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SURFACE REPRESENTATION AND BRIESKORN MANIFOLDS

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Introduction and motivation to contribution

Let $M(p,q,r)$ be the intersection of the unit sphere $S^5 = \{(z_1,z_2,z_3) \in \mathbb{C}^3 \mid |\, z_1 \,|^2 + |\, z_2 \,|^2 + |\, z_3 \,|^2 = 1\}$ with the hypersurface (complex variety of Pham and Brieskorn)

$$z_1^p + z_1^q + z_1^r = 0,$$

where p,q,r are integers ≥ 2. $M(p,q,r)$ is known as the Brieskorn manifold. It is a smooth, compact 3-manifold and a typical example of a Seifert manifold. In [5] we showed that $M(p,q,r)$ admits a spherical CR structure. A non- degenerate CR- structure on a smooth manifold is a contact subbundle of codimension one together with a complex structure on it. A CR manifold is a smooth odd dimensional manifold that admits a nondegenerate CR-structure. In particular, a nondegenerate CR- structure on a $(2n+1)$-manifold M is spherical if and only if M is locally isomorphic to S^{2n+1}. The sphere S^{2n+1} is a principal fibre bundle over the complex n-dimensional projective space. The complex projective space admits a homogeneous Kähler metric. Since a Kähler manifold is a symplectic manifold it is therefore interesting to study CR manifolds and in particular spherical CR manifolds.

More precisely, a spherical CR structure on a smooth $(2n+1)$- manifold is a maximal collection of charts modelled on the standard $(2n+1)$-sphere whose coordinate changes lie in the group $Aut_{CR}(S^{2n+1})$ of CR- automorphisms of S^{2n+1}.

In this paper, we show that Brieskorn manifolds admit a spherical CR structure which belong to the same connected component as the $\mathbb{R}$ and $\mathbb{C}$-Fuchsian representations.

Sections 2 and 3 are about known results by Goldman-Kapovich-Leeb [2] and Milnor [9] respectively. The last section is devoted to our main results.

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1. Goldman-Kapovich-Leeb's result

1.1. **Complex hyperbolic surface.** Let Σ denote a closed oriented (Riemann) surface of genus $g \geq 2$ and let $\rho : \pi_1(\Sigma) \to \mathrm{PU}(n+1,1)$ be a representation of its fundamental group in the group of biholomorphic isometries of complex hyperbolic $(n+1)$-space. There is an invariant associated with ρ which is defined as follows: Let $\tilde{f} : \tilde{\Sigma} \to \mathbb{H}^{n+1}_{\mathbb{C}}$ be a ρ-equvariant smooth mapping of the universal covering of Σ. Then Toledo's invariant $\tau(\rho; \tilde{f})$ is defined as the (normalized) integral of the pull-back of the Kähler form ω on $\mathbb{H}^{n+1}_{\mathbb{C}}$.

$$\tau(\rho : \tilde{f}) := \frac{1}{2\pi} \int_{\Sigma} \tilde{f}^* \omega$$

1.2. $\mathbb{R}$ **and** $\mathbb{C}$ **-Fuchsian representations.** There are two kinds of totally geodesic submanifolds in $\mathbb{H}^2_{\mathbb{C}}$ of real dimension two:

- Complex geodesic (copies of $\mathbb{H}^1_{\mathbb{C}}$) which is the intersection of a complex line with
  $H^2_{\mathbb{C}}$. It carries the Poincare model of the hyperbolic plane which gives a natural inclusion $Isom(\mathbb{H}^1_{\mathbb{C}}) \subset Isom(\mathbb{H}^2_{\mathbb{C}})$.
- Totally real geodesic 2-planes (copies of $\mathbb{H}^2_{\mathbb{R}}$) which is the intersection of the R^2-plane with $\mathbb{H}^2_{\mathbb{C}}$. It carries the Klein-Beltrami model of the hyperbolic plane which gives a natural inclusion $Isom(\mathbb{H}^2_{\mathbb{R}}) \subset Isom(\mathbb{H}^2_{\mathbb{C}})$.

A representation of $\pi_1(\Sigma)$ into the isometry group of the hyperbolic plane is called Fuchsian if it is discrete and faithful.

A representation of $\pi_1(\Sigma)$ into $Isom(\mathbb{H}^2_{\mathbb{C}})$ restricted to a Fuchsian representation such that $\mid \tau(\rho) \mid = 2g-2$ is called $\mathbb{C}$-Fuchsian . Similarly, a representation of $\pi_1(\Sigma)$ into $Iso(\mathbb{H}^2_{\mathbb{C}})$ restricted to a Fuchsian representation such that $\tau(\rho) = 0$ is called $\mathbb{R}$-Fuchsian .

The following are the results of Goldman-Kapovich-Leeb:

Theorem 1. *For every genus $g \geq 2$ and every even integer τ satisfying*

$$2 - 2g \leq \tau \leq 2g - 2 \tag{1.1}$$

there exists a convex-cocompact discrete and faithful representation $\rho : \pi_1(\Sigma) = \pi \longrightarrow \mathrm{PU}(2,1)$, $(\rho(\pi) = \Gamma)$ with $\tau(\rho) = \tau$. Furthermore, the complex hyperbolic surface $M = \mathbb{H}^2_{\mathbb{C}}/\Gamma$ is diffeomorphic to the total space of an oriented $\mathbb{R}^2$-bundle ξ over Σ with the Euler number

$$e(\xi) = \chi(\Sigma) + \mid \tau(\rho)/2 \mid \tag{1.2}$$

1.3. **The boundary.** If we take the convex core $C(\Gamma)$ of $\mathbb{H}^2_{\mathbb{C}}/\Gamma$ by pushing forward the boundary $\partial C(\Gamma)$ and taking the quotient we obtain $S^3 - L(\Gamma)/\Gamma$.

Corollary 1. *The manifold $S^3 - L(\Gamma)/\Gamma$ is diffeomorphic to the total space of an S^1- bundle over the surface Σ which has the same Euler number as the $\mathbb{R}^2$ fibration of $M = \mathbb{H}^2_{\mathbb{C}}/\Gamma$.*

This implies the S^1 bundle admits a spherical CR structure by definition.

2. Milnor's result

2.1. **Milnor's classification of** $M(p,q,r)$**.** Put $\kappa = p^{-1}+q^{-1}+r^{-1}-1$, Milnor [9] has shown that $M(p,q,r)$ is diffeomorphic to a coset space of the form $\Pi\backslash G$ where G is a simply connected 3-dimensional Lie group and Π is a discrete subgroup. The classification is as follows:

(1) $\kappa > 0$. $G = \mathrm{SU}(2)$ and Π is a finite subgroup.
(2) $\kappa = 0$. $G = \mathcal{N}$, the Heisenberg Lie group and Π is a discrete uniform subgroup.
(3) $\kappa < 0$. $G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$, the universal covering of $\mathrm{PSL}(2,\mathbb{R})$ and Π is a cocompact subgroup.

As mentioned in the introduction, we have shown in [5] that every $M(p,q,r)$ according to the above classification admits a spherical CR structure.

We would like to list the possibilities of $\kappa = p^{-1}+q^{-1}+r^{-1}-1$. Note that (p,q,r) is listed such that $p \le q \le r$.

- When $\kappa > 0$, (p,q,r) must be one of the triples
$$(2,3,3),(2,3,4),(2,3,5) \text{ or } (2,2,r)$$
for some $r \ge 2$.
- When $\kappa = 0$, the triple (p,q,r) must be either
$$(2,3,6),(2,4,4) \text{ or } (3,3,3).$$
- In the last case when $\kappa < 0$ we have the infinitely remaining triples.

We are interested in the case when $\kappa < 0$. The theorem below is another result by Milnor.

Theorem 2. *If the least common multiples of (p,q),(p,r) and of (q,r) are all equal*

$$m = l.c.m(p,q) = l.c.m(p,r) = l.c.m(q,r), \tag{2.1}$$

then the Brieskorn manifold $M(p,q,r)$ fibers as a smooth circle bundle with chern number $-pqr/m^2$ over a Riemann surface of Euler charactersitic $pqr(p^{-1}+q^{-1}+r^{-1}-1)/m$.

$M(p,q,r)$ is a principal circle bundle over a Riemann surface. The free action is defined as follows for $t \in S^1$:

$$t(z_1,z_2,z_3) = (t^{\frac{m}{p}} z_1, t^{\frac{m}{q}} z_2, t^{\frac{m}{r}} z_3)$$

3. Main results

From the results of Goldman-Kapovich-Leeb and Milnor we have two S^1 bundles. Two S^1 bundles over the same surface are equivalent if and only if their Euler numbers are equal. Hence we would like to show that $e(M(p,q,r)) = e(S^3 - L(\Gamma)/\Gamma)$ if we take same surface Σ.

Lemma 1. *$M(p,q,r)$ is diffeomorphic to $S^3 - L(\Gamma)/\Gamma$*

Proof. First, we substitute $e(M(p,q,r))$ and $\chi(\Sigma)$ into the equation 1.2. We have

$$-pqr/m^2 = pqr(p^{-1}+q^{-1}+r^{-1}-1)/m + \mid \tau(\rho)/2 \mid$$

The next step is to find a Toledo invariant τ satisfying the above equation.

To find such a τ we have to substitute the possible values of the triples (p,q,r) satisfying the condition 2.1 to calculate $\mid \tau(\rho)/2 \mid$ satisfying the condition that for every $g \geq 2$

$$2-2g \leq \tau \leq 2g-2$$

When $p=q=r=m$, such that the difference $\mid \tau(\rho)/2 \mid= e(M(p,q,r)) - \chi(\Sigma)$ is m^2-4m and $\mid \tau \mid= 2m^2-8m$, only $p,q,r=4$ satisfy the conditions 2.1 and 1.1 with $\mid \tau \mid= 0$ and $g=3$. □

We recall that in the case when $\kappa < 0$, the triples (p,q,r) are infinitely many. For this reason, we would like to consider other possiblities for example when $p \neq q \neq r$ also $p \neq q = r$. Although there are infinitely many triples that satisy the condition 2.1 we know of only a few of them that satisfy the condition 1.1.

Theorem 3. *When $p \neq q \neq r$ and $p \neq q = r$,the following triples (p,q,r) satisfy the condition 2.1 such that τ is an even integer with $g \geq 2$ satisfying the condition 1.1.*

	(p,q,r)		(p,q,r)		(p,q,r)	
	$\pm\tau$	g	$\pm\tau$	g	$\pm\tau$	g
p = 2	(2,5,10)		(2,6,6)		(2,8,8)	
	2	2	0	2	4	3
p= 3	(3,6,6)					
	6	4				

We recall that in [5] we showed that when $\kappa < 0$, $M(p,q,r)$ admits spherical CR structure such that $L(\Gamma)$ is a geometric circle S^1 and Γ is not necessarily discrete. Xia [10] has shown that when Σ is compact the Toledo invariant distinguishes different connected components of the space of all representations(not necessarily discrete and faithful)of $\pi_1(\Sigma)$. We can therefore conclude from the values of the Toledo invariant τ that for $M(4,4,4)$ and $M(2,6,6)$ our representation belongs to the same connected component as the $\mathbb{R}$-Fuchsian but are not conjugate. Similarly, for $M(2,5,10), M(2,8,8)$ and $M(3,6,6)$ our representation belongs to the same connected component as the $\mathbb{C}$-Fuchsian but are not conjugate.

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