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A CLASSIFICATION OF RADIAL SELF-SIMILAR SOLUTIONS OF THE HEAT EQUATION

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Abstract. In this note we investigate some properties of solutions of ordinary differential equations

$$u'' + (\frac{r}{2} + \frac{N-1}{r})u' + \frac{1}{p-1}u + k|u|^{p-1}u = 0, \quad r > 0$$
$$u'(0) = 0$$
$$u(0) = \alpha.$$

Here $k = 1$ or -1.

We explain some recent results and what should be done sooner or later.

1. Introduction

During the past decades, the global existence and nonexistence and the asymptotic behavior of solutions of a heat equation

$$v_t = \Delta v + k|v|^{p-1}v \qquad \text{in} \qquad \mathbf{R}^N \times (0, \infty) \tag{1.1}$$

have been studied. Here $k = 1$ or -1. In the studies radial self-similar solutions play an important role and many parts have been devoted to a full investigation of such a special solution. Such a solution satisfies a simple ordinary differential equation

$$\begin{aligned} u'' + (\frac{r}{2} + \frac{N-1}{r})u' + \frac{1}{p-1}u + k|u|^{p-1}u = 0, \quad r > 0 \\ u'(0) = 0 \quad u(0) = \alpha. \end{aligned} \tag{1.2}$$

Pretty much is known but despite of its simple structure, many valuable information is not covered yet.

The main purpose of this note is to bring your attention to some open problems concerning to these equations which can be easily accessible. We need only a basic knowledge of differential equations and your good insight.

We here consider two types of heat equations one with source term ($k = 1$ case) and the other with absorption term ($k = -1$ case). In the first case the global existence and nonexistence and the blow-up phenomena are the main concern. In

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the second case, the asymptotic behaviors are the main issue. These are very different stories but eventually we are reduced to the studies of equation (2). A difference in sign yields a much difference in structure but many resemblance as we will see below.

Related to these problems, we may consider a degenerate parabolic equation and we obtain parallel results, see [4], [5].

2. The case $u_t = \Delta u + |u|^{p-1}u$

This section is concerned with equation

$$\Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u + |u|^{p-1}u = 0 \qquad \text{in} \qquad \mathbf{R}^N, \tag{2.1}$$

where $n \geq 1$ and $p > 1$. This equation was introduced by Haraux and Weissler [1] to study forward self-similar solutions of the heat equation

$$v_t = \Delta v + |v|^{p-1}v \qquad \text{in} \qquad \mathbf{R}^N \times (0, \infty) \tag{2.2}$$

Any solution of the form

$$v(x,t) = t^{-1/(p-1)}u(t^{-1/2}x) \tag{2.3}$$

is called a forward self-similar solution, which plays an important role for the study of structure of solutions to equation (2.2). Substituting (2.3) into (2.2), we see that u must satisfy equation (2.1).

In this note, we restrict our attention to radial solutions of (2.1), that is, a solution of the form $u = u(r)$, $r = |x|$. Then $u(r)$ must satisfy the initial value problem

$$\begin{aligned} u_{rr} + (\frac{r}{2} + \frac{N-1}{r})u_r + \frac{1}{p-1}u + |u|^{p-1}u = 0, \quad r > 0 \\ u'(0) = 0, \qquad u(0) = \alpha. \end{aligned} \tag{2.4}$$

The above initial value problem has a unique global solution $u = u(r; \alpha, p)$ in $C^2([0,\infty))$. Moreover for any $\alpha > 0$ and $p > 1$, the limit

$$L(\alpha) = \lim_{r\to\infty} r^{2/(p-1)}u(r; \alpha, p)$$

exists and is finite. If $L(\alpha) = 0$, the non-zero limit

$$A = \lim_{r\to\infty} exp(r^2/4)r^{n-2/(p-1)}u(r; \alpha, p) \tag{2.5}$$

exists, see [2] and references therein.

We say that a solution of (2.4) is a rapidly decaying solution if $L(\alpha) = 0$ and a slowly decaying solution otherwise.

The existence of rapidly decaying solutions was investigated by Weissler [6]. It was shown that if $(n-2)p < (n+2)$, then there exist infinitely many decaying solution, conversely, if $(n-2) \geq (n+2)$, then $u(r;\alpha)$ has no zero in $(0,\infty)$ and decays more slowly than (2.5) for every $\alpha > 0$.

In case $n = 1$, the uniqueness of a positive rapidly decaying solutions with prescribed numbers of zeros was proved by Weissler [6]. For any $n \geq 1$, the uniqueness of a positive rapidly decaying solution was investigated by Yanagida [7] under the condition $(n-2)p \leq n$. Thus the case $n < (n-2)p < n+2$ remains open. In the followings, we introduce some results of Yanagida [7] for your reference.

The equation (2.4) can be written as

$$\{\rho(r)u_r\}_r + \rho(r)\{\frac{1}{p-1}u + |u|^{p-1}u\} = 0. \tag{2.6}$$

Set

$$U(r) = \frac{\partial}{\partial \alpha}u(r;\alpha).$$

Differentiating (2.6) with α, we see that $U(r)$ satisfies

$$L[U(r)] = 0, \quad r > 0, \quad U(0) = 1, \quad U_r(0) = 0, \tag{2.7}$$

where L is a linear operator defined by

$$L[U] = \{\rho(r)U_r\}_r + \rho(r)\{\frac{1}{p-1} + p|u|^{p-1}\}U.$$

By (6), $u(r;\alpha)$ satisfies

$$L[u(r;\alpha)] = (p-1)\rho(r)|u(r;\alpha)|^{p-1}u(r;\alpha). \tag{2.8}$$

Set

$$V(r) = \frac{r}{2}u_r(r;\alpha) + \frac{1}{p-1}u(r;\alpha),$$

then $V(r)$ satisfies

$$L[V(r)] = -\rho(r)V(r). \tag{2.9}$$

By applying the Sturm comparison theorem, we see from (2.7), (2.8) and (2.9) that $U(r)$ oscillates faster than $u(r;\alpha)$ and more slowly than $V(r)$. In fact, let x_j, y_j and z_j denote j-th zeros of $U(r), V(r)$ and $u(r;\alpha)$ respectively, then the following results hold, see [7].

Lemma 1. *Let $(n-2)p \leq n$. Then $z_{j-1} < y_j < z_j$ and $z_{j-1} < x_j, z_j$ for every $j \geq 1$. Here we put $z_0 = 0$.*

The assumption $(n-2)p \leq n$ is required only to show the above Lemma, which implies

Proposition 2. *Let $(n-2)p \leq n$. Then the number of zeros of $u(r;\alpha)$ monotonivally increases as α increases.*

and

Lemma 3. *Let $(n-2)p \le n$. Then $U(r), V(r)$ has one and only one zero in (z_l, ∞), where z_l is the last zero of $u(r;\alpha)$.*

Lemma 3, in turn, implies that if α increases, then the $(l+1)$-th zero of $u(r;\alpha)$ appears from ∞.

Proposition 4. *Let $(n-2)p \le n$. If $\epsilon > 0$ is sufficiently small, then $u(r, \alpha+\epsilon)$ has at least $l+1$ zeros for $r > 0$.*

Hence we may conclude that

Theorem 5. *Let $(n-2)p \le n$. For each nonnegative integer i, there exists at most one $\alpha > 0$ such that $u(r;\alpha)$ becomes a rapidly decaying solution with exactly i zeros for $r > 0$.*

Based on the above results, the authors provide a complete information about the structure of solutions in the parameter space of n, p and α, see [2].

Theorem 6.
Set

$$p_k = 1 + \frac{2}{n+2k}, \qquad k = 0, 1, 2, \cdots,$$

$p_1 = \infty$ for $n = 1, 2$ and $p_{-1} = n/(n-2)$ for $n > 2$.

(i) For each k, there exists a C^1 function $\alpha = \alpha_k(p) > 0$ defined for $p \in (p_k, p_{-1}]$ such that $u(r; \alpha_k(p), p)$ is a rapidly decaying solution with k zeros in $(0, \infty)$.

(ii) For $p \in (p_k, p_{k-1}]$, the sequence $\{\alpha_i(p)\}$ satisfies

$$0 < \alpha_k(p) < \alpha_{k+1}(p) < \alpha_{k+2}(p) < \cdots \to \infty,$$

and $u(r;\alpha,p)$ is a slowly decaying solution with k zeros in $(0,\infty)$ for any $\alpha \in (0, \alpha_k(p))$, and is a slowly decaying solution with $i+1$ zeros in $(0,\infty)$ for any $\alpha \in (\alpha_i(p), \alpha_{i+1}(p))$, where $i = k, k+1, k+2, \cdots$.

3. The case $u_t = \Delta u - |u|^{p-1}u$

Let us consider the semilinear parabolic equation

$$v_t = \Delta v - |v|^{p-1}v \qquad \text{in} \qquad \mathbf{R}^N \times (0, \infty), \tag{3.1}$$

where $1 < p < (n+2)/n$. We first observe that if $v(x,t)$ solves (2), then for every $\lambda > 0$, the rescaled functions

$$v_\lambda(x,t) = \lambda^{2/(p-1)} v(\lambda x, \lambda^2 t) \tag{3.2}$$

defines a one parameter family of solutions to (3.1). A solution v is said to be self-similar when $v_\lambda(x,t) = v(x,t)$ for every $\lambda > 0$. It can be easily verified that v is a self-similar solution to (2) if and only if v has the form

$$v(x,t) = t^{-1/(p-1)} u(x/\sqrt{t}), \tag{3.3}$$

where u satisfies

$$\begin{aligned} u_{rr} + (\frac{r}{2} + \frac{N-1}{r})u_r + \frac{1}{p-1}u - |u|^{p-1}u = 0, \quad r > 0 \\ u'(0) = 0, \qquad u(0) = \alpha. \end{aligned} \tag{3.4}$$

The asymptotic behavior of solutions of (3.1) is usually determined by the limiting profile of (3.2) as $\lambda \to \infty$, which becomes a self-similar solution (see [3]). Henceforth the classification of self-similar solutions is valuable and we put a remark on the nature of, in particular, radial solutions.

We see that $u(r;\alpha^*) = \alpha^* = (\frac{1}{p-1})^{1/(p-1)}$ is a constant solution. When $\alpha > \alpha^*$, the corresponding solution is monotonely increasing to infinity since it is convex at every critical point. $u(r;\alpha_0)$ is the minimal positive solution of (3.4) and decays at the exponential rate as $r \to \infty$, which corresponds to the rapidly decaying solution in section 2 and it will be called now a fast orbit for some $\alpha_0 > 0$. As well-known $u(r;\alpha)$, $\alpha_0 < \alpha < \alpha^*$, is the unique positive solution of (5.1) satisfying the decay rate

$$\lim_{r\to\infty} r^{2/(p-1)} u(r;\alpha) = C(\alpha) > 0 \tag{3.5}$$

and thus called a slow orbit, which corresponds to a slowly decaying solution. ¿From the uniqueness of solutions, $u(r;-\alpha) = -u(r;\alpha)$ and we are left to the case $|\alpha| < \alpha_0$. But not much is known for this case.

For the weight function $K(x) = exp(|x|^2/4)$, the operator

$$Lu = \frac{1}{K}\nabla \cdot (K\nabla u)$$

is a self-adjoint operator on $L^(K)$. The eigenvalues of L are the numbers $\{\lambda_k\}_{k\geq 1}$, where $\lambda_k = \frac{N+k-1}{2}$. The next results are the only available ones, see [3] for details.

Proposition 7. *Let $\lambda = 1/(p-1)$. If $\lambda > \lambda_{2k+1}$, then there exist at least $2k$ nontrivial fast orbits for (3.4). And if $\lambda_{2k+1} < \lambda \leq \lambda_{2k+3}$ and $0 < \alpha < \alpha_0$, then $u(r;\alpha)$ vanishes at most $k+1$ times for $r > 0$. Moreover if $u(r;\alpha)$ vanishes exactly $k+1$ times for $r > 0$, then $u(r;\alpha)$ becomes a slow orbit.*

In particular, Proposition 7 implies that if $N/2 < 1/(p-1) \leq 1 + N/2$ and $0 < |\alpha| < \alpha_0$, then $u(r;\alpha)$ is a slow orbit since $u(r;\alpha)$ changes sign at least once for $r > 0$.

Moreover we can show that $u(r;\alpha)$ is in fact a slow orbit if $|\alpha|$ is relatively small.

Proposition 8. *Let $1/(p-1) > \lambda_{2k+1}$, $k \geq 0$ and $0 < |\alpha| < \alpha^*$. If $|\alpha|^{p-1} \leq 1/(p-1) - \lambda_{2k+1}$, then $u(r;\alpha)$ vanishes at least $k+1$ times for $r > 0$.*

This result follows from the standard comparison argument, see [3] for a complete proof.

Notice that the inequality $1/(p-1) > \lambda_{2k+1}$ is equivalent to $p < p_k$ in Theorem 6. Thus we may imagine a similar picture to this case but we do not know any proof. The complete understanding of the nature of solutions of (3.4) is very important in order to determine the large time behavior of solutions of (3.1) specially when the initial data changes sign because the asymptotic profile may be determined by its lap number.

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