polytope and  $\mathbf{0}$  is in the interior of  $P$  (which implies that  $b_i > 0$  for all  $i$ ), then  $P^*$  is also a convex polytope given by

$$P^* = \text{conv} \left( \frac{\mathbf{a}_1}{b_1}, \dots, \frac{\mathbf{a}_m}{b_m} \right),$$

where  $\text{conv}(\cdot)$  denotes the convex hull of a set of points.  $P^*$  is called the *polar* (or *dual*) *polytope* of  $P$ . If  $P$  is simple, then  $P^*$  is *simplicial* (all proper faces are simplices), and vice versa. In this case we have that  $F_{i_1} \cap \dots \cap F_{i_k}$  is an  $(n - k)$ -face of  $P$  if and only if  $\text{conv} \left( \frac{\mathbf{a}_{i_1}}{b_{i_1}}, \dots, \frac{\mathbf{a}_{i_k}}{b_{i_k}} \right)$  is a  $k$ -face of  $P^*$ .

A set of vectors  $\mathbf{g}_1, \dots, \mathbf{g}_m \in \mathbb{R}^{m-n-1}$  satisfying the conditions of Lemma I.6 is called a *Gale diagram* of  $P^*$ .

## I.2. FROM QUADRICS TO POLYTOPES

Here we briefly review a construction of [BM] which associates a simple polytope to every link (I.7).

For every set of vectors  $\mathbf{g}_1, \dots, \mathbf{g}_m \in \mathbb{R}^{m-n-1}$  there is the associated link:

$$\mathcal{L} := \left\{ \mathbf{z} \in \mathbb{C}^m : \begin{array}{l} \sum_{k=1}^m \mathbf{g}_k |z_k|^2 = \mathbf{0}, \\ \sum_{k=1}^m |z_k|^2 = 1. \end{array} \right\}$$



We observe that if the set  $\mathbf{g}_1, \dots, \mathbf{g}_m$  is obtained from a simple polytope (I.1) as described in Construction I.4, and therefore  $\mathcal{L}$  is an embedding of the corresponding moment-angle manifold  $\mathcal{Z}_P$ , then the following two conditions are satisfied:

- (i)  $\mathbf{0} \in \text{conv}(\mathbf{g}_1, \dots, \mathbf{g}_m)$ ;
- (ii) if  $\mathbf{0} \in \text{conv}(\mathbf{g}_{i_1}, \dots, \mathbf{g}_{i_k})$  then  $k \geq m - n$ .

(Indeed, (i) expresses the fact that  $\mathcal{L}$  is nonempty, and (ii) easily follows from Lemma I.6.) The converse statement also holds:

**Proposition I.7** ([BM, Lemma 0.12]). *Let  $\mathcal{L}$  be a link satisfying conditions (i) and (ii) above. Then there is a simple polytope (I.1) such that  $\mathcal{L}$  is an embedding of the moment-angle manifold  $\mathcal{Z}_P$ . In particular the intersection of quadrics defining  $\mathcal{L}$  is nondegenerate.*

*Proof.* The torus  $\mathbb{T}^m$  acts on  $\mathcal{L}$  coordinatewise, and the quotient  $\mathcal{L}/\mathbb{T}^m$  is the set  $Q$  of nonnegative solutions of the system

$$(I.9) \quad \sum_{k=1}^m \mathbf{g}_k y_k = \mathbf{0}, \quad \sum_{k=1}^m y_k = 1.$$

Condition (ii) implies that this system has maximal rank  $m - n$  [BM, Lemma 0.3], and therefore  $Q$  is the intersection of an  $n$ -dimensional affine plane with  $\mathbb{R}_{\geq}^m$ . Since  $Q$  is bounded (as  $\mathcal{L}$  is compact) and contains a nonzero point  $\mathbf{y}$  (by condition (i)), it is a convex  $n$ -dimensional polytope. Writing a general solution of system (I.9) we obtain a presentation (I.1) for the polytope  $Q$ , with  $\mathbf{a}_1, \dots, \mathbf{a}_m$  being a Gale transform of  $\mathbf{g}_1, \dots, \mathbf{g}_m$  and  $\sum_{k=1}^m b_k = 1$ . This implies that  $\mathbf{g}_1, \dots, \mathbf{g}_m$  is a Gale diagram of the dual polytope  $Q^*$ . Now condition (ii) and Lemma I.6 imply that at most  $n$  facets of  $Q$  may meet, hence  $Q$  is simple.  $\square$

**Example I.8.** Let  $m - n - 1 = 1$ , so that  $\mathbf{g}_i \in \mathbb{R}$ . Condition (ii) implies that all  $\mathbf{g}_i$  are nonzero; assume that there are  $p$  positive and  $q = m - p$  negative numbers among them. Then condition (i) implies that  $p > 0$  and  $q > 0$ . Therefore  $\mathcal{L}$  is diffeomorphic to

$$\left\{ \begin{array}{l} \mathbf{z} \in \mathbb{C}^m : \quad |z_1|^2 + \dots + |z_p|^2 - |z_{p+1}|^2 - \dots - |z_m|^2 = 0, \\ \quad \quad \quad |z_1|^2 + \dots + |z_m|^2 = 1. \end{array} \right\}$$

The first equation specifies a cone over  $S^{2p-1} \times S^{2q-1}$ , so that  $\mathcal{L} \cong S^{2p-1} \times S^{2q-1}$ . The corresponding polytope is either a simplex  $\Delta^{m-2}$  with one redundant inequality (if  $p = 1$  or  $q = 1$ ) or a product  $\Delta^{p-1} \times \Delta^{q-1}$ .

## Lecture II. Other constructions of the moment-angle manifold

In the previous lecture we defined the moment-angle manifold  $\mathcal{Z}_P$  corresponding to a simple polytope  $P$ , and embedded it into  $\mathbb{C}^m$  as a nondegenerate intersection of  $m - n$  real quadrics. There are several other constructions of the moment-angle manifold, some of which admit interesting generalisations and lead to new applications.

### II.1. IDENTIFICATION SPACE

Denote by  $[m]$  the  $m$ -element set  $\{1, \dots, m\}$ . For any subset  $I \subset [m]$  define the corresponding *coordinate subgroup*  $T^I$  in the torus  $\mathbb{T}^m$  as

$$(II.1) \quad T^I := \{\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{T}^m : t_j = 1 \text{ for } j \notin I\}.$$

In particular,  $T^\emptyset$  is the trivial subgroup  $\{1\}$ .

We consider the map  $\mathbb{R}_{\geq}^m \times \mathbb{T} \rightarrow \mathbb{C}$  defined by  $(y, t) \mapsto yt$ . Taking product we obtain a map  $\mathbb{R}_{\geq}^m \times \mathbb{T}^m \rightarrow \mathbb{C}^m$ . The preimage of a point  $\mathbf{z} \in \mathbb{C}^m$  under this map is  $\mathbf{y} \times T^{I(\mathbf{z})}$ , where  $y_i = |z_i|$  for  $1 \leq i \leq m$  and  $I(\mathbf{z}) \subset [m]$  is the set of zero coordinates of  $\mathbf{z}$ . Therefore,  $\mathbb{C}^m$  can be identified with the quotient space

$$\mathbb{R}_{\geq}^m \times \mathbb{T}^m / \sim \quad \text{where } (\mathbf{y}, \mathbf{t}_1) \sim (\mathbf{y}, \mathbf{t}_2) \text{ if } \mathbf{t}_1^{-1} \mathbf{t}_2 \in T^{I(\mathbf{y})}.$$

The space  $\mathcal{Z}_P$  was originally defined in [DJ] as a similar identification space. Given  $p \in P$ , set  $I_p = \{i \in [m] : p \in F_i\}$  (the set of facets containing  $p$ ).

**Proposition II.1.** *The moment-angle manifold  $\mathcal{Z}_P$  is  $\mathbb{T}^m$ -equivariantly homeomorphic to the quotient*

$$P \times \mathbb{T}^m / \sim \quad \text{where } (p, \mathbf{t}_1) \sim (p, \mathbf{t}_2) \text{ if } \mathbf{t}_1^{-1} \mathbf{t}_2 \in T^{I_p}.$$

*Proof.* It follows from (I.3) that  $\mathcal{Z}_P$  is  $\mathbb{T}^m$ -homeomorphic to  $i_P(P) \times \mathbb{T}^m / \sim$ , and a point  $p \in P$  is mapped by  $i_P$  to  $\mathbf{y} \in \mathbb{R}_{\geq}^m$  with  $I_p = I(\mathbf{y})$ .  $\square$

**Corollary II.2.** *The  $\mathbb{T}^m$ -equivariant topological type of the manifold  $\mathcal{Z}_P$  depends only on the combinatorial type (the face poset) of the polytope  $P$ .*

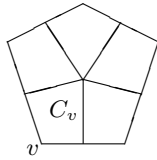
### II.2. MOMENT-ANGLE COMPLEX

We consider the unit polydisc in  $\mathbb{C}^m$ :

$$\mathbb{D}^m = \{(z_1, \dots, z_m) \in \mathbb{C}^m : |z_i| \leq 1, \quad i = 1, \dots, m\}.$$

The quotient  $\mathbb{D}^m / \mathbb{T}^m$  is the standard *unit  $m$ -cube*  $\mathbb{I}^m = [0, 1]^m$ , and we have an identification  $\mathbb{I}^m \times \mathbb{T}^m / \sim \cong \mathbb{D}^m$  like that considered in the previous subsection.

Let  $P$  be an  $n$ -dimensional combinatorial simple polytope, and let  $V(P)$  be its set of vertices (it is helpful to have a geometric presentation (I.1) in mind, although only



the combinatorial structure of  $P$  is relevant here). Every such  $P$  may be represented as a union  $\bigcup C_v$  of combinatorial  $n$ -cubes, with one such cube  $C_v$  for vertex  $v \in V(P)$ . See the figure on the left where a pentagon is split into 5 quadrilaterals (combinatorial 2-cubes); the details of this self-evident construction can be found in [BP2, §4.2].

Every vertex  $v \in P$  can be written as an intersection of  $n$  facets:  $v = F_{i_1} \cap \dots \cap F_{i_n}$ , and we denote  $I_v := \{i_1, \dots, i_n\}$ . Consider the subset  $C_v \times \mathbb{T}^m / \sim \subset P \times \mathbb{T}^m / \sim \cong \mathcal{Z}_P$ . We have

$$C_v \times \mathbb{T}^m / \sim = (C_v \times T^{I_v} / \sim) \times T^{[m] \setminus I_v} \cong \mathbb{D}^n \times \mathbb{T}^{m-n}.$$

We therefore may identify  $\mathcal{Z}_P$  with the union

$$(II.2) \quad \bigcup_{v \in V(P)} B_v \subset \mathbb{D}^m,$$

where

$$B_v := \{(z_1, \dots, z_m) \in \mathbb{D}^m : |z_j| = 1 \text{ if } v \notin F_j\} \cong \mathbb{D}^n \times \mathbb{T}^{m-n}.$$

Now let  $\mathcal{K}_P$  be the boundary  $\partial P^*$  of the dual simplicial polytope. It can be viewed as a simplicial complex on the set  $[m]$ , whose simplices are subsets  $I = \{i_1, \dots, i_k\}$  such that  $F_{i_1} \cap \dots \cap F_{i_k} \neq \emptyset$  in  $P$ . (Note that a redundant inequality in (I.1) whose corresponding facet  $F_i$  is empty gives rise to a *ghost vertex* of  $\mathcal{K}_P$ , i.e. a one-element subset  $\{i\}$  of  $[m]$  which is not a vertex of  $\mathcal{K}_P$ .) Then we may rewrite (II.2) as  $\mathcal{Z}_P \cong \bigcup_{I \in \mathcal{K}_P} B_I$ , where

$$B_I := \{(z_1, \dots, z_m) \in \mathbb{D}^m : |z_j| = 1 \text{ for } j \notin I\}.$$

This construction admits the following generalisation.

**Definition II.3** ([BP2]). Let  $\mathcal{K}$  be a simplicial complex on the set  $[m]$ . The corresponding *moment-angle complex*  $\mathcal{Z}_{\mathcal{K}}$  is defined as

$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} B_I \subset \mathbb{D}^m.$$

Note that  $\dim \mathcal{Z}_{\mathcal{K}} = m + \dim \mathcal{K} + 1$ .

**Example II.4.** 1. Let  $\mathcal{K} = \partial \Delta^n$ , the boundary of an  $n$ -simplex (which is dual to  $P = \Delta^n$ ) and  $m = n + 1$ . Then

$$\mathcal{Z}_{\mathcal{K}} = (\mathbb{D} \times \dots \times \mathbb{D} \times \mathbb{T}) \cup (\mathbb{D} \times \dots \times \mathbb{T} \times \mathbb{D}) \cup \dots \cup (\mathbb{T} \times \mathbb{D} \times \dots \times \mathbb{D}) = \partial \mathbb{D}^{n+1} \cong S^{2n+1}.$$

2. Let  $\mathcal{K}$  be a disjoint union of  $m$  points (this example is not of the form  $\mathcal{K}_P$  if  $m \neq 2$ ). Then  $\mathcal{Z}_{\mathcal{K}}$  is given as the following  $m + 1$ -dimensional subspace in  $\mathbb{D}^m$ :

$$\mathcal{Z}_{\mathcal{K}} = (\mathbb{D} \times \mathbb{T} \times \dots \times \mathbb{T}) \cup (\mathbb{T} \times \mathbb{D} \times \dots \times \mathbb{T}) \cup \dots \cup (\mathbb{T} \times \mathbb{T} \times \dots \times \mathbb{D}).$$

**Lemma II.5** ([BP2, Lemma 6.13]). *If  $\mathcal{K}$  is a triangulation (simplicial subdivision) of an  $(n - 1)$ -dimensional sphere, then  $\mathcal{Z}_{\mathcal{K}}$  is a closed topological manifold of dimension  $m + n$ .*

*Remark.* If  $\mathcal{K} = \mathcal{K}_P$  for a simple polytope  $P$ , then  $\mathcal{Z}_{\mathcal{K}} \cong \mathcal{Z}_P$  and therefore the manifold  $\mathcal{Z}_{\mathcal{K}}$  can be smoothed by Theorem I.2. However, there are many sphere triangulations  $\mathcal{K}$  which are not of the form  $\mathcal{K}_P$  for any  $P$ . The question of whether the corresponding  $\mathcal{Z}_{\mathcal{K}}$  can be smoothed is open in general (see [PU] for a construction of smooth structures on some nonpolytopal  $\mathcal{Z}_{\mathcal{K}}$ ).

### II.3. COORDINATE SUBSPACE ARRANGEMENT COMPLEMENT

A *coordinate subspace* in  $\mathbb{C}^m$  is determined by a subset  $I = \{i_1, \dots, i_m\} \subset [m]$ :

$$L_I := \{z \in \mathbb{C}^m : z_{i_1} = \dots = z_{i_k} = 0\}.$$

Every simplicial complex  $\mathcal{K}$  on  $[m]$  defines an *arrangement* of coordinate subspaces in  $\mathbb{C}^m$  and its *complement*

$$(II.3) \quad U(\mathcal{K}) := \mathbb{C}^m \setminus \bigcup_{I \notin \mathcal{K}} L_I.$$

**Example II.6.** 1. If  $\mathcal{K} = \partial\Delta^{m-1}$  then  $U(\mathcal{K}) = \mathbb{C}^m \setminus \{z : z_1 = \dots = z_m = 0\} = \mathbb{C}^m \setminus \{\mathbf{0}\}$ .

2. If  $\mathcal{K}$  is a disjoint union of  $m$  points, then

$$U(\mathcal{K}) = \mathbb{C}^m \setminus \bigcup_{1 \leq i < j \leq m} \{z : z_i = z_j = 0\}$$

is the complement to all coordinate planes of codimension 2.

**Proposition II.7.** *Every complement to a set of coordinate subspaces in  $\mathbb{C}^m$  has the form  $U(\mathcal{K})$  for some  $\mathcal{K}$ .*

*Proof.* Let  $U \subset \mathbb{C}^m$  be such a complement. Then we have  $U = U(\mathcal{K})$  where  $\mathcal{K} = \{I \subset [m] : L_I \cap U \neq \emptyset\}$ .  $\square$

Observe that  $Z_{\mathcal{K}} \subset U(\mathcal{K})$  for every  $\mathcal{K}$ .

**Theorem II.8** ([BP2, Th. 8.9]). *There is a  $\mathbb{T}^m$ -equivariant deformation retraction  $U(\mathcal{K}) \rightarrow Z_{\mathcal{K}}$ .*

**Example II.9.** Let  $\mathcal{K} = \partial\Delta^{m-1}$ . Then  $U(\mathcal{K}) = \mathbb{C}^m \setminus \{\mathbf{0}\}$  retracts onto  $Z_{\mathcal{K}} = S^{2m-1}$ .

### II.4. LEVEL SET FOR A MOMENT MAP

The moment-angle manifold  $Z_P$  is closely related to the construction of *Hamiltonian toric manifolds* via symplectic reduction.

Recall that a *symplectic manifold*  $(W, \omega)$  is a smooth (but not necessarily compact) manifold  $W$  with a closed 2-form  $\omega$  which is nondegenerate at every point. Assume that a torus  $T$  acts on  $W$  preserving the symplectic form  $\omega$ . Denote by  $\mathfrak{t}$  the Lie algebra of  $T$  (this algebra is commutative and therefore trivial, but the construction may be extended to noncommutative Lie group actions). For any  $u \in \mathfrak{t}$  denote by  $\xi_u$  the corresponding  $T$ -invariant vector field on  $W$ . The  $T$ -action is *Hamiltonian* if the 1-form  $\omega(\cdot, \xi_u)$  is exact for every  $u \in \mathfrak{t}$ . In other words, there is a function  $H_u$  on  $W$ , called a *Hamiltonian*, such that  $\omega(\xi, \xi_u) = dH_u(\xi)$  for every vector field  $\xi$  on  $W$ . The *moment map*

$$\mu: W \rightarrow \mathfrak{t}^*, \quad (x, u) \mapsto H_u(x)$$

is therefore defined. Its image  $\mu(W)$  is a convex polyhedron (a convex polytope if  $W$  is compact) by a theorem of Atiyah and Guillemin–Sternberg [Gu].

**Example II.10.** A basic example is  $W = \mathbb{C}^m$  with the symplectic form  $\omega = 2 \sum_{k=1}^m dx_k \wedge dy_k$  where  $z_k = x_k + iy_k$ . The coordinatewise action of  $\mathbb{T}^m$  is Hamiltonian and the moment map  $\mu: \mathbb{C}^m \rightarrow \mathbb{R}^m$  is given by  $\mu(z_1, \dots, z_m) = (|z_1|^2, \dots, |z_m|^2)$ .

A simple polytope (I.1) is called *Delzant* if the vectors  $\mathbf{a}_i$  have integral coordinates and for every vertex  $v = F_{i_1} \cap \dots \cap F_{i_n}$  the set  $\{\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}\}$  is a basis of the integral lattice  $\mathbb{Z}^n \subset \mathbb{R}^n$ .

Assume now that  $P$  is a Delzant polytope. We also assume for simplicity that there are no redundant inequalities in (I.1) (redundant inequalities may be also taken into account by simple modifications to the constructions below). Let  $\Lambda: \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $\mathbf{e}_i \mapsto \mathbf{a}_i$ , be the transpose of  $A_P$ . Since  $P$  is Delzant, it restricts to a map of integral lattices  $\mathbb{Z}^m \rightarrow \mathbb{Z}^n$  and defines a map of tori  $\mathbb{T}^m \rightarrow \mathbb{T}^n$ , which we continue denoting  $\Lambda$ . Consider  $K := \text{Ker}(\Lambda: \mathbb{T}^m \rightarrow \mathbb{T}^n)$ . Because of the Delzant condition,  $K$  is isomorphic to an  $(m - n)$ -torus.

**Example II.11.** By restricting the  $\mathbb{T}^m$ -action of Example II.10 to  $K$  we obtain a Hamiltonian action whose moment map is given by the composition

$$\mu_K: \mathbb{C}^m \longrightarrow \mathbb{R}^m \xrightarrow{\Gamma} \mathbb{R}^{m-n},$$

where  $\Gamma = (\gamma_{jk})$  is defined by (I.4). The quadratic forms  $\sum_{k=1}^m \gamma_{jk} |z_k|^2$  for  $1 \leq j \leq m - n$  constitute a basis in the space of Hamiltonian functions, and (I.5) implies the following

**Proposition II.12.** *If  $P$  is a Delzant polytope then the moment-angle manifold  $\mathcal{Z}_P$  is identified with the level set  $\mu_K^{-1}(\mathbf{c})$  of the moment map  $\mu_K$  for the Hamiltonian action of  $K$  on  $\mathbb{C}^m$ , where  $\mathbf{c} = (c_1, \dots, c_{m-n})$  and  $c_j = \sum_{k=1}^m \gamma_{jk} b_k$ .*

Note that  $\mathbf{c}$  is a *regular value* of the moment map  $\mu_K$  by Theorem I.2.

**Lemma II.13.** *If  $P$  is Delzant then the action of  $K \subset \mathbb{T}^m$  on  $\mathcal{Z}_P$  is free.*

*Proof.* A point  $\mathbf{z} \in \mathbb{C}^m$  has a nontrivial isotropy subgroup with respect to the  $\mathbb{T}^m$ -action only if some of the coordinates of  $\mathbf{z}$  vanish. These  $\mathbb{T}^m$ -isotropy subgroups are of the form  $T^{I(\mathbf{z})}$ , see (II.1), where  $I(\mathbf{z})$  is the set of zero coordinates of  $\mathbf{z}$ . If  $\mathbf{z} \in i_Z(\mathcal{Z}_P)$  then  $\bigcap_{i \in I(\mathbf{z})} F_i \neq \emptyset$ , and the restriction of  $\Lambda: \mathbb{T}^m \rightarrow \mathbb{T}^n$  to every such  $T^{I(\mathbf{z})}$  is an injection by the Delzant condition. Therefore,  $K = \text{Ker } \Lambda$  intersects every  $\mathbb{T}^m$ -isotropy subgroup only at the unit.  $\square$

**Construction II.14** (Symplectic reduction). The manifold  $\mu_K^{-1}(\mathbf{c}) \cong \mathcal{Z}_P$  fails to be symplectic as the restriction of  $\omega$  to  $\mu_K^{-1}(\mathbf{c})$  is degenerate. However it may be shown [Gu] that the quotient  $\mu_K^{-1}(\mathbf{c})/K$  supports a nondegenerate 2-form  $\omega'$  satisfying the condition  $p^*\omega' = i_Z^*\omega$ , where  $p: \mu_K^{-1}(\mathbf{c}) \rightarrow \mu_K^{-1}(\mathbf{c})/K$  is the projection. Therefore  $(\mu_K^{-1}(\mathbf{c})/K, \omega')$  is a symplectic manifold of dimension  $2n$ . It has a residual Hamiltonian action of the  $n$ -torus  $\mathbb{T}^m/K$ . The manifold  $M_P := \mu_K^{-1}(\mathbf{c})/K$  is referred to as a *Hamiltonian toric manifold*. The passage from  $(\mathbb{C}^m, \omega)$  to  $(M_P, \omega')$  is known as the *symplectic reduction* of  $\mathbb{C}^m$  by the action of  $K$ .

Hamiltonian toric manifolds are closely related to nonsingular projective toric varieties in algebraic geometry.

A *toric variety* is a normal algebraic variety  $X$  on which an algebraic torus  $(\mathbb{C}^\times)^n$  acts with a dense orbit, see [Da].

A set of vectors  $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{R}^n$  defines a convex polyhedral *cone*

$$\sigma = \{\mu_1 \mathbf{a}_1 + \dots + \mu_k \mathbf{a}_k: \mu_i \in \mathbb{R}_{\geq 0}\}.$$

A cone is *rational* if its generating vectors can be chosen from the integral lattice  $\mathbb{Z}^n \subset \mathbb{R}^n$ , and is *strongly convex* if it does not contain a line. A cone is *simplicial* (respectively, *regular*) if it is generated by a part of basis of  $\mathbb{R}^n$  (respectively,  $\mathbb{Z}^n$ ).

A *fan* is a finite collection  $\Sigma = \{\sigma_1, \dots, \sigma_s\}$  of strongly convex cones in some  $\mathbb{R}^n$  such that every face of a cone in  $\Sigma$  belongs to  $\Sigma$  and the intersection of any two cones in  $\Sigma$  is a face of each. A fan  $\Sigma$  is *rational* (respectively, *simplicial*, *regular*) if every cone in  $\Sigma$  is rational (respectively, simplicial, regular). A fan  $\Sigma = \{\sigma_1, \dots, \sigma_s\}$  is called *complete* if  $\sigma_1 \cup \dots \cup \sigma_s = \mathbb{R}^n$ .

**Example II.15.** Let (I.1) be a simple polytope. The *normal fan*  $\Sigma_P$  of  $P$  consists of cones spanned by those sets of vectors  $\mathbf{a}_{j_1}, \dots, \mathbf{a}_{j_k}$  for which intersection  $F_{j_1} \cap \dots \cap F_{j_k}$  is nonempty. It is a complete simplicial fan in  $\mathbb{R}^n$ . If  $P$  is Delzant, then  $\Sigma_P$  is rational and regular.

As is well known in algebraic geometry, toric varieties are classified by rational fans [Da]. Under this correspondence, complete fans give rise to compact varieties, normal fans of polytopes to projective varieties, regular fans to nonsingular varieties, and simplicial fans to varieties with mild (orbifold-type) singularities.

There is the following algebraic version of symplectic reduction, which is now commonly referred to as the ‘Cox construction’, although it takes origin in the work of several authors [Co].

**Construction II.16.** Assume that  $\Sigma$  is a complete rational simplicial fan in  $\mathbb{R}^n$  with  $m$  one-dimensional cones generated by primitive vectors  $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{Z}^m$ . The *underlying simplicial complex* of  $\Sigma$  is defined as

$$\mathcal{K}_\Sigma := \{I = \{i_1, \dots, i_k\} \subset [m] : \mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_k} \text{ span a cone of } \Sigma\}.$$

It may be viewed geometrically as the intersection of  $\Sigma$  with a unit sphere.

Let  $\Lambda_{\mathbb{C}}: (\mathbb{C}^\times)^m \rightarrow (\mathbb{C}^\times)^n$  be the map of algebraic tori corresponding to the map  $\mathbb{Z}^m \rightarrow \mathbb{Z}^n$ ,  $\mathbf{e}_i \mapsto \mathbf{a}_i$ . Set  $G := \text{Ker } \Lambda_{\mathbb{C}}$ . This is an  $(m - n)$ -dimensional algebraic subgroup in  $(\mathbb{C}^\times)^m$ , hence, it is isomorphic to a product of  $(\mathbb{C}^\times)^{m-n}$  and a finite group (the finite group is trivial if the fan is regular). The group  $G$  acts almost freely (with finite isotropy subgroups) on the open set  $U(\mathcal{K}_\Sigma)$  of (II.3); moreover, this action is free if  $\Sigma$  is a regular fan. This is proved in the same way as in Lemma II.13.

The toric variety associated to the fan  $\Sigma$  is defined as the quotient  $X_\Sigma := U(\mathcal{K}_\Sigma)/G$ . It is a complex algebraic variety of dimension  $n$ . The variety  $X_\Sigma$  is nonsingular whenever  $\Sigma$  is regular; otherwise it has only orbifold-type singularities (locally isomorphic to a quotient of  $\mathbb{C}^n$  by a finite group). The quotient algebraic torus  $(\mathbb{C}^\times)^m/G \cong (\mathbb{C}^\times)^n$  acts on  $X_\Sigma$  with a dense orbit.

The variety  $X_\Sigma$  is projective if and only if  $\Sigma$  is the normal fan of a polytope  $P$ ; in this case we shall denote the variety by  $X_P$ .

The Cox construction extends to noncomplete and nonsimplicial fans (in the latter case the ordinary quotient needs to be replaced by the *categorical* one), but we shall not need this generality here.

Now if  $P$  is a Delzant polytope, then the nonsingular projective toric variety  $X_P$  is symplectic and is  $\mathbb{T}^n$ -equivariantly symplectomorphic (for an appropriate choice of the symplectic form) to the Hamiltonian toric manifold  $M_P$ . In other words, the quotient of the open set  $U(\mathcal{K}_{\Sigma_P}) \subset \mathbb{C}^m$  by the action of a noncompact group  $G$  can be identified with the quotient of the compact subset  $i_Z(\mathcal{Z}_P) \subset U(\mathcal{K}_{\Sigma_P})$  by a compact subgroup  $K \subset G$  [Gu, App. 1].

### Lecture III. Topology of moment-angle complexes

The topology of moment-angle manifolds  $\mathcal{Z}_P$  and complexes  $\mathcal{Z}_{\mathcal{K}}$  is quite complicated even for relatively small and easily described polytopes  $P$  and complexes  $\mathcal{K}$ . We give evidences to this by describing the cohomology ring of  $\mathcal{Z}_{\mathcal{K}}$  and then providing explicit homotopy and diffeomorphism types for some series of  $P$  and  $\mathcal{K}$ .

#### III.1. THE COHOMOLOGY RING

We continue denoting by  $\mathcal{K}$  a simplicial complex on  $[m]$ . We denote by  $\mathbb{Z}[v_1, \dots, v_m]$  the polynomial ring and by  $\Lambda[u_1, \dots, u_m]$  the exterior ring with integer coefficients. Given a subset  $I = \{i_1, \dots, i_k\} \subset [m]$  we denote by  $v^I$  the square-free monomial  $v_{i_1} \cdots v_{i_k}$ . We use ‘dg ring’ as an abbreviation for ‘differential graded ring’ and similarly for abelian groups ( $\mathbb{Z}$ -modules).

**Definition III.1.** The *face ring* (also known as the *Stanley–Reisner ring*) of  $\mathcal{K}$  is the following quotient of the polynomial ring on  $m$  generators:

$$\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \dots, v_m]/(v^I : I \notin \mathcal{K}).$$

We make  $\mathbb{Z}[\mathcal{K}]$  a graded ring by setting  $\deg v_i = 2$  for all  $i$ .

**Example III.2.** 1. If  $\mathcal{K} = \partial\Delta^{m-1}$  then  $\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \dots, v_m]/(v_1 \cdots v_m)$ .  
2. If  $\mathcal{K}$  is  $m$  points, then  $\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \dots, v_m]/(v_i v_j \text{ for } 1 \leq i < j \leq m)$ .

We abbreviate  $\mathbb{Z}[v_1, \dots, v_m]$  to  $\mathbb{Z}[m]$  to make formulae shorter. The face ring  $\mathbb{Z}[\mathcal{K}]$  is a  $\mathbb{Z}[m]$ -module via the quotient projection. Its *free resolution* is an exact sequence of finitely generated  $\mathbb{Z}[m]$ -modules

$$0 \longrightarrow R^{-m} \longrightarrow \dots \longrightarrow R^{-1} \longrightarrow R^0 \longrightarrow \mathbb{Z}[\mathcal{K}] \longrightarrow 0$$

in which all  $R^{-i}$  are free modules. The  $(-i)$ th Tor group  $\text{Tor}_{\mathbb{Z}[m]}^{-i}(\mathbb{Z}[\mathcal{K}], \mathbb{Z})$  is defined as the  $(-i)$ th cohomology group of the complex

$$0 \longrightarrow R^{-m} \otimes_{\mathbb{Z}[m]} \mathbb{Z} \longrightarrow \dots \longrightarrow R^{-1} \otimes_{\mathbb{Z}[m]} \mathbb{Z} \longrightarrow R^0 \otimes_{\mathbb{Z}[m]} \mathbb{Z} \longrightarrow 0.$$

The groups  $\text{Tor}_{\mathbb{Z}[m]}^{-i}(\mathbb{Z}[\mathcal{K}], \mathbb{Z})$  acquire an internal grading from the grading in  $\mathbb{Z}[m]$  and  $\mathbb{Z}[\mathcal{K}]$ . We define

$$\text{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z}) := \bigoplus_{i=0}^m \text{Tor}_{\mathbb{Z}[m]}^{-i}(\mathbb{Z}[\mathcal{K}], \mathbb{Z}),$$

which therefore has two gradings; the *total degree* is the sum of these two gradings. Moreover,  $\text{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z})$  has a canonical multiplication turning it into a graded ring with respect to the total degree (see [BP2]).

**Theorem III.3.** *The cohomology ring of the moment-angle complex  $\mathcal{Z}_{\mathcal{K}}$  is given by the isomorphisms*

$$\begin{aligned} H^*(\mathcal{Z}_{\mathcal{K}}; \mathbb{Z}) &\cong \text{Tor}_{\mathbb{Z}[v_1, \dots, v_m]}(\mathbb{Z}[\mathcal{K}], \mathbb{Z}) \\ &\cong H[\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d], \end{aligned}$$

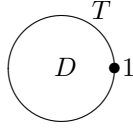
where the latter ring is the cohomology of the dg ring whose grading and differential are given by

$$\deg u_i = 1, \deg v_i = 2; \quad du_i = v_i, dv_i = 0.$$

This theorem was proved in [BP1] (with coefficients in a field), see also [BP2]. We give a sketch of the proof for  $\mathbb{Z}$  coefficients, following [Pa]. This proof first appeared in [BBP]. Another proof of the integral version appeared in [Fr].

*Sketch of proof of Theorem III.3.* We only prove that  $H^*(\mathcal{Z}_{\mathcal{K}}; \mathbb{Z})$  is isomorphic to  $H[\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d]$ ; the fact that the latter ring is isomorphic to the Tor is a standard application of the Koszul resolution. The proof is split into 4 steps.

*Step 1: cellular decomposition of  $\mathcal{Z}_{\mathcal{K}}$ .* We consider the following decomposition of the disc  $\mathbb{D}$  into 3 cells: the point  $1 \in \mathbb{D}$  is a 0-cell; the complement to 1 in the boundary circle is a 1-cell, which we denote  $T$ ; and the interior of  $\mathbb{D}$  is a 2-cell, which we denote  $D$ . By taking product we obtain a cellular decomposition of  $\mathbb{D}^m$



whose cells are parametrised by pairs of subsets  $I, J \subset [m]$  with  $I \cap J = \emptyset$ : the set  $I$  parametrises the  $T$ -cells in the product and  $J$  parametrises the  $D$ -cells. We denote the cell of  $\mathbb{D}^m$  corresponding to a pair  $I, J$  by  $\varkappa(I, J)$ ; it is a product of  $|I|$  cells of  $T$  type and  $|J|$  cells of  $D$  type. Then  $\mathcal{Z}_{\mathcal{K}}$  includes as a cellular subcomplex in  $\mathbb{D}^m$ ; we have  $\varkappa(I, J) \subset \mathcal{Z}_{\mathcal{K}}$  whenever  $J \in \mathcal{K}$ .

We denote by  $C^*(\mathcal{Z}_{\mathcal{K}})$  the cellular cochain group of  $\mathcal{Z}_{\mathcal{K}}$ . It has a basis of cochains  $\varkappa(I, J)^*$  dual to the corresponding cells.

*Step 2: dg ring model for  $C^*(\mathcal{Z}_{\mathcal{K}})$ .* We consider the following quotient dg ring:

$$R^*(\mathcal{K}) := \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}] / (u_i v_i, v_i^2, \text{ for } 1 \leq i \leq m).$$

It has a finite rank as an abelian group, unlike  $\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}]$ . Namely, the monomials  $u^I v^J$  with  $I \cap J = \emptyset$  and  $J \in \mathcal{K}$  constitute a basis of  $R^*(\mathcal{K})$ . Define the map

$$g: R^*(\mathcal{K}) \longrightarrow C^*(\mathcal{Z}_{\mathcal{K}}), \quad u^I v^J \mapsto \varkappa(I, J)^*.$$

It is an isomorphism of dg groups by inspection. Therefore we have an additive isomorphism  $H[R^*(\mathcal{K})] \cong H^*(\mathcal{Z}_{\mathcal{K}})$ .

*Step 3:  $H[\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}], d] \cong H[R^*(\mathcal{K}), d]$ , i.e.  $(u_i v_i, v_i^2, 1 \leq i \leq m)$  is an acyclic ideal.* We have a pair of maps of dg groups:

$$\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}] \begin{array}{c} \xrightarrow{\varrho} \\ \xleftarrow{\iota} \end{array} R^*(\mathcal{K})$$

where  $\varrho$  is the quotient projection (a ring map) and  $\iota$  sends  $u^I v^J$  to itself (it is a monomorphism of dg groups, but not a ring map). We have  $\varrho \cdot \iota = \text{id}$  and there is an explicitly defined map  $s$  satisfying the identity  $ds + sd = \text{id} - \iota \cdot \varrho$  (a cochain homotopy between  $\text{id}$  and  $\iota \cdot \varrho$ ). It follows that  $\varrho$  induces an isomorphism in cohomology.

*Step 4:  $g: R^*(\mathcal{K}) \rightarrow C^*(\mathcal{Z}_{\mathcal{K}})$  is a ring isomorphism.* We already know from Step 2 that  $g$  is an isomorphism of dg groups. A ring structure in  $C^*(\mathcal{Z}_{\mathcal{K}})$  is defined by a choice of a cellular approximation for the diagonal map  $\Delta: \mathcal{Z}_{\mathcal{K}} \rightarrow \mathcal{Z}_{\mathcal{K}} \times \mathcal{Z}_{\mathcal{K}}$ . Consider the map  $\tilde{\Delta}: \mathbb{D} \rightarrow \mathbb{D} \times \mathbb{D}$ , defined in polar coordinates  $z = \rho e^{i\varphi} \in \mathbb{D}$ ,  $0 \leq \rho \leq 1$ ,  $0 \leq \varphi < 2\pi$  as follows:

$$\rho e^{i\varphi} \mapsto \begin{cases} (1 + \rho(e^{2i\varphi} - 1), 1) & \text{for } 0 \leq \varphi \leq \pi, \\ (1, 1 + \rho(e^{2i\varphi} - 1)) & \text{for } \pi \leq \varphi < 2\pi. \end{cases}$$

This is a cellular map (with respect to the cellular decomposition of Step 1) homotopic to the diagonal  $\Delta: \mathbb{D} \rightarrow \mathbb{D} \times \mathbb{D}$ . Taking an  $m$ -fold product, we obtain a cellular approximation  $\tilde{\Delta}: \mathbb{D}^m \rightarrow \mathbb{D}^m \times \mathbb{D}^m$  which restricts to a cellular approximation for



the diagonal map of  $\mathcal{Z}_{\mathcal{K}}$  for arbitrary  $\mathcal{K}$ . The ring structure in  $C^*(\mathcal{Z}_{\mathcal{K}})$  defined by the composition

$$C^*(\mathcal{Z}_{\mathcal{K}}) \otimes C^*(\mathcal{Z}_{\mathcal{K}}) \xrightarrow{\times} C^*(\mathcal{Z}_{\mathcal{K}} \times \mathcal{Z}_{\mathcal{K}}) \xrightarrow{\tilde{\Delta}^*} C^*(\mathcal{Z}_{\mathcal{K}})$$

and induces the cup product in the cohomology of  $\mathcal{Z}_{\mathcal{K}}$ .

We therefore need to check that  $g: R^*(\mathcal{K}) \rightarrow C^*(\mathcal{Z}_{\mathcal{K}})$  is a multiplicative map with respect to the ring structures in  $R^*(\mathcal{K})$  and  $C^*(\mathcal{Z}_{\mathcal{K}})$ . To do this we note that both ring structures are functorial with respect to inclusions of simplicial complexes, and  $R^*(\Delta^{m-1}) \rightarrow C^*(\mathbb{D}^m)$  is a ring isomorphism by inspection (both rings are isomorphic to  $\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[m]/(u_i v_i, v_i^2, 1 \leq i \leq m)$ ). The multiplicativity of  $g$  for arbitrary  $\mathcal{K}$  follows by considering the commutative diagram

$$\begin{array}{ccc} R^*(\Delta^{m-1}) & \xrightarrow{\text{ring iso}} & C^*(\mathbb{D}^m) \\ \text{ring epi} \downarrow & & \downarrow \text{ring epi} \\ R^*(\mathcal{K}) & \xrightarrow[\text{add. iso}]{g} & C^*(\mathcal{Z}_{\mathcal{K}}). \end{array}$$

□

The bigrading in the Tor defines a bigrading in  $H^*(\mathcal{Z}_{\mathcal{K}})$ , and we may define

$$H^{-i,2j}(\mathcal{Z}_{\mathcal{K}}) := \text{Tor}_{\mathbb{Z}[v_1, \dots, v_m]}^{-i,2j}(\mathbb{Z}[\mathcal{K}], \mathbb{Z}).$$

This bigrading may be also induced from the dg ring  $\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}[\mathcal{K}]$  by setting bideg  $u_i = (-1, 2)$  and bideg  $v_i = (0, 2)$ . Moreover, this bigrading may be refined to a  $\mathbb{Z} \oplus \mathbb{Z}^m$ -multigrading by setting

$$\text{mdeg } u_i = (-1, 2\mathbf{e}_i), \quad \text{mdeg } v_i = (0, 2\mathbf{e}_i)$$

where  $\mathbf{e}_i \in \mathbb{Z}^m$  is the  $i$ th basis vector. We denote by  $H^{-i,2\mathbf{a}}(\mathcal{Z}_{\mathcal{K}})$  the component of multidegree  $(-i, 2\mathbf{a})$  for  $\mathbf{a} \in \mathbb{Z}^m$ . Since  $H^*(\mathcal{Z}_{\mathcal{K}}) \cong H[R^*(\mathcal{K})]$  (see the proof of Theorem III.3),  $H^{-i,2\mathbf{a}}(\mathcal{Z}_{\mathcal{K}})$  is nonzero only if  $\mathbf{a} \in \{0, 1\}^m$ , and such vectors  $\mathbf{a}$  may be identified with subsets  $J \subset [m]$  by considering the unit coordinates of  $\mathbf{a}$ . We therefore have

$$H^k(\mathcal{Z}_{\mathcal{K}}) = \bigoplus_{-i+2j=k} H^{-i,2j}(\mathcal{Z}_{\mathcal{K}}) = \bigoplus_{-i+2|J|=k} H^{-i,2J}(\mathcal{Z}_{\mathcal{K}}) \quad \text{for } i, j, k \geq 0, J \subset [m].$$

Given  $J \subset [m]$  denote by  $\mathcal{K}_J$  the corresponding *full subcomplex* of  $\mathcal{K}$  (the restriction of  $\mathcal{K}$  to  $J$ ).

**Corollary III.4.** *We have*

$$H^k(\mathcal{Z}_{\mathcal{K}}) \cong \bigoplus_{J \subset [m]} \tilde{H}^{k-|J|-1}(\mathcal{K}_J) \quad \text{and} \quad H^{-i,2J}(\mathcal{Z}_{\mathcal{K}}) \cong \tilde{H}^{|J|-i-1}(\mathcal{K}_J),$$

where  $\tilde{H}^p(\mathcal{K}_J)$  denotes the  $p$ th reduced simplicial cohomology group of  $\mathcal{K}_J$ .

*Proof.* The second formula follows from the fact that the differential  $d$  preserves the  $\mathbb{Z}^m$  part of the  $\mathbb{Z} \oplus \mathbb{Z}^m$ -multigrading in  $R^*(\mathcal{K})$ , and the cohomology of  $R^{*,2J}(\mathcal{K})$  is isomorphic to  $\tilde{H}^*(\mathcal{K}_J)$  with a shift in dimension. The first formula is obtained by summation. □

*Remark.* We also obtain that

$$\mathrm{Tor}_{\mathbb{Z}[v_1, \dots, v_m]}^{-i}(\mathbb{Z}[\mathcal{K}], \mathbb{Z}) \cong \bigoplus_{J \subset [m]} \tilde{H}^{|J|-i-1}(\mathcal{K}_J),$$

which is known in combinatorial commutative algebra as the *Hochster formula*.

The multiplication in  $H^*(\mathcal{Z}_{\mathcal{K}})$  may be also described in terms of full subcomplexes of  $\mathcal{K}$ : the product of  $\alpha \in H^{-i, 2J}(\mathcal{Z}_{\mathcal{K}})$  and  $\beta \in H^{-k, 2L}(\mathcal{Z}_{\mathcal{K}})$  is zero if  $J \cap L \neq \emptyset$ , and otherwise  $\alpha \cdot \beta$  is given by a certain element in  $H^{|J|+|L|-i-k-1}(\mathcal{K}_{J \sqcup L})$ , see [Pa, §5.1].

**Corollary III.5.** *We have*

$$H^k(\mathcal{Z}_P) \cong \bigoplus_{J \subset [m]} \tilde{H}^{k-|J|-1}(P_J) \quad \text{and} \quad H^{-i, 2J}(\mathcal{Z}_P) \cong \tilde{H}^{|J|-i-1}(P_J),$$

where  $P_J = \bigcup_{j \in J} F_j \subset P$ .

*Proof.* By considering the barycentric subdivision of  $\mathcal{K} = \mathcal{K}_P$  (which is also the barycentric subdivision of  $\partial P$ ) we observe that the union of facets  $\bigcup_{j \in J} F_j$  retracts onto  $\mathcal{K}_J$ .  $\square$

### III.2. SOME HOMOTOPY AND DIFFEOMORPHISM TYPES

We start with an example of the cohomology ring calculation using Theorem III.3.

**Example III.6.** 1. Let  $P$  be a pentagon. Then  $\dim \mathcal{Z}_P = 7$  and

$$\mathbb{Z}[\mathcal{K}_P] = \mathbb{Z}[v_1, \dots, v_5] / (v_i v_j : j - i = 2 \pmod{5}).$$

We have the following nontrivial cohomology groups

$$H^0(\mathcal{Z}_P) \cong \mathbb{Z}, \quad \text{generated by } 1 \in R^*(\mathcal{K}_P)$$

$$H^3(\mathcal{Z}_P) \cong \mathbb{Z}^5, \quad \text{generated by } [u_i v_j] \in R^*(\mathcal{K}_P) \text{ for } j - i = 2 \pmod{5}$$

$$H^4(\mathcal{Z}_P) \cong \mathbb{Z}^5, \quad \text{generated by } [u_i u_j v_k] \in R^*(\mathcal{K}_P) \text{ for } k - i = 3, k - j = 2 \pmod{5}$$

$$H^7(\mathcal{Z}_P) \cong \mathbb{Z}, \quad \text{generated by } [u_1 u_2 u_3 v_4 v_5] \in R^*(\mathcal{K}_P).$$

The multiplication is also easily determined (e.g.,  $[u_2 v_4] \cdot [u_2 u_3 v_5] = 0$  and  $[u_2 u_3 v_5] \cdot [u_4 v_1] = [u_1 u_2 u_3 v_4 v_5]$ ), and we obtain the following isomorphism of rings:

$$H^*(\mathcal{Z}_P) \cong H^*((S^3 \times S^4)^{\#5}),$$

where  $M^{\#k}$  denotes the connected sum of  $k$  copies of the manifold  $M$ .

2. A similar calculation shows that if  $P$  is an  $m$ -gon, then

$$(III.1) \quad H^*(\mathcal{Z}_P) \cong H^*\left(\#_{k=3}^{m-1} (S^k \times S^{m+2-k})^{\#(k-2)\binom{m-2}{k-1}}\right).$$

In fact, the cohomology ring isomorphism of (III.1) is induced by a diffeomorphism, so that the moment-angle manifolds corresponding to polygons are connected sums of sphere products, with 2 spheres in each product. This description of the diffeomorphism type of  $\mathcal{Z}_P$  admits the following generalisation to a series of higher-dimensional polytopes.

Let  $P$  be given by (I.1) and let  $\mathbf{v} \in P$  be a vertex. Choose a hyperplane  $\{\mathbf{x} : (\mathbf{a}, \mathbf{x}) + b = 0\}$  separating  $\mathbf{v}$  from the other vertices of  $P$ , i.e.  $(\mathbf{a}, \mathbf{v}) + b < 0$  and  $(\mathbf{a}, \mathbf{v}') + b > 0$  for any other vertex  $\mathbf{v}' \in P$ . We refer to the polytope  $P'$  obtained by adding the inequality  $(\mathbf{a}, \mathbf{x}) + b \geq 0$  to (I.1) as a *vertex cut* of  $P$ .

**Theorem III.7** (essentially McGavran, see [BM, Th. 6.3]). *Let  $P$  be a polytope obtained from a simplex  $\Delta^n$  by applying vertex cut operation  $m - n - 1$  times. Then  $\mathcal{Z}_P$  is diffeomorphic to the following connected sum of sphere products:*

$$\#_{k=3}^{m-n+1} (S^k \times S^{m+n-k})^{\#(k-2)\binom{m-n}{k-1}}.$$

For  $n = 2$  we obtain the diffeomorphism behind cohomology isomorphism (III.1). There are also other polytopes  $P$  for which  $\mathcal{Z}_P$  is diffeomorphic to a connected sum of sphere products, see [GL]. However, in general the topology of  $\mathcal{Z}_P$  is much more complicated, as is shown by the next example.

**Example III.8** (Baskakov [Ba], see also [Pa, §5.3]). Let  $P$  be a 3-dimensional polytope obtained from the cube by cutting off two non coplanar edges. (The edge cut operation is defined similarly to the vertex cut, by choosing a hyperplane separating the edge from the other vertices of the polytope.) The corresponding  $\mathcal{Z}_P$  is an 11-dimensional manifold. It has a nontrivial *triple Massey product* of 3-dimensional cohomology classes. It follows that  $\mathcal{Z}_P$  is not *formal* in the sense of the rational homotopy theory; in particular,  $\mathcal{Z}_P$  cannot be diffeomorphic to a connected sum of sphere products.

Now let us consider some non polytopal examples.

**Example III.9.** Let  $\mathcal{K}$  consist of  $m$  disjoint points. Then  $\mathcal{Z}_{\mathcal{K}}$  is the space of Example II.6.2 and it is homotopy equivalent to the complement  $U(\mathcal{K})$  to all codimension-two coordinate planes in  $\mathbb{C}^m$ , see Example II.6. We have

$$\mathbb{Z}[\mathcal{K}] = \mathbb{Z}[v_1, \dots, v_m] / (v_i v_j : 1 \leq i < j \leq m).$$

The subspace of  $(k + 1)$ -dimensional cocycles in  $R^*(\mathcal{K})$  is generated by the monomials

$$u_{i_1} u_{i_2} \cdots u_{i_{k-1}} v_{i_k}, \quad k \geq 2 \text{ and } i_p \neq i_q \text{ for } p \neq q,$$

and has dimension  $m \binom{m-1}{k-1}$ . The subspace of coboundaries is generated by the elements of the form

$$d(u_{i_1} \cdots u_{i_k})$$

and is  $\binom{m}{k}$ -dimensional. Therefore

$$\dim H^0(\mathcal{Z}_{\mathcal{K}}) = 1,$$

$$\dim H^1(\mathcal{Z}_{\mathcal{K}}) = H^2(\mathcal{Z}_{\mathcal{K}}) = 0,$$

$$\dim H^{k+1}(\mathcal{Z}_{\mathcal{K}}) = m \binom{m-1}{k-1} - \binom{m}{k} = (k-1) \binom{m}{k}, \quad 2 \leq k \leq m,$$

and multiplication in the cohomology of  $\mathcal{Z}_{\mathcal{K}}$  is trivial. We therefore have an isomorphism of rings

$$(III.2) \quad H^*(\mathcal{Z}_{\mathcal{K}}) \cong H^* \left( \bigvee_{k=2}^m (S^{k+1})^{\vee(k-1)\binom{m}{k}} \right),$$

where  $X^{\vee k}$  denotes the wedge of  $k$  copies of the space  $X$ . In fact, this cohomology ring isomorphism is induced by a homotopy equivalence:

**Theorem III.10** (Grbić–Theriault [GT, Cor. 9.5]). *Let  $\mathcal{K}$  be the  $i$ -dimensional skeleton of a simplex  $\Delta^{m-1}$ , so that  $U(\mathcal{K})$  is the complement to all codimension*

$(i+2)$  coordinate planes in  $\mathbb{C}^m$ . Then  $U(\mathcal{K})$  has the homotopy type of the wedge of spheres

$$\bigvee_{k=i+2}^m (S^{i+k+1})^{\vee \binom{m}{k} \binom{k-1}{i+1}}.$$

For  $i = 0$  we obtain the homotopy equivalence behind cohomology isomorphism (III.2). It is also shown in [GT] that  $U(\mathcal{K})$  has the homotopy type of a wedge of spheres for a wider class of simplicial complexes, including *shifted complexes*.

**Example III.11.** Let  $P$  be a polytope obtained from  $\Delta^n$  by applying the vertex cut operation  $p-1$  times, and let  $\mathcal{K}$  consist of  $p$  disjoint points. Then the numbers of spheres and their dimensions in the connected sum  $\mathcal{Z}_P$  correspond to the numbers of spheres and their dimensions in the wedge  $\mathcal{Z}_{\mathcal{K}}$ :

$$\mathcal{Z}_P \cong \#_{k=3}^{p+1} (S^k \times S^{p+2n-k})^{\# \binom{k-2}{k-1} \binom{p}{k-1}}, \quad \mathcal{Z}_{\mathcal{K}} \cong \bigvee_{k=3}^{p+1} (S^k)^{\vee \binom{k-2}{k-1} \binom{p}{k-1}}.$$

For instance, for  $n = 3$  and  $p = 4$  we get

$$\begin{aligned} \mathcal{Z}_P &\cong (S^3 \times S^7)^{\#6} \# (S^4 \times S^6)^{\#8} \# (S^5 \times S^5)^{\#3}, \\ \mathcal{Z}_{\mathcal{K}} &\cong (S^3)^{\vee6} \vee (S^4)^{\vee8} \vee (S^5)^{\vee3}. \end{aligned}$$

The nature of this correspondence is yet to be fully understood.

### Lecture IV. Complex-analytic structures on moment-angle manifolds

Here we review a construction of Bosio–Meersseman [BM], which endows an even-dimensional moment-angle manifold  $\mathcal{Z}_P$  with a non Kähler complex-analytic structure of an *LVM-manifold*. We finish by obtaining some new information about the Dolbeault cohomology and Hodge numbers of these complex structures on  $\mathcal{Z}_P$ .

#### IV.1. LVM-MANIFOLDS

Let  $P$  be a simple polytope (I.1) and  $\mathcal{Z}_P$  the corresponding moment-angle manifold. For simplicity we shall identify  $\mathcal{Z}_P$  with its embedding  $i_{\mathcal{Z}}(\mathcal{Z}_P)$  in  $\mathbb{C}^m$ . As detailed in Lecture I, we may describe  $\mathcal{Z}_P$  as an intersection of  $m - n - 1$  homogeneous real quadrics with a unit sphere in  $\mathbb{C}^m$ :

$$\mathcal{Z}_P = \left\{ \mathbf{z} \in \mathbb{C}^m : \begin{array}{l} \sum_{k=1}^m \mathbf{g}_k |z_k|^2 = \mathbf{0}, \\ \sum_{k=1}^m |z_k|^2 = 1, \end{array} \right\}$$

where  $\mathbf{g}_1, \dots, \mathbf{g}_m$  is a set of vectors in  $\mathbb{R}^{m-n-1}$  satisfying conditions (i) and (ii) of Section I.2.

Assume that  $m - n - 1$  is even and let  $m - n - 1 = 2s$ . The transpose of the  $2s \times m$  matrix  $\Gamma^* = (\mathbf{g}_1, \dots, \mathbf{g}_m) = (\gamma_{jk})$  defines an inclusion of a  $2s$ -dimensional subspace in  $\mathbb{R}^m$ , which we denote  $V$ . We choose a complex  $s \times m$  matrix  $\Omega = (\omega_{jk})$  such that the image of the  $\mathbb{R}$ -linear map  $\mathbb{C}^s \xrightarrow{\Omega^t} \mathbb{C}^m \xrightarrow{\text{Re}} \mathbb{R}^m$  is exactly  $V$ . Let  $\omega_k \in \mathbb{C}^s$  denote the  $k$ th column of  $\Omega$ , so that  $\Omega = (\omega_1, \dots, \omega_m)$ . A sample choice of  $\Omega$  is as follows:  $\omega_{jk} = \gamma_{2j-1,k} + i\gamma_{2j,k}$  for  $1 \leq j \leq s$  and  $1 \leq k \leq m$ .

We may now use the complex vectors  $\omega_k \in \mathbb{C}^s$  instead of the real vectors  $\mathbf{g}_k \in \mathbb{R}^{2s}$  in the presentation of  $\mathcal{Z}_P$ :

$$(IV.1) \quad \mathcal{Z}_P = \left\{ \mathbf{z} \in \mathbb{C}^m : \begin{array}{l} \sum_{k=1}^m \omega_k |z_k|^2 = \mathbf{0}, \\ \sum_{k=1}^m |z_k|^2 = 1, \end{array} \right\}$$

Now define the manifold  $N$  as the projectivisation of the intersection of homogeneous quadrics in (IV.1):

$$(IV.2) \quad N := \{ \mathbf{z} \in \mathbb{C}P^{m-1} : \omega_1 |z_1|^2 + \dots + \omega_m |z_m|^2 = \mathbf{0} \}, \quad \omega_k \in \mathbb{C}^s.$$

We therefore have a principal  $S^1$ -bundle  $\mathcal{Z}_P \rightarrow N$ .

**Theorem IV.1** ([Me]). *The manifold  $N$  has a holomorphic atlas describing it as a complex manifold of complex dimension  $m - 1 - s$ .*

*Sketch of proof.* Consider a holomorphic action of  $\mathbb{C}^s$  on  $\mathbb{C}^m$  given by

$$(IV.3) \quad \begin{array}{l} \mathbb{C}^s \times \mathbb{C}^m \longrightarrow \mathbb{C}^m \\ (\mathbf{w}, \mathbf{z}) \mapsto (z_1 e^{\langle \omega_1, \mathbf{w} \rangle}, \dots, z_m e^{\langle \omega_m, \mathbf{w} \rangle}), \end{array}$$

where  $\mathbf{w} = (w_1, \dots, w_s) \in \mathbb{C}^s$ , and  $\langle \omega_k, \mathbf{w} \rangle = \omega_{1k} w_1 + \dots + \omega_{sk} w_s$ .

An argument similar to that of the proof of Lemma II.13 shows that the restriction of the action (IV.3) to  $U(\mathcal{K}_P) \subset \mathbb{C}^m$  is free. (Here  $U(\mathcal{K}_P)$  is the complement of the coordinate subspace arrangement determined by  $\mathcal{K}_P$ , see Section II.3.) Using a holomorphic atlas transverse to the orbits of the free action of  $\mathbb{C}^s$  on the complex manifold  $U(\mathcal{K}_P)$  we obtain that the quotient  $U(\mathcal{K}_P)/\mathbb{C}^s$  has a structure of a complex manifold.

On the other hand, it may be shown that the function  $|z_1|^2 + \dots + |z_m|^2$  on  $\mathbb{C}^m$  has a unique minimum when restricted to an orbit of the free action of  $\mathbb{C}^s$  on  $U(\mathcal{K}_P)$ . The set of these minima can be described as

$$T := \{\mathbf{z} \in \mathbb{C}^m \setminus \{\mathbf{0}\} : \omega_1|z_1|^2 + \dots + \omega_m|z_m|^2 = \mathbf{0}\}.$$

It follows that the quotient  $U(\mathcal{K}_P)/\mathbb{C}^s$  may be identified with  $T$ , and therefore  $T$  acquires a structure of a complex manifold of dimension  $m - s$ .

This construction may be projectivised by replacing  $\mathbb{C}^m$  by  $\mathbb{C}P^{m-1}$  and  $U(\mathcal{K}_P)$  by the complement to an arrangement in  $\mathbb{C}P^{m-1}$ . Therefore,  $N$  also becomes a complex manifold.  $\square$

The manifold  $N$  endowed with the complex structure of Theorem IV.1 is referred to as an *LVM-manifold*. These manifolds were described by Meersseman [Me] as a generalisation of the construction of Lopez de Medrano–Verjovsky [LV].

*Remark.* The embedding of  $T$  in  $\mathbb{C}^m$  and of  $N$  in  $\mathbb{C}P^{m-1}$  given by (IV.2) is not holomorphic.

#### IV.2. $\mathcal{Z}_P$ AS AN LVM-MANIFOLD

By using the principal  $S^1$ -bundle  $\mathcal{Z}_P \rightarrow N$  and playing with redundant inequalities one may also endow  $\mathcal{Z}_P$  (if its dimension  $m + n$  is even) or  $\mathcal{Z}_P \times S^1$  (if  $m + n$  is odd) with a structure of an LVM-manifold. We first summarise the effects that a redundant inequality in (I.1) has on different spaces appeared above.

**Proposition IV.2.** *The following conditions are equivalent:*

- (a)  $\langle \mathbf{a}_i, \mathbf{x} \rangle + b_i \geq 0$  is a redundant inequality in (I.1) (i.e.  $F_i = \emptyset$ );
- (b)  $\mathcal{Z}_P \subset \{\mathbf{z} \in \mathbb{C}^m : z_i \neq 0\}$ ;
- (c)  $U(\mathcal{K}_P)$  has a factor  $\mathbb{C}^\times$  on the  $i$ th coordinate;
- (d)  $\mathbf{0} \notin \text{conv}(\omega_k : k \neq i)$ .

*Proof.* The equivalence (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) follows directly from the definition of  $\mathcal{Z}_P$  and  $U(\mathcal{K}_P)$ . The equivalence (a) $\Leftrightarrow$ (d) follows from Lemma I.6.  $\square$

**Theorem IV.3** ([BM, Th. 12.2]). *Let  $\mathcal{Z}_P$  be the moment angle manifold corresponding to an  $n$ -dimensional simple polytope (I.1) defined by  $m$  inequalities.*

- (a) *If  $m + n$  is even then  $\mathcal{Z}_P$  has a complex structure as an LVM-manifold.*
- (b) *If  $m + n$  is odd then  $\mathcal{Z}_P \times S^1$  has a complex structure as an LVM-manifold.*

*Proof.* (a) Since  $m + n$  is even,  $m - n - 1$  is odd. We add one redundant inequality  $1 \geq 0$  to (I.1), and denote the resulting moment-angle manifold by  $\mathcal{Z}'_P$ . We have  $\mathcal{Z}'_P \cong \mathcal{Z}_P \times S^1$ , and it follows from Construction I.4 that  $\mathcal{Z}'_P$  is given by

$$\left\{ \begin{array}{l} \mathbf{z} \in \mathbb{C}^{m+1} : \quad \mathbf{g}_1|z_1|^2 + \dots + \mathbf{g}_m|z_m|^2 = \mathbf{0}, \\ \quad \quad \quad |z_1|^2 + \dots + |z_m|^2 - |z_{m+1}|^2 = 0, \\ \quad \quad \quad |z_1|^2 + \dots + |z_m|^2 + |z_{m+1}|^2 = 1, \end{array} \right\}$$

where  $\Gamma^* = (\mathbf{g}_1 \dots \mathbf{g}_m)$  is the  $(m - n - 1) \times m$  matrix of coefficients of the homogeneous quadrics for  $\mathcal{Z}_P$ . The corresponding matrix for  $\mathcal{Z}'_P$  is therefore

$$\Gamma^{*'} = \begin{pmatrix} \mathbf{g}_1 & \dots & \mathbf{g}_m & 0 \\ 1 & \dots & 1 & -1 \end{pmatrix}.$$

Its height is  $m - n$  and therefore even, so that we may replace it by an  $s \times (m + 1)$  complex matrix  $\Omega = (\omega_1 \dots \omega_{m+1})$  where  $m - n = 2s$ , and define

$$(IV.4) \quad N' := \{z \in \mathbb{C}P^m : \omega_1 |z_1|^2 + \dots + \omega_{m+1} |z_{m+1}|^2 = \mathbf{0}\}.$$

Then  $N'$  has a complex structure as an LVM-manifold by Theorem IV.1. On the other hand,

$$N' \cong \mathcal{Z}'_P / S^1 = (\mathcal{Z}_P \times S^1) / S^1 \cong \mathcal{Z}_P,$$

and  $\mathcal{Z}_P$  also has a complex structure.

(b) The proof here is similar, but we have to add two redundant inequalities to (I.1). Then  $\mathcal{Z}'_P \cong \mathcal{Z}_P \times S^1 \times S^1$  is given by

$$\left\{ \begin{array}{l} z \in \mathbb{C}^{m+2}: \quad \mathbf{g}_1 |z_1|^2 + \dots + \mathbf{g}_m |z_m|^2 = \mathbf{0}, \\ |z_1|^2 + \dots + |z_m|^2 - |z_{m+1}|^2 = 0, \\ |z_1|^2 + \dots + |z_m|^2 - |z_{m+2}|^2 = 0, \\ |z_1|^2 + \dots + |z_m|^2 + |z_{m+1}|^2 + |z_{m+2}|^2 = 1. \end{array} \right\}$$

The matrix of coefficients of the homogeneous quadrics is therefore

$$\Gamma^{*'} = \begin{pmatrix} \mathbf{g}_1 & \dots & \mathbf{g}_m & 0 & 0 \\ 1 & \dots & 1 & -1 & 0 \\ 1 & \dots & 1 & 0 & -1 \end{pmatrix}.$$

We replace it by an  $s \times (m+2)$  complex matrix  $\Omega = (\omega_1 \dots \omega_{m+2})$  where  $m - n + 1 = 2s$ , and define

$$(IV.5) \quad N' := \{z \in \mathbb{C}P^{m+1} : \omega_1 |z_1|^2 + \dots + \omega_{m+2} |z_{m+2}|^2 = \mathbf{0}\}.$$

Then  $N'$  has a complex structure as an LVM-manifold by Theorem IV.1. On the other hand,

$$N' \cong \mathcal{Z}'_P / S^1 = (\mathcal{Z}_P \times S^1 \times S^1) / S^1 \cong \mathcal{Z}_P \times S^1,$$

and  $\mathcal{Z}_P \times S^1$  also has a complex structure.  $\square$

We demonstrate this construction on the two classical examples of non Kähler complex manifolds.

**Example IV.4** (Hopf manifold). An example of a non Kähler compact complex manifold is provided by the quotient of  $\mathbb{C}^m \setminus \{\mathbf{0}\}$  by the action of  $\mathbb{Z}$  generated by the coordinatewise multiplication by a complex number  $\tau$  such that  $|\tau| \neq 1$ . The complex manifolds obtained in this way are known as the *Hopf manifolds*; they are all diffeomorphic to  $S^{2m-1} \times S^1$ . If  $m = 1$  then the Hopf manifold is a complex torus; otherwise it is not Kähler as its second cohomology group is zero. The Hopf manifolds may be obtained as particular cases of moment-angle manifolds with the complex structures described above.

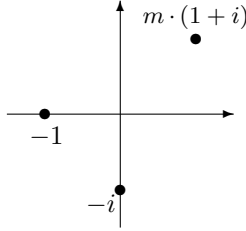
Let  $P$  be an  $n$ -simplex, so that  $m = n + 1$ ,  $\mathcal{Z}_P \cong S^{2n+1}$  is given by a single equation  $|z_1|^2 + \dots + |z_m|^2 = 1$  in  $\mathbb{C}^m$ , and  $\Gamma^*$  is empty. Since  $m + n$  is odd, we need to consider  $\mathcal{Z}'_P \cong \mathcal{Z}_P \times S^1 \times S^1$  given by

$$\left\{ \begin{array}{l} z \in \mathbb{C}^{m+2}: \quad |z_1|^2 + \dots + |z_m|^2 - |z_{m+1}|^2 = 0, \\ |z_1|^2 + \dots + |z_m|^2 - |z_{m+2}|^2 = 0, \\ |z_1|^2 + \dots + |z_m|^2 + |z_{m+1}|^2 + |z_{m+2}|^2 = 1, \end{array} \right\}$$

Then we replace the  $2 \times m + 2$  matrix

$$\Gamma^{*'} = \begin{pmatrix} 1 & 1 & \dots & 1 & -1 & 0 \\ 1 & 1 & \dots & 1 & 0 & -1 \end{pmatrix}.$$

by the  $1 \times (m + 2)$  complex matrix  $\Omega = (\omega_1 \dots \omega_{m+2})$  where  $\omega_k = 1 + i$  for  $1 \leq k \leq m$ ,  $\omega_{m+1} = -1$ , and  $\omega_{m+2} = -i$ . The corresponding configuration of points in  $\mathbb{C}$  is shown on the figure; note that it satisfies conditions (i) and (ii) of Section I.2. Then the manifold  $N'$  defined by (IV.5) acquires a structure of a complex manifold of dimension  $m$ , and we have  $N' \cong \mathcal{Z}_P \times S^1$ . We therefore obtain a complex structure on  $S^{2n+1} \times S^1$ ; this complex structure may be shown to be equivalent to that of a Hopf manifold.



**Example IV.5** (Calabi–Eckmann manifold). Another example of non Kähler complex manifold is due to Calabi–Eckmann. It is obtained by endowing the fibre  $S^1 \times S^1$  of the product of two Hopf bundles  $S^{2p+1} \times S^{2q+1} \rightarrow \mathbb{C}P^p \times \mathbb{C}P^q$  with a structure of a complex torus; therefore the total space  $S^{2p+1} \times S^{2q+1}$  also acquires a complex structure. Like in the case of Hopf manifolds, these complex structures are particular cases of the complex structures on  $\mathcal{Z}_P$  described in Theorem IV.3.

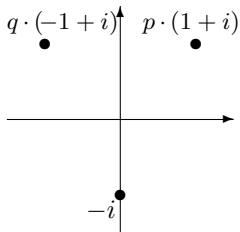
Let  $P = \Delta^{p-1} \times \Delta^{q-1}$  be a product of two simplices ( $p > 1$  and  $q > 1$ ), so that  $m = p + q = n + 2$ , and  $\mathcal{Z}_P \cong S^{2p-1} \times S^{2q-1}$  is given by the equations of Example I.8. Since  $m + n$  even, we need to consider  $\mathcal{Z}'_P \cong \mathcal{Z}_P \times S^1$  given by

$$\left\{ \begin{array}{l} \mathbf{z} \in \mathbb{C}^{m+1}: \quad |z_1|^2 + \dots + |z_p|^2 - |z_{p+1}|^2 - \dots - |z_m|^2 = 0, \\ |z_1|^2 + \dots + |z_m|^2 - |z_{m+1}|^2 = 0, \\ |z_1|^2 + \dots + |z_m|^2 + |z_{m+1}|^2 = 1. \end{array} \right.$$

Then we replace the  $2 \times (m + 1)$  matrix

$$\Gamma^{*'} = \begin{pmatrix} 1 & \dots & 1 & -1 & \dots & -1 & 0 \\ 1 & \dots & 1 & 1 & \dots & 1 & -1 \end{pmatrix}.$$

by the  $1 \times (m + 1)$  complex matrix  $\Omega = (\omega_1 \dots \omega_{m+1})$  where  $\omega_k = 1 + i$  for  $1 \leq k \leq p$ ,  $\omega_k = -1 + i$  for  $p + 1 \leq k \leq m$ , and  $\omega_{m+1} = -i$ . The corresponding configuration of points in  $\mathbb{C}$  is shown on the figure; it also satisfies conditions (i) and (ii) of Section I.2. Then the manifold  $N'$  defined by (IV.4) acquires a structure of a complex manifold of dimension  $m - 1$ , and we have  $N' \cong \mathcal{Z}_P$ . We therefore obtain a complex structure on  $S^{2p-1} \times S^{2q-1}$ ; this complex structure may be shown to be equivalent to that of a Calabi–Eckmann manifold.



*Remark.* There is also a more direct method of giving  $\mathcal{Z}_P$  a complex structure, without referring to projectivised quadrics and LVM manifolds, see [PU]. It identifies  $\mathcal{Z}_P$  with the quotient of  $U(\mathcal{K}_P)$  by a holomorphic action of  $\mathbb{C}^\ell$ , and may be generalised to some non polytopal moment-angle manifolds  $\mathcal{Z}_\mathcal{K}$  (namely those for which  $\mathcal{K}$  is the underlying complex of a complete simplicial fan).



## IV.3. DOLBEAULT COHOMOLOGY AND HODGE NUMBERS

Here we use the construction [MV] of holomorphic principal bundles over projective toric varieties and a spectral sequence due to Borel [Bo] to describe the Dolbeault cohomology of  $\mathcal{Z}_P$  in the case when  $P$  is a Delzant polytope.

Given a complex  $n$ -dimensional manifold  $M$ , there is a decomposition  $\Omega_{\mathbb{C}}^*(M) = \bigoplus \Omega^{p,q}(M)$  of the space of complex differential forms on  $M$  into a direct sum of the subspaces of  $(p,q)$ -forms for  $0 \leq p, q \leq n$ , and the Dolbeault differential  $\bar{\partial}: \Omega^{p,q}(M) \rightarrow \Omega^{p,q+1}(M)$ . The dimensions  $h^{p,q}$  of the Dolbeault cohomology groups  $H_{\bar{\partial}}^{p,q}(M)$  are known as the *Hodge numbers* of  $M$ . They are important invariants of the complex structure of  $M$ .

Assume now that  $P$  given by (I.1) is a Delzant polytope (see Section II.4). Then there is a principal  $T^{m-n}$ -bundle  $\mathcal{Z}_P \rightarrow X_P$  (see Lemma II.13), where  $X_P$  is the nonsingular projective toric variety corresponding to  $P$ .

Assume now that  $m - n$  is even (otherwise we add one redundant inequality to (I.1)), and let  $m - n = 2\ell$ . A construction of [MV] defines a structure of a holomorphic principal bundle on  $\mathcal{Z}_P \rightarrow X_P$ , with fibre a compact complex  $\ell$ -dimensional torus  $T_{\mathbb{C}}^{\ell}$ . A spectral sequence of Borel [Bo] enables us to calculate the Dolbeault cohomology of the total space of a holomorphic bundle with a Kählerian fibre in terms of the Dolbeault cohomology of the fibre and base. In the case of the bundle  $\mathcal{Z}_P \rightarrow X_P$  the Dolbeault cohomology of the fibre and base are well-known and easily described.

The Dolbeault cohomology of  $T_{\mathbb{C}}^{\ell}$  is isomorphic to an exterior algebra on  $2\ell$  generators:

$$(IV.6) \quad H_{\bar{\partial}}^{*,*}(T_{\mathbb{C}}^{\ell}) \cong \Lambda[\xi_1, \dots, \xi_{\ell}, \eta_1, \dots, \eta_{\ell}],$$

where  $\xi_i \in H_{\bar{\partial}}^{1,0}(T_{\mathbb{C}}^{\ell})$  are classes of holomorphic 1-forms, and  $\eta_i \in H_{\bar{\partial}}^{0,1}(T_{\mathbb{C}}^{\ell})$  are classes of antiholomorphic 1-forms, for  $1 \leq i \leq \ell$ . In particular, the Hodge numbers are given by  $h^{p,q}(T_{\mathbb{C}}^{\ell}) = \binom{\ell}{p} \binom{\ell}{q}$ .

The Dolbeault cohomology of a nonsingular projective toric variety  $X_P$  is given by a result of Danilov–Jurkiewicz [Da]:

$$(IV.7) \quad H_{\bar{\partial}}^{*,*}(X_P) \cong \mathbb{C}[v_1, \dots, v_m] / (\mathcal{I}_{\mathcal{K}_P} + \mathcal{J}_{\Sigma_P}),$$

where  $v_i \in H_{\bar{\partial}}^{1,1}(X_P)$  for  $1 \leq i \leq m$ ,  $\mathcal{I}_{\mathcal{K}_P} = (v^I: I \notin \mathcal{K}_P)$  is the Stanley–Reisner ideal (see Definition III.1), and  $\mathcal{J}_{\Sigma_P}$  is generated by the linear combinations  $\sum_{k=1}^m a_{kj} v_k$  for  $1 \leq j \leq n$ , where  $a_{kj}$  is the  $j$ th coordinate of  $\mathbf{a}_k$ . We have  $h^{p,p}(X_P) = h_p(P)$ , where  $(h_0(P), h_1(P), \dots, h_n(P))$  is the  $h$ -vector of  $P$  [BP2, §1.2], and  $h^{p,q}(X_P) = 0$  for  $p \neq q$ . Since  $X_P$  is Kähler, its de Rham cohomology algebra (with coefficients in  $\mathbb{C}$ ) is obtained from the Dolbeault cohomology by passing to the total degree.

**Theorem IV.6** ([PU]). *Let  $P$  be a  $n$ -dimensional Delzant polytope defined by  $m$  inequalities (I.1) of which at most one is redundant, and  $m - n = 2\ell$ . Let  $\mathcal{Z}_P$  be the moment-angle manifold with a complex structure of an LVM-manifold. Then the Dolbeault cohomology group  $H_{\bar{\partial}}^{p,q}(\mathcal{Z}_P)$  is isomorphic to the  $(p,q)$ -th cohomology group of the differential bigraded algebra*

$$[\Lambda[\xi_1, \dots, \xi_{\ell}, \eta_1, \dots, \eta_{\ell}] \otimes H_{\bar{\partial}}^{*,*}(X_P), d]$$

whose bigrading is defined by (IV.6) and (IV.7), and differential  $d$  of bidegree  $(0, 1)$  is defined on the generators as

$$dv_i = d\eta_j = 0, \quad d\xi_j = w_j, \quad 1 \leq i \leq m, \quad 1 \leq j \leq \ell,$$

where the  $w_j$  are certain linearly independent elements in  $H_{\bar{\partial}}^{1,1}(X_P)$  whose explicit form depends on the complex structure of  $Z_P$ .

**Corollary IV.7.** *Let  $Z_P$  be as in Theorem IV.6, and let  $k \leq 1$  be the number of redundant inequalities in (I.1). Then*

- (a)  $h^{p,0}(Z_P) = 0$  for  $p > 0$ ;
- (b)  $h^{0,q}(Z_P) = \binom{\ell}{q}$  for  $q \geq 0$ ;
- (c)  $h^{1,q}(Z_P) = (\ell - k) \binom{\ell}{q-1}$  for  $q \geq 1$ .

*Remark.* Note that  $h^{1,0}(Z_P) < h^{0,1}(Z_P)$ , which implies that  $Z_P$  is not Kählerian.

**Example IV.8.** Let  $Z_P \cong S^1 \times S^{2n+1}$  be the Hopf manifold of Example IV.4. The corresponding fan is the normal fan  $\Sigma_P$  of the standard  $n$ -dimensional simplex  $P$  with one redundant inequality. We have  $X_P = \mathbb{C}P^n$ , and (IV.7) describes its cohomology as the quotient of  $\mathbb{C}[v_1, \dots, v_{n+2}]$  by the two ideals:  $\mathcal{I}$  generated by  $v_1$  and  $v_2 \cdots v_{n+2}$ , and  $\mathcal{J}$  generated by  $v_2 - v_{n+2}, \dots, v_{n+1} - v_{n+2}$ . The differential algebra of Theorem IV.6 is therefore given by  $[\Lambda[\xi, \eta] \otimes \mathbb{C}[t]/t^{n+1}, d]$ , and  $dt = d\eta = 0$ ,  $d\xi = \alpha t$  for some  $\alpha \neq 0$ . The nontrivial cohomology classes are represented by the cocycles  $1, \eta, \xi t^n$  and  $\xi \eta t^n$ , which gives the following nonzero Hodge numbers of  $Z_P$ :  $h^{0,0} = h^{0,1} = h^{n+1,n} = h^{n+1,n+1} = 1$ .

It is also interesting to compare Theorem IV.6 with the following description of the ordinary cohomology of  $Z_P$ .

**Theorem IV.9** ([BP2, Th. 7.36]). *The cohomology algebra of  $Z_P$  (with coefficients in a field) is isomorphic to the cohomology of the differential graded algebra*

$$[\Lambda[u_1, \dots, u_{m-n}] \otimes H^*(X_P), d],$$

where  $\deg u_i = 1$ ,  $\deg v_i = 2$ , and differential  $d$  is defined on the generators as

$$dv_j = 0, \quad du_j = \gamma_{j1}v_1 + \dots + \gamma_{jm}v_m, \quad 1 \leq j \leq m,$$

where  $\Gamma = (\gamma_{jk})$  is given by (I.4).

Theorem IV.9 may be deduced from the general Theorem III.3 using homological methods. In the case when  $P$  is Delzant the above theorem, like Theorem IV.6, is the collapse result for a spectral sequence (this time the Leray–Serre spectral sequence of the principal  $T^{m-n}$ -bundle  $Z_P \rightarrow X_P$ ).

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