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GLOBAL ATTRACTORS FOR NONLINEAR WAVE EQUATIONS WITH NONLINEAR DISSIPATIVE TERMS

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Abstract. We show the existence, size and some absorbing properties of global attractors of the nonlinear wave equations with nonlinear dissipations like $\rho(x, u_t) = a(x)|u_t|^r u_t$.

1. Introduction

In this paper we are concerned with global attractors for the nonlinear wave equations with nonlinear dissipative term:

$$u_{tt} - \Delta u + \rho(x, u_t) + g(x, u) = f(x) \quad \text{in} \quad \Omega \times R^+, \tag{1.1}$$

$$u(x, 0) - u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{and} \quad u(x, t)|_{\partial\Omega} = 0, \tag{1.2}$$

where $\rho(x, v)$ is a dissipation like $a(x)|v|^r v$ and $g(x, u)$ is a sourcing term like $|u|^\alpha u - |u|^\beta u, \alpha > \beta \geq 0$.

When $\rho(x, v) = v$, a linear dissipation, the existence of global attractor is well discussed and it is a standard result that if $0 < \alpha < 2/(N-2)^+$, the problem admits a global attractor in the energy space $H_1^0(\Omega) \times L^2(\Omega)$. The proof is based on the exponential decay of energy of solutions for the case $g(x, u) = f(x) = 0$ and the compactness considered by Bavin and Vishik [2], Arrieta, Carvaho and Hale [1], Feireisl [6] etc. Finite dimensionality of global attractor is proved in Eden, Milani and Nickolenco [5]. See Ball [3] where the existence of global attractor is proved without uniueness assumption and many references are cited.

Some standard results are generalized by several authors to the nonlinear dissipative cast $\rho(v)$ with $0 < \epsilon_0 \leq \rho'(v) < k_1 < \infty$. These are also based, at least in spirit, on the exponential decay for the equation with $g(x, u) = f(x) = 0$. For such nonlinearity finite dimensionality of the attractors is also investigated. See Raugel [16], Lasiecka and Ruzmaikina [11]. See also Chueshow, Eller and Lasiecka [4], where nonlinear boundary dissipation is considered.

However, when the dissipation has a stronger nonlinearity such as $\rho'(0) = 0$ or $\rho'(0) = \infty$ there seems to be few results. The object of this paper is to show the existence of global attractor in $H_1^0(\Omega) \times L^2(\Omega)$ for the essentially nonlinear case like $\rho(x, v) = a(x)|v|^r v, \ \ r \neq 0$, and further give some characterizations of it. First we consider the case where the dissipation is effective in the whole domain Ω and

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next treat a more delicate case where the dissipation is effective possibly near some part of the boundary $\partial\Omega$. We call the latter case as nonlinear localized dissipation. When the dissipative mechanism does not follow the Hooke's law the most general dissipation would be of the form $\rho(x,u_t)$ where $\rho(x,v)$ is a monotone increasing function in v. So, it seems an interesting problem and also reasonalble to consider the nonlinear dissipative term like $a(x)|u_t|^r u_t$ as a typical model.

Lasiecka and Ruzmaikina [11] proved the existence of global attractor under the assumption that $\rho(v)$ is strictly increasing and $\rho'(v) \geq m_0 > 0$ for $|v| \geq 1$, which is applied to the case $\rho(v) = |v|^r v,\ \ 0 \leq r \leq 4/(N-2)^+$. But, our result includes the case $-1 < r < 0$ and gives preciser informations on the size and the absorbing rate. Feireisl and Zuazua [7] treated the equation with nonlinear localized dissipation $\rho(x,v) = a(x)\rho_0(v)$ and the terms $g(x,u) = g(u),\ \ f(x) = 0$ and proved the existence of global attractor under the assumption that $\rho_0(v)$ is strictly increasing in v and $0 < m_0 \leq \rho'(v) \leq m_1 < \infty$ for $|v| >> 1$. But, no estimate on the size and the absorbing rate is given there.

2. Notaions and main results

Let us state precise assumptions on the terms $\rho(x,u),\ \ g(x,u)$ and $f(x)$. We first assume the following.

Hyp.A. $rho(x,v)$ is measurable in $x \in \Omega$ for any $v \in R$ and differentiable in $v \neq 0$ for a.e. $x \in \Omega$, and satisfies

$$k_0|v|^{r+2} \leq \rho(x,v)v \leq k_1|v|^{r+2} \ \text{ if } \ |v| \leq 1, \tag{2.3}$$

$$\begin{aligned} &k_0|v_1 - v_2|^{r^+ + 2} \leq (\rho(x,v_1) - \rho(x,v_2))(v_1 - v_2) \\ &\leq k_1(|v_1 - v_2|^2 + |v_1 - v_2|^{r+2}) \ \text{ if } \ |v_1|, |v_2| \leq 1 \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &k_0|v_1 - v_2|^{p+2} \leq (\rho(x,v_1) - \rho(x,v_2))(v_1 - v_2) \\ &\leq k_1(|v_1|^p + |v_2|^2)|v_1 - v_2|^2 \ \text{ if max } \ \{|v_1|, |v_2|\} \geq 1, \end{aligned} \tag{2.5}$$

where $k_0, k_1 > 0,\ \ -1 < r < \infty$ and $0 \leq p \leq \frac{4}{(N-2)^+}$.

Hyp.B. $g(x,u)$ is measurable in $x \in \Omega$ for all $v \in R$ and continuous in $v \in R$ for a.e. $x \in \Omega$, satisfying:

$$g(x,0) = 0,\ \ g(x,u)u + \bar{L} \geq \mu(\int_0^u g(x,\eta)d\eta + L) \geq 0 \tag{2.6}$$

for some $\mu > 0$ and $L,\ \ \bar{L} \geq 0$ and

$$|g(x,u_1) - g(x,u_2)| \leq k_1(1 + |u_1|^\alpha + |u_2|^\alpha)|u_1 - u_2| \ \text{ for } \ u_1, u_2 \in R, \tag{2.7}$$

with some $k_0, k_1 > 0$ and $0 \leq \alpha < 2/(N-2)^+$.

We set

$$G(x,u) = \int_0^u g(x,\eta)d\eta.$$

Hyp.C. $f \in L^2(\Omega)$.

It is well known that under Hyp.A, Hyp.B and Hyp.C the problem (1.1)-(1.2) admits a unique solution $u \in C([0,\infty); H_1^0(\Omega)) \cap C([0,\infty); L^2(\Omega))$ for each $(u_0, u_1) \in H_1^0(\Omega) \times L^2(\Omega)$. We denote the solution $u(t)$ by $U(t)(u_0, u_1)$. Since our system is autonomus $U(t)$ is a continuous group as operator in $H_1^0 \times L^2$. Our result is real as follows.

Theorem1.1. Under Hyp.A, Hyp.B and Hyp.C the problem has a global attractor in the space $H_1^0(\Omega) \times L^2(\Omega)$. Further A is include in a ball $B(\bar{R})$ in $H_1^0(\Omega) \times L^2(\Omega)$ centered at 0 with the radius $\bar{R} = C(M_0^2 + L + \bar{L})$ and for any bounded set $B_0 \subset H_1^0(\Omega) \times L^2(\Omega)$ we have the absorbing property

$$dist(U(t)B_0, B(\bar{R})) \leq C(B_0)(1+t)^{-1/2\gamma} \tag{2.8}$$

where γ is defined by

$$\gamma = -r/2(r+1) \quad \text{if} \quad -1 < r \leq 0,$$
$$r/2 \quad \text{if} \quad r \geq 0.$$

We note that when $f = 0$ and $L = \bar{L} = 0$, the ball $B(\bar{R})$ is reduced to 0 and the estimate gives a well-known decay rate of solutions. For the proof of Theorem 1.1 we use an idea in our earlier paper [13] where the same algebraic decay or stability of the bounded solution in proved for the case $g(x,u) = 0$. In Lasiecka and Ruzmaikina [11] the existence of global attractor is proved for a similar problem, but, the size of the absorbing set nor the absorbing rate as (1.8) is not derived there.

Secondly, we consider a more delicate case where $\rho(x,v)$ is nonlinear and possibly vanishes on some large area in Ω. To state our assumption on the dissipation $\rho(x,v)$ precisely, we define a set of the boundary $\Gamma(x_0)$ introduced by Russell [17]:

$$\Gamma(x_0) = \{x \in \partial\Omega | (x - x_0) \cdot \nu(x) > 0\},$$

where $x_0 \in R^N$ and $\nu(x)$ is the outward normal vector at $x \in \partial\Omega$. Let $x(x)$ be a nonnegative bounded function on Ω satisfying:

There exist $x_0 \in R^N$ and a relatively open set $w \subset \bar{\Omega}$ such that

$$\bar{\Gamma(x_0)} \subset w \quad \text{and} \quad a(x) \geq \epsilon_0 > 0 \quad \text{for} \quad x \in w.$$

Theorem1.2. Under Hyp.A, Hyp.B and Hyp.C, the problem has a global attractor A in the space $H_0^1(\Omega) \times L^2(\Omega)$. Further, under Hyp.B, (1), A is included in a ball $B(\bar{r})$ centered at 0 with a radius $\bar{R} = \bar{R}(M_0, K_0, d_0, d_1) > 0$ and it holds that

$$dist(U(t)B_0, B(\bar{R})) \leq C(B_0)e^{-\lambda t},$$

where λ depends on B_0 except for special cases.

Under Hyp.B, (2), A is included in a ball $B(\bar{R})$ centered at 0 with a radius $\bar{R}$ given

$$\bar{R}^2 = C(M_0^2 + K_0^2), C > 0$$

and it holds that

$$dist(U(t)B_0, B(\bar{R})) \leq C(B_0)(1+t)^{-1/2\gamma} \tag{2.9}$$

where γ is the same as in Theorem 1.1.

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