Trends in Mathematics - New Series
Information Center for Mathematical Sciences
Volume 11, Number 1, 2009, pages 1–32

TRANSFER MAPS FOR MOTIVIC COHOMOLOGY AND NESTERENKO-SUSLIN THEOREM

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Abstract. In this expository article, we first introduce the Milnor's K-groups and the Goodwillie groups. Then, transfer maps, sometimes called norm maps, for Milnor's K-theory and for the Goodwillie group are introduced. The natural transfer maps for the Goodwillie groups, which are easily defined, actually agree with the classical but difficult transfer maps for the Milnor's K-theory. A by-product of this result will be a proof that Milnor's K-groups of fields is isomorphic to the Goodwillie groups. This result is analogous to one of the most important theorem in algebraic K-theory that motivic cohomology groups of fields, when the degree is equal to the weight, are isomorphic to Milnor's K-groups of fields, which was proved by Nesterenko and Suslin (1989). The present article is expository and reasonably self-contained.

1. Introduction

For a finite field extension L/k, we have two natural homomorphisms, namely the inclusion map $i_{L/k} : k^\times \to L^\times$ and the norm map $N_{L/k} : L^\times \to k^\times$. The two homomorphisms are related by the formula $N_{L/k} \circ i_{L/k} = [L : k]$. Very interestingly, for $\alpha \in L^\times$, $N_{L/k}(\alpha)$ can be described quite differently in the following two ways.

(i) Let $\sigma_1, \dots, \sigma_d$ $(d = [L : k])$ be the embeddings of L into an algebraic closure $\bar{k} \supset L$ of k, which fixes k. Then $\prod_{i=1}^{d} \sigma_i(\alpha)$ is in k and denoted by $N_{L/k}(\alpha)$.

(ii) The multiplication by α induces an invertible k-linear map from L onto itself. So, it is associated with a matrix $C_\alpha \in GL_n(k)$ when a basis for the k-vector space L is chosen. The determinant of C_α, which is denoted by $N_{L/k}(\alpha)$, depends only on α and is independent of the choice of basis vectors.

In this expository article, we want to explain how to generalize the notion of norm maps to "l-variables". One of the generalizations involves l-tuples of invertible elements of L and the other one deals with l-tuples of commuting invertible matrices over L. These generalizations are called the transfer maps. For the case of l-tuples of elements in $L^\times$, the transfer maps are defined through Milnor's K-theory and is somewhat difficult to describe because, although these elements give commuting k-linear automorphisms on L, it is hard to devise a process, which was the determinant in case of a single element, to push them back to elements of k. Bass and Tate ([1]) was able to create a method for this case and its validity was verified by Kato ([2]). For the case of l-tuples of commuting invertible matrices over L, the transfer maps are defined via the Goodwillie group. In this case, we just observe that commuting matrices in $GL_n(L)$ give rise to commuting matrices in $GL_{dn}(k)$ by simply regarding L as d-dimensional k-vector space and declare that this process gives rise to the transfer maps.

We find that these two transfer maps are compatible. After all, the transfer maps will be the key ingredients relating Milnor's K-theory and the Goodwillie group (Theorem 6.7). The Goodwillie group can be used to describe the motivic cohomology of fields when the degree is equal to the weight. Its natural generalization called Goodwillie-Lichtenbaum complex actually gives the motivic cohomology of arbitrary regular schemes ([9]). Consequently, Theorem 6.7 may be viewed as a variantion of Nesterenko-Suslin theorem ([6]) which states that Milnor's K-theory and the motivic cohomology of fields, defined in terms of higher Chow groups, are isomorphic when the degree is equal to the weight. The main result of the present article is that $\bigoplus_{l\geq 0} K_l^M(k)$ and $\bigoplus_{l\geq 0} GW_l(k)$ are isomorphic as graded rings with their respective product structures.

Most of the definitions and results are from [4] and [5], but they are presented with more details.

2. Milnor's K-groups and some computations of symbols

Let k be a field. Then we define Milnor's K-groups $K_n^M(k)$ of k as follows.

Definition 2.1. *The n-th Milnor's K-group $K_n^M(k)$ is the additive quotient group of the tensor product $(k^\times)^{\otimes n} = k^\times \otimes k^\times \otimes \cdots \otimes k^\times$ (n-times) by the subgroup generated by the elements of the form $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in (k^\times)^{\otimes n}$ where $a_i + a_j = 1$ for some $1 \leq i < j \leq n$. We denote by $\{a_1, a_2, \ldots, a_n\}$, called a Milnor symbol, the image of $a_1 \otimes a_2 \otimes \cdots \otimes a_n \in (k^\times)^{\otimes n}$ in $K_n^M(k)$.*

In particular, $K_1^M(k) \simeq k^\times$, but with an additively written group operation $\{a\} + \{b\} = \{ab\}$ and we set $K_0^M(k) = \mathbb{Z}$. The following properties are basic relations for symbols in $K_2^M(k)$.

Lemma 2.2. *Suppose that a, b and c are arbitrary nonzero elements of k.*
(i) (Multilinearity) $\{ab, c\} = \{a, c\} + \{b, c\}$. In particular, $\{a, 1\} = 0$ for any $a \in k^\times$.
(ii) (Skew-symmetry) $\{a, b\} = -\{b, a\}$.
(iii) $\{a, -a\} = 0$.

Proof. (i) Multilinearity is immediate from the definition of tensor products.

(iii) $0 = \{a, 1-a\} + \{a^{-1}, 1-a^{-1}\} = \{a, 1-a\} + \{a, \left(1-a^{-1}\right)^{-1}\}$
$= \{a, (1-a)\left(1-a^{-1}\right)^{-1}\} = \{a, -a\}$.

(ii) $0 = \{ab, -ab\} = \{a, -ab\} + \{b, -ab\} = \{a, -a\} + \{a, b\} + \{b, a\} + \{b, -b\} = \{a, b\} + \{b, a\}$ by (c).

□

Thanks to Lemma 2.2, $\bigoplus\limits_{l \geq 0} K_l^M(k)$ may be given a anti-commutative graded ring structure by setting $K_0^M(k) = \mathbb{Z}$ and defining the product by the rule $\{a_1, \ldots, a_p\} \cdot \{b_1, \ldots, b_q\} = \{a_1, \ldots, a_p, b_1, \ldots, b_q\}$. Another immediate consequence of Definition 2.1 is the following lemma, which looks simple but quite useful.

Lemma 2.3. *Suppose that $1 \leq n \leq l$ and $a_1, \ldots, a_n$ are nonzero elements of k such that $\{a_1, \ldots, a_n\} = 0$ in $K_n^M(k)$. Then, we have $\{a_1, \ldots, a_n, b_{n+1}, \ldots, b_l\} = 0$ in $K_l^M(k)$ for arbitrary nonzero elements $b_{n+1}, \ldots, b_l$ of k.*

Proof. If $n \geq 2$, then simply observe that elements of the form $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ $(a_i + a_j = 1)$ in $(k^\times)^{\otimes n}$ is contained in the subgroup quotient of $(k^\times)^{\otimes l}$ by which is $K_l(k)$. If $n = 1$, then $\{a\} = 0$ for $a \in k^\times$ only when $a = 1$. But then the assertion follows from Lemma 2.2 (i) for $l = 2$ and successively by the above case $n = 2$ for $l > 2$. □

Lemma 2.4. *Suppose that $n \geq 2$ and $a_1, \ldots, a_n$ are nonzero elements of k. If $a_1 + a_2 + \cdots + a_n = 1$, then $\{a_1, a_2, \ldots, a_n\} = 0$ in $K_n^M(k)$.*

Proof. If one of $a_1, \ldots, a_n$ is 1, then the symbol in question is trivially 0, so we may assume that none of $a_1, \ldots, a_n$ is 1. We proceed by induction on $n \geq 2$.

The case $n = 2$ is immediate from the definition of the Milnor's K-group.

Since $a_1 + a_2 + \cdots + a_n = 1$, we deduce that $\dfrac{1-a_1}{a_2} + \dfrac{a_3}{-a_2} + \dfrac{a_4}{-a_2} + \cdots + \dfrac{a_n}{-a_2} = 1$.
Hence, by our inductive hypothesis for $n-1$, we have $\{a_1, \dfrac{1-a_1}{a_2}, \dfrac{a_3}{-a_2}, \dfrac{a_4}{-a_2}, \ldots, \dfrac{a_n}{-a_2}\} =$

0. Now, expand out the symbol in the left hand side using multilinearity of Milnor symbols. But, when expanded out, all the terms except $-\{a_1, a_2, \dots, a_n\}$ are of the form $\pm\{a_1, 1-a_1, \dots\}$ or $\pm\{, \dots, a_2, \dots, -a_2, \dots\}$ which vanish. Therefore, we conclude that $\{a_1, a_2, \dots, a_n\} = 0$ in $K_n^M(k)$. □

The following lemma is not an essential relation, but we include it for the sake of keeping collection of interesting elementary results.

Lemma 2.5. *Suppose that $n \geq 2$ and $a_1, a_2, \dots, a_n$ are non-zero elements of k. If $a_1 + a_2 + \cdots + a_n = 0$, then $\{a_1, a_2, \dots, a_n\} = 0$ in $K_n^M(k)$.*

Proof. The case $n = 2$ is already treated in Lemma 2.2 (c). If $n \geq 3$, then since $\frac{a_1}{-a_n} + \frac{a_2}{-a_n} + \cdots + \frac{a_{n-1}}{-a_n} = 1$, we have $\{\frac{a_1}{-a_n}, \frac{a_2}{-a_n}, \dots, \frac{a_{n-1}}{-a_n}, a_n\} = 0$ by Lemma 2.4. Expand out the symbol in the left hand side using multilinearity of Milnor symbols and observe that all the terms except $\{a_1, a_2, \dots, a_n\}$ are of the form $\pm\{\dots, -a_n, \dots, a_n\}$ which vanish by Lemma 2.2 and Lemma 2.3. Therefore, we have $\{a_1, a_2, \dots, a_n\} = 0$. □

Lemma 2.6. *Let c and d be arbitrary nonzero elements of a field k. Then, we have the following relations in $K_2^M(k)$.*
(i) $\{c, d\} = \{\frac{c}{d}, d - c\} + \{-1, d\}$.
(ii) $\{c, d\} = \{-\frac{c}{d}, d + c\}$.

Proof. (i) $0 = \{\frac{c}{d}, 1 - \frac{c}{d}\} = \{\frac{c}{d}, \frac{d-c}{d}\} = \{\frac{c}{d}, d - c\} - \{c, d\} + \{d, d\}$. But, $\{d, d\} = \{-1, d\} + \{-d, d\} = \{-1, d\}$ by Lemma 2.2 (c).
(ii) $0 = \{-\frac{c}{d}, 1 + \frac{c}{d}\} = \{-\frac{c}{d}, \frac{c+d}{d}\} = \{-\frac{c}{d}, c + d\} - \{-\frac{c}{d}, d\} = \{-\frac{c}{d}, c + d\} - \{c, d\} + \{-d, d\}$
$= \{-\frac{c}{d}, c + d\} - \{c, d\}$, again by Lemma 2.2 (c). □

Now we prove the following key relation, which will be used later in the proof of Lemma 6.4

Proposition 2.7. *Suppose that $l \geq 1$ and that $l + 1$ elements $x_0 = 0, x_1, x_2, \dots, x_l$ of a field k is such that $x_i - x_j \neq 0$, whenever $i \neq j$ modulo $l + 1$, where the indices are considered modulo $l + 1$. Then we have*

$$\sum_{i=0}^{l} (-1)^{l(i+1)} \{x_{i+1} - x_i, x_{i+2} - x_i, x_{i+3} - x_i, \dots, x_{i+l} - x_i\} = \{-1, \dots, -1\} \quad \text{in } K_l^M(k).$$

Proof. We proceed by induction on l. The case $l = 1$ is straightforward since $-\{x_1 - x_0\} + \{x_0 - x_1\} = \{\frac{x_0 - x_1}{x_1 - x_0}\} = \{-1\}$ in $K_1(k)$.

Now let us suppose our proposition is true for $1, 2, \ldots, l-1$. To prove the proposition we aim to prove the following equality for $n = 2, 3, \ldots, l$;

(1)
$$\begin{aligned}\{x_1, x_2, x_3, x_4, \ldots, x_l\} &= \{-x_1, x_2-x_1, x_3-x_1, x_4-x_1, \ldots, x_n-x_1, x_{n+1}, \ldots, x_l\} \\ &- \{-x_2, x_1-x_2, x_3-x_2, x_4-x_2, \ldots, x_n-x_2, x_{n+1}, \ldots, x_l\} \\ &+ \{-x_3, x_1-x_3, x_2-x_3, x_4-x_3, \ldots, x_n-x_3, x_{n+1}, \ldots, x_l\} \\ &- \cdots + (-1)^{n+1}\{-x_n, x_1-x_n, x_2-x_n, x_3-x_n, x_4-x_n, \ldots, x_{n-1}-x_n, x_{n+1}, \ldots, x_l\} \\ &+ \{-1, -1, -1, -1 \ldots, -1, x_{n+1}, \ldots, x_l\}.\end{aligned}$$

Once the relation (1) is proved, by letting $n = l$, we deduce that

$$\begin{aligned}&\{x_1, x_2, x_3, x_4, \ldots, x_l\} + (-1)^l\{x_2-x_1, x_3-x_1, x_4-x_1, \ldots, x_l-x_1, x_0-x_1\} \\ &+ \{x_3-x_2, x_4-x_2, x_5-x_2, \ldots, x_1-x_2\} + (-1)^l\{x_4-x_3, x_5-x_3, x_6-x_3, \ldots, x_2-x_3\} \\ &+ \cdots + \{x_0-x_l, x_1-x_l, x_2-x_l, \ldots, x_{l-1}-x_l\} = \{-1, -1, -1, \ldots, -1\}\end{aligned}$$

and the proof of the proposition is complete.

We will prove the equality (1) inductively on $n = 2, 3, \ldots, l$. This will be a small inductive circuit inside the main induction on l for our proposition itself.

For $n = 2$, let us use Lemma 2.6(i) and multilinearity of Milnor symbol (Lemma 2.2 (i)) to expand out $\{x_1, x_2, \ldots, x_l\}$ $(= \{x_1-x_0, x_2-x_0, \ldots, x_l-x_0\})$ as follows.

$$\begin{aligned}&\{x_1, x_2, x_3, x_4, \ldots, x_l\} = \{\frac{x_1}{x_2}, x_2-x_1, x_3, \ldots, x_l\} + \{-1, x_2, x_3, \ldots, x_l\} \\ &= \{x_1, x_2-x_1, x_3, \ldots, x_l\} - \{x_2, x_2-x_1, x_3, \ldots, x_l\} + \{-1, x_2, x_3, \ldots, x_l\} \\ &= \{-x_1, x_2-x_1, x_3, \ldots, x_l\} - \{-x_2, x_1-x_2, x_3, \ldots, x_l\} \\ &\quad + \{-1, x_2-x_1, x_3, \ldots, x_l\} + \{-1, x_1-x_2, x_3, \ldots, x_l\} + \{-x_2, -1, x_3, \ldots, x_l\} \\ &\quad + \{-1, -1, x_3, \ldots, x_l\} + \{-1, x_2, x_3, \ldots, x_l\} \\ &= \{-x_1, x_2-x_1, x_3, \ldots, x_l\} - \{-x_2, x_1-x_2, x_3, \ldots, x_l\} + \{-1, -1, x_3, \ldots, x_l\}.\end{aligned}$$

This proves (1) for $n = 2$. In the last equality, we noted that any Milnor symbol which has -1 as a coordinate is 2-torsion.

Now we assume that the relation (1) is true for $n-1$, i.e.,

$$\begin{aligned}\{x_1,x_2,x_3,x_4,\dots,x_l\} &= \{-x_1,x_2-x_1,x_3-x_1,x_4-x_1,\dots,x_{n-1}-x_1,x_n,\dots,x_l\}\\ &-\{-x_2,x_1-x_2,x_3-x_2,x_4-x_2,\dots,x_{n-1}-x_2,x_n,\dots,x_l\}\\ &+\{-x_3,x_1-x_3,x_2-x_3,x_4-x_3,\dots,x_{n-1}-x_3,x_n,\dots,x_l\}\\ &-\cdots+(-1)^n\{-x_{n-1},x_1-x_{n-1},x_2-x_{n-1},x_3-x_{n-1},x_4-x_{n-1},\dots,\\ &\quad x_{n-2}-x_{n-1},x_n,\dots,x_l\}+\{-1,-1,-1,-1\dots,-1,x_n,\dots,x_l\}.\end{aligned}$$

We apply Lemma 2.6 (ii) and multilinearity of Milnor symbol (Lemma 2.2 (i)) to the right hand side of this equality to expand out $\{x_1,x_2,\dots,x_l\}$ further.

$$\begin{aligned}&\{x_1,x_2,x_3,x_4,\dots,x_l\}\\ &=\{\frac{-x_1}{-x_n},x_2-x_1,x_3-x_1,\dots,x_{n-1}-x_1,x_n-x_1,x_{n+1},\dots,x_l\}\\ &\quad-\{\frac{-x_2}{-x_3},x_1-x_2,x_3-x_2,\dots,x_{n-1}-x_2,x_n-x_2,x_{n+1},\dots,x_l\}\\ &\quad+\{\frac{-x_3}{-x_n},x_1-x_3,x_2-x_3,\dots,x_{n-1}-x_3,x_n-x_3,x_{n+1},\dots,x_l\}\\ &\quad-\cdots+(-1)^n\{\frac{-x_{n-1}}{-x_n},x_1-x_{n-1},x_2-x_{n-1},x_3-x_{n-1},\dots,x_{n-2}-x_{n-1},x_n\\ &\quad-x_{n-1},x_{n+1}\dots,x_l\}+\{-1,-1,-1,\dots,-1,x_n,\dots,x_l\}\\ &=\{-x_1,x_2-x_1,x_3-x_1,\dots,x_{n-1}-x_1,x_n-x_1,x_{n+1},\dots,x_l\}\\ &\quad-\{-x_2,x_1-x_2,x_3-x_2,\dots,x_{n-1}-x_2,x_n-x_2,x_{n+1},\dots,x_l\}\\ &\quad+\{-x_3,x_1-x_3,x_2-x_3,\dots,x_{n-1}-x_3,x_n-x_3,x_{n+1},\dots,x_l\}\\ &\quad-\cdots+(-1)^n\{-x_{n-1},x_1-x_{n-1},x_2-x_{n-1},x_3-x_{n-1},\dots,x_{n-2}-x_{n-1},x_n\\ &\quad-x_{n-1},x_{n+1}\dots,x_l\}+\{-1,-1,-1,\dots,-1,x_n,\dots,x_l\}\\ &-\{-x_n,x_2-x_1,x_3-x_1,\dots,x_{n-1}-x_1,x_n-x_1,x_{n+1},\dots,x_l\}\\ &\quad+\{-x_n,x_1-x_2,x_3-x_2,\dots,x_{n-1}-x_2,x_n-x_2,x_{n+1},\dots,x_l\}\\ &\quad-\{-x_n,x_1-x_3,x_2-x_3,\dots,x_{n-1}-x_3,x_n-x_3,x_{n+1},\dots,x_l\}\\ &\quad+\cdots+(-1)^n\{-x_n,x_1-x_{n-1},x_2-x_{n-1},x_3-x_{n-1},\dots,x_{n-2}-x_{n-1},x_n\\ &\quad-x_{n-1},x_{n+1}\dots,x_l\}\end{aligned}$$

Therefore, in each inductive step to prove (1) from $n-1$ to n, it suffices to show the following equality:

$$
\begin{aligned}
(2) \quad & -\{-x_n, x_2-x_1, x_3-x_1, x_4-x_1, \dots, x_n-x_1, x_{n+1}, \dots, x_l\} \\
& + \{-x_n, x_1-x_2, x_3-x_2, x_4-x_2, \dots, x_n-x_2, x_{n+1}, \dots, x_l\} \\
& - \{-x_n, x_1-x_3, x_2-x_3, x_4-x_3, \dots, x_n-x_3, x_{n+1}, \dots, x_l\} \\
& +\cdots+(-1)^n\{-x_n, x_1-x_n, x_2-x_n, x_3-x_n, x_4-x_n, \dots, x_{n-1}-x_n, x_{n+1}, \dots, x_l\} \\
& = \{-x_n, -1, -1, -1 \dots, -1, x_{n+1}, \dots, x_l\}
\end{aligned}
$$

Let us prove this equality, for example, in case where $n = 3$ as follows.

$$
\begin{aligned}
& \{x_1, x_2, x_3, x_4, \dots, x_l\} \\
& = \{\frac{-x_1}{-x_3}, x_2-x_1, x_3-x_1, x_4, \dots, x_l\} - \{\frac{-x_2}{-x_3}, x_1-x_2, x_3-x_2, x_4, \dots, x_l\} \\
& + \{-1, -1, x_3, x_4, \dots, x_l\} = \{-x_1, x_2-x_1, x_3-x_1, x_4, \dots, x_l\} \\
& - \{-x_2, x_1-x_2, x_3-x_2, x_4, \dots, x_l\} + \{-x_3, x_1-x_2, \frac{x_3-x_2}{x_3-x_1}, x_4, \dots, x_l\} \\
& + \{-x_3, -1, x_3-x_1, x_4, \dots, x_l\} + \{-1, -1, x_3, x_4, \dots, x_l\}.
\end{aligned}
$$

But, $\{-x_3, x_1-x_2, \frac{x_3-x_2}{x_3-x_1}, x_4, \dots, x_l\} = \{-x_3, x_1-x_2, 1+\frac{x_1-x_2}{x_3-x_1}, x_4, \dots, x_l\}$ $= \{-x_3, x_1-x_3, 1+\frac{x_1-x_2}{x_3-x_1}, x_4, \dots, x_l\} = \{-x_3, x_1-x_3, \frac{x_3-x_2}{x_3-x_1}, x_4, \dots, x_l\}$ $= \{-x_3, x_1-x_3, x_3-x_2, x_4, \dots, x_l\} = \{-x_3, x_1-x_3, x_2-x_3, x_4, \dots, x_l\}+\{-x_3, x_1-x_3, -1, x_4, \dots, x_l\}$.

Hence, $\{x_1, x_2, x_3, x_4, \dots, x_l\}$

$$
\begin{aligned}
& = \{-x_1, x_2-x_1, x_3-x_1, x_4, \dots, x_l\} - \{-x_2, x_1-x_2, x_3-x_2, x_4, \dots, x_l\} \\
& +\{-x_3, x_1-x_3, x_2-x_3, x_4, \dots, x_l\}+\{-x_3, -1, -1, x_4, \dots, x_l\}+\{-1, -1, x_3, x_4, \dots, x_l\} \\
& = \{-x_1, x_2-x_1, x_3-x_1, x_4, \dots, x_l\} - \{-x_2, x_1-x_2, x_3-x_2, x_4, \dots, x_l\} \\
& + \{-x_3, x_1-x_3, x_2-x_3, x_4, \dots, x_l\} + \{-1, -1, -1, x_4, \dots, x_l\}
\end{aligned}
$$

This proves (2) for $n = 3$ when $l \geq 3$.

To prove (2) for arbitrary $n \geq 4$ and $l \geq n$, by Lemma 2.3, it suffices to prove the following relation in $K^M_{n-1}(k)$:

$$
\begin{aligned}
(3) \qquad & -\{x_2-x_1, x_3-x_1, x_4-x_1, \dots, x_n-x_1\} \\
& \qquad\qquad + \{x_1-x_2, x_3-x_2, x_4-x_2, \dots, x_n-x_2\} \\
& \qquad\qquad - \{x_1-x_3, x_2-x_3, x_4-x_3, \dots, x_n-x_3\} \\
& \quad + \dots + (-1)^n\{x_1-x_l, x_2-x_l, x_3-x_l, x_4-x_l, \dots, x_{n-1}-x_n\} \\
& \qquad\qquad\qquad\qquad = \{-1,-1,-1\dots,-1\}
\end{aligned}
$$

Once the relation (3) is proved for $n \geq 4$, the relation (1) is established and we're done.

For example, when $n = 4$, the following calculation provides the equality (3).

$$
\begin{aligned}
& -\{x_2-x_1, x_3-x_1, x_4-x_1\} + \{x_1-x_2, x_3-x_2, x_4-x_2\} - \{x_1-x_3, x_2-x_3, x_4-x_3\} \\
&= -\{x_1-x_2, \frac{x_3-x_1}{x_4-x_1}, x_4-x_3\} + \{x_1-x_2, -1, x_4-x_1\} + \{-1, x_3-x_1, x_4-x_1\} \\
&\quad + \{x_1-x_2, \frac{x_3-x_2}{x_4-x_2}, x_4-x_3\} + \{x_1-x_2, -1, x_4-x_2\} \\
&\quad\quad - \{x_1-x_2, \frac{x_2-x_3}{x_1-x_3}, x_4-x_3\} + \{x_1-x_3, -1, x_4-x_3\} \\
&= \{x_1-x_2, \frac{x_1-x_4}{x_2-x_4}, x_4-x_3\} + \{x_1-x_2, x_4-x_1, -1\} + \{x_1-x_2, x_4-x_2, -1\} \\
&\quad + \{x_3-x_1, x_4-x_1, -1\} + \{x_1-x_3, x_4-x_3, -1\} \\
&= \{x_1-x_2, 1+\frac{x_1-x_2}{x_2-x_4}, x_4-x_3\} + \{x_4-x_2, x_1-x_4, -1\} \\
&\quad + \{x_1-x_3, x_4-x_1, -1\} + \{-1, x_4-x_1, -1\} + \{x_1-x_3, x_4-x_3, -1\} \\
&= \{x_4-x_2, x_1-x_4, x_4-x_3\} \\
&\quad + \{x_4-x_2, x_1-x_4, -1\} + \{x_4-x_1, x_3-x_4, -1\} + \{x_4-x_1, -1, -1\} \\
&= \{x_2-x_4, x_1-x_4, x_3-x_4\} + \{-1, x_1-x_4, x_4-x_3\} + \{x_2-x_4, x_1-x_4, -1\} \\
&\quad + \{x_4-x_2, x_1-x_4, -1\} + \{x_4-x_3, x_1-x_4, -1\} + \{x_4-x_1, -1, -1\} \\
&= -\{x_1-x_4, x_2-x_4, x_3-x_4\} + \{-1,-1,-1\}.
\end{aligned}
$$

For the general case $n \le l$, let us write $y_i = x_{i+1} - x_1$ for $i = 1, 2, \dots, n-1$. Then (3) can be re-written as the following equality;

$$\begin{aligned}
&- \{y_1 - y_0, y_2 - y_0, y_3 - y_0, \dots, y_{n-1} - y_0\} \\
&\quad + \{y_0 - y_1, y_2 - y_1, y_3 - y_1, \dots, y_{n-1} - y_1\} \\
&\qquad - \{y_0 - y_2, y_1 - y_2, y_3 - y_2, \dots, y_{n-1} - y_2\} \\
&\quad + \cdots + (-1)^n \{y_0 - y_{n-1}, y_1 - y_{n-1}, y_2 - y_{n-1}, y_3 - y_{n-1}, \dots, y_{n-2} - y_{n-1}\} \\
&\qquad\qquad = \{-1, -1, -1 \dots, -1\}
\end{aligned}$$

But, this equality follows immediately from our main inductive hypothesis for our proposition, with x_i replaced by y_i, since $n - 1 \le l - 1$ and the proposition holds for $1, 2, \dots, l-1$ by inductive hypothesis. □

3. Transfer maps for Milnor's K-groups

Let us recall a definition of the transfer map for Milnor's K-groups (See [1] or [2] §1.2). To understand the way it is defined, we will first want to study Weil's reciprocity law.

Let k be any field and let $K = k(X)$ be the field of rational functions of the projective line $\mathbb{P}^1_k$ over k.

A discrete valuation v of the field $K = k(X)$, which vanishes on k, is called a discrete valuation of K/k. For each discrete valuation v of K/k, let π_v be a uniformizing parameter and $k_v = R_v/(\pi_v)$ be the residue field of the valuation ring $R_v = \{r \in K | v(r) \ge 0\}$.

For each $f, g \in k(X)^\times$, we define the tame symbol $\partial_v(f,g) \in k_v^\times$ by $\partial_v(f,g) = (-1)^{v(f)v(g)} \left(\overline{f^{v(g)}/g^{v(f)}}\right)$, where $\overline{x}$ means $x \pmod{(\pi_v)}$ when $x \in R_v$. Note that $f^{v(g)}/g^{v(f)} \in R_v^\times$ since its value under v is 0.

Then it is easy to show that ∂_v is bi-multiplicative and vanishes when $f + g = 1$. Hence, ∂_v is a homomorphism from $K_2^M(K)$ into $k_v^\times$.

There are two types of discrete valuations of K/k. For each irreducible polynomial π_v of the polynomial ring $k[X]$, we can associate the valuation v of K such that $v(\pi_v) = 1$. The other type is v_∞ such that $\pi_v = 1/X$, i.e., $v(f) = -\deg f(X)$ whenever $f(X)$ is a polynomial in $k[X]$.

Suppose for a moment that k is algebraically closed. In this case, we may write $f = a(X-\alpha_1)^{r_1}(X-\alpha_2)^{r_2}\dots(X-\alpha_m)^{r_m}$, $g = b(X-\alpha_1)^{s_1}(X-\alpha_2)^{s_2}\dots(X-\alpha_m)^{s_m}$ for some $a, b \in k^\times$, $\alpha_j \in k$ and $r_j, s_j \in \mathbb{Z}$ for $j = 1, \dots, m$. Then $\partial_v(f,g) = 1$ unless v is the discrete valuation associated with $\pi_v = X - \alpha_j$ for some $j = 1, \dots, m$ or $v = v_\infty$.

If $\pi_v = X - \alpha_j$ for some $j = 1, \dots, m$, then

$$\partial_v(f,g) = (-1)^{r_j s_j} \overline{\left(\frac{a^{s_j}(X-\alpha_1)^{r_1 s_j}(X-\alpha_2)^{r_2 s_j} \dots (X-\alpha_j)^{r_j s_j} \dots (X-\alpha_m)^{r_m s_j}}{b^{r_j}(X-\alpha_1)^{s_1 r_j}(X-\alpha_2)^{s_2 r_j} \dots (X-\alpha_j)^{r_j s_j} \dots (X-\alpha_m)^{s_m r_j}} \right)}$$

$$= (-1)^{r_j s_j} \frac{a^{s_j}(\alpha_j-\alpha_1)^{r_1 s_j}(\alpha_j-\alpha_2)^{r_2 s_j} \dots (\alpha_j-\alpha_{j-1})^{r_{j-1} s_j}(\alpha_j-\alpha_{j+1})^{r_{j+1} s_j} \dots (\alpha_j-\alpha_m)^{r_m s_j}}{b^{r_j}(\alpha_j-\alpha_1)^{s_1 r_j}(\alpha_j-\alpha_2)^{s_2 r_j} \dots (\alpha_j-\alpha_{j-1})^{s_{j-1} r_j}(\alpha_j-\alpha_{j+1})^{s_{j+1} r_j} \dots (\alpha_j-\alpha_m)^{s_m r_j}}$$

On the other hand,

$$\partial_\infty(f,g) = (-1)^{(-r_1-r_2-\dots-r_m)(-s_1-s_2-\dots-s_m)} \frac{a^{-s_1-s_2-\dots-s_m}}{b^{-r_1-r_2-\dots-r_m}}$$

$$= (-1)^{(r_1+r_2+\dots+r_m)(s_1+s_2+\dots+s_m)} \frac{b^{r_1+r_2+\dots+r_m}}{a^{s_1+s_2+\dots+s_m}}.$$

From these calculations, it is immediate that $\prod_v \partial_v(g,f) = 1$ for $f, g \in K^\times$ where v runs through the discrete valuations of K/k.

This formula, after a slight modification, is still valid when k is not necessarily algebraically closed and is called the Weil's reciprocity law.

Lemma 3.1. *(Weil Reciprocity Law) Let k be a field and let $K = k(X)$ be the field of rational functions with X as an indeterminate. For each discrete valuation v of K/k, let $N_{k_v/k} : k_v^\times \to k^\times$ be the norm map from the residue field $k_v = R_v/(\pi_v)$ to k. Then, for $f, g \in K^\times$, we have*

$$\prod_v N_{k_v/k}(\partial_v(f,g)) = 1,$$

where v runs over the discrete valuation v of K/k.

Proof. Since norms are multiplicative and tame symbols are bimultiplicative, it suffices to prove the lemma for irreducible polynomials $f, g \in k[X]$. When f, g are relatively prime in $k[X]$, let v_f, v_g be the discrete valuation of K such that $v_f(f) = v_g(g) = 1$. We also let a and b be the leading coefficients of f and g, respectively, and let m, n be the degrees of f and g, respectively. Then

$$\partial_{v_f}(f,g) = (-1)^{v_f(f)v_f(g)} \overline{\left(\frac{f^{v_f(g)}}{g^{v_f(g)}} \right)} = g^{-1} \pmod{(f)}.$$

Similarly, $\partial_{v_g}(f,g) = f \pmod{(g)}$ and $\partial_\infty(f,g) = (-1)^{(-\deg f)(-\deg g)} \left(\frac{a^{-n}}{b^{-m}} \right) = (-1)^{\deg f \deg g} \left(\frac{b^m}{a^n} \right)$, where $v_i nfty$ is the discrete valuation of $K = k(X)$ such that $v_\infty(1/X) = 1$. Also, note that $\partial_v(f,g) = 1$ if v is not equal to one of v_f, v_g and v_∞.

As polynomials over the algebraic closure $\bar{k}$, we may write $f(X) = a(X - \alpha_1)\dots(X-\alpha_m)$ and $g(X) = b(X-\beta_1)\dots(X-\beta_n)$. Then $N_{v_f}(f \pmod{(g)})$ is the product of the conjugates of $f \pmod{(g)} \in k[X]/(g)$ over k and thus $N_{v_f}(f \pmod{(g)}) = f(\beta_1)f(\beta_2)\dots f(\beta_n) = a^n \prod_{i=1,\dots,m;j=1,\dots,n}(\beta_j - \alpha_i)$. Similarly, $N_{v_g}(g \pmod{(f)}) = b^m \prod_{i=1,\dots,m;j=1,\dots,n}(\alpha_i - \beta_j)$. We also note that N_{v_∞} is the identity map since $k_{v_\infty} = k$. Therefore,

$$\prod_v N_{k_v/k}(\partial_v(f,g)) = N_{v_f}(\partial_{v_f}(f,g))N_{v_g}(\partial_{v_g}(f,g))N_{v_\infty}(\partial_{v_\infty}(f,g))$$
$$= a^n \prod_{i,j}(\beta_j - \alpha_i) \cdot \frac{1}{b^m \prod_{i,j}(\alpha_i - \beta_j)} \cdot (-1)^{\deg f \deg g} \left(\frac{b^m}{a^n}\right) = 1$$

and the proof is complete.

This computation also show why the formula is called a reciprocity law. It basically says that, for irreducible polynomials $f, g \in k[X]$, we see that the determinant of the linear map induced by the multiplication by f modulo g is equal to the determinant of the linear map induced by the multiplication by g modulo f up to the sign $(-1)^{\deg f \deg g}$.

For example, let us take $k = \mathbb{Q}$, $f = X^2+2$ and $g = X^3+2$. Modulo g, take the basis $\{1, X, X^2\}$. Multiplication by $f = X^2+2$ sends $1, X, X^2$ to $2+X^2$, $-2+X$ and $-2X+2X^2$, respectively, and the associated matrix with respect to this basis is equal to $\begin{pmatrix} 2 & 0 & 1 \\ -2 & 2 & 0 \\ 0 & -2 & 2 \end{pmatrix}$, whose determinant is 12. On the other hand, $\{1, X\}$ forms a basis modulo f and multiplication by $g = X^3+2$ sends 1 and X to $-2-2X$ and $4-2X$, respectively. Therefore, the associated matrix is equal to $\begin{pmatrix} -2 & -2 \\ 4 & -2 \end{pmatrix}$, whose determinant is also 12, which is not a coincidence. □

We already had the norm maps from $K_1^M(k_v) = k_v^\times$ to $K_1^M(k) = k^\times$ for the discrete valuations v of $k(X)/k$ when we deduced the Weil reciprocity law. But, conversely, we can observe that the Weil reciprocity law is a defining property of the norm maps for the simple extensions of k. This is a key observation and we are actually going to define the higher norm maps, i.e. the transfer maps, $N_{k_v/k} : K_l^M(k_v) \to K_l^M(k)$ to be the unique homomorphisms satisfying a higher form of the Weil reciprocity law.

We first define the higher tame symbol $\partial_v : K_{l+1}^M(K) \to K_l^M(k_v)$ as follows.

Definition 3.2. *The tame symbol $\partial_v : K_{l+1}^M(K) \to K_l^M(k_v)$ is the unique epimorphism such that*

$$\partial_v(\{u_1, \dots, u_l, y\}) = v(y)\{\overline{u_1}, \dots, \overline{u_l}\}$$

whenever $u_1, \dots, u_l$ *are units of the valuation ring* R_v.

For any discrete valuation v of the field extension K/k and a uniformizing parameter $\pi_v \in R_v$ for v, every element a of $K = k(v)$ can be written as $u\pi_v^i$ for some u is a unit of R_v and $i \in \mathbb{Z}$. Therefore, any Milnor symbol in $K_{l+1}^M(K)$ can be written as $x = \{u_1\pi_v^{i_1}, \dots, u_l\pi_v^{i_l}, u_{l+1}\pi_v^{i_{l+1}}\}$, where $u_1, \dots, u_l, u_{l+1}$ are units of R_v and $i_1, \dots, i_l, i_{l+1}$ are integers. Using the relations $\{-\pi_v, \pi_v\} = 0$ and multilinearity (Lemma 2.2), it is immediate that $\{u\pi_v^i, \pi_v^j\} = \{(-1)^i u, \pi^j\}$. So,

$$\begin{aligned} x &= \{u_1\pi_v^{i_1}, \dots, u_l\pi_v^{i_l}, u_{l+1}\} + \{u_1\pi_v^{i_1}, \dots, u_l\pi_v^{i_l}, \pi_v^{i_{l+1}}\} \\ &= \{u_1\pi_v^{i_1}, \dots, u_{l-1}\pi_v^{i_{l-1}}, u_l, u_{l+1}\} + \{u_1\pi_v^{i_1}, \dots, \pi_v^{i_l}, u_{l+1}\} \\ &\quad + \{(-1)^{i_1}u_1, \dots, (-1)^{i_l}u_l, u_{l+1}\pi_v^{i_{l+1}}\} \\ &= \{u_1\pi_v^{i_1}, \dots, u_{l-1}\pi_v^{i_{l-1}}, u_l, u_{l+1}\} + \{(-1)^{i_1}u_1, \dots, (-1)^{i_{l-1}}u_{l-1}\pi_v^{i_l}, u_{l+1}\} \\ &\quad + \{(-1)^{i_1}u_1, \dots, (-1)^{i_l}u_l, u_{l+1}\pi_v^{i_{l+1}}\} \end{aligned}$$

Inductively, we deduce that

$$\begin{aligned} x &= \{u_1\pi_v^{i_1}, u_2, u_3, \dots, u_{l+1}\} + \{(-1)^{i_1}u_1, u_2\pi_v^{i_2}, u_3, \dots, u_{l+1}\} \\ &\quad + \dots + \{(-1)^{i_1}u_1, \dots, (-1)^{i_{l-1}}u_{l-1}\pi_v^{i_l}, u_{l+1}\} + \{(-1)^{i_1}u_1, \dots, (-1)^{i_l}u_l, u_{l+1}\pi_v^{i_{l+1}}\} \\ &= (-1)^l\{u_2, u_3, \dots, u_{l+1}, u_1\pi_v^{i_1}\} + (-1)^{l-1}\{(-1)^{i_1}u_1, u_3, \dots, u_{l+1}, u_2\pi_v^{i_2}\} \\ &\quad + \dots + \{(-1)^{i_1}u_1, \dots, (-1)^{i_l}u_l, u_{l+1}\pi_v^{i_{l+1}}\}. \end{aligned}$$

Hence, $$\begin{aligned} \partial_v(x) &= (-1)^l i_1\{\overline{u_2}, \overline{u_3}, \dots, \overline{u_{l+1}}\} + (-1)^{l-1} i_2\{(-1)^{i_1}\overline{u_1}, \overline{u_3}, \dots, \overline{u_{l+1}}\} \\ &\quad + (-1)^{l-2} i_3\{(-1)^{i_1}\overline{u_1}, (-1)^{i_2}\overline{u_2}, \overline{u_4}, \dots, \overline{u_{l+1}}\} \\ &\quad + \dots + i_{l+1}\{(-1)^{i_1}\overline{u_1}, (-1)^{i_2}\overline{u_2}, \dots, (-1)^{i_l}\overline{u_l}\} \end{aligned}$$

In particular, when $l = 1$, we have $\partial_v(\{u_1\pi_v^{i_1}, u_2\pi_v^{i_2}\}) = -i_1\{\overline{u_2}\} + i_2\{(-1)^{i_1}\overline{u_1}\} = \{(-1)^{i_1 i_2}\overline{u_1}^{i_2}/\overline{u_2}^{i_1}\}$. If we identify $K_1(k_v)$ with $k_v^\times$, then we may write $\partial_v(\{a, b\}) = (-1)^{v(a)v(b)}\overline{\left(\frac{a^{v(b)}}{b^{v(a)}}\right)} \in k_v^\times$.

Let v_∞ be the discrete valuation of $K = k(X)$, which vanishes on k, such that $v_\infty(X) = -1$. In particular, $v_\infty(f) = -\deg(f)$ if $f \in k[X]$ is a polynomial of degree $\deg(f)$. Every simple algebraic extension L of k is isomorphic to k_v for some discrete valuation $v \neq v_\infty$ which corresponds to a prime ideal $\mathfrak{p}$ of $k[X]$. Conversely, every discrete valuation $v \neq v_\infty$ of K/k gives rise to a simple algebraic extension L of k. This fact motivates the following definition of the transfer maps for simple extensions, which is due to Bass and Tate ([1]). The reader can notice that the product in the original Weil reciprocity law is replaced with the sum due to the additive convention for the Milnor symbols.

Definition 3.3. *The transfer maps* $N_{k_v/k} : K_l^M(k_v) \to K_l^M(k)$ *are the unique homomorphisms such that, for every* $w \in K_{l+1}^M(k(X))$, $\sum_v N_{k_v/k}(\partial_v w) = 0$ *where the sum is taken over all discrete valuations of* $k(X)/k$ *including* v_∞ *on* $k(X)$*. For* $v = v_\infty$*, we take* $N_{v_\infty} = \mathrm{Id}$.

Kato ([2] §1.7) has shown that these maps, if defined as compositions of transfer maps for simple extensions for a given tower of simple extensions, depend only on the field extension L/k, i.e., that it enjoys functoriality. See also [8]. Therefore, the transfer map is well-defined for arbitrary finite extension L/k and functorial for compositions of finite field extensions. The following formula for Milnor's K-groups will be useful in the proof of Theorem 6.7.

Lemma 3.4. *(Projection formula) For* $x \in K_p^M(k)$ *and* $y \in K_q^M(L)$*, we have* $N_{L/k}(\{x, y\}) = \{x, N_{L/k}(y)\}$.

Proof. It suffices to prove the formula for the case of a simple extension. Then it is obtained immediately from Definition 3.3 by regarding $\bigoplus_{q\geq 0} K_*^M(k(X))$ and $\bigoplus_{q\geq 0} K_q^M(k_v)$ as graded modules over the graded ring $\bigoplus_{p\geq 0} K_p^M(k)$ and noting that each ∂_v is a homomorphism of graded modules of degree -1 and thus each N_v is a homomorphism of graded modules of degree 0 (c.f. 4.1.3 of [8]). □

We would like to mention some simple lemmas before we go onto the next section.

Lemma 3.5. *Let* v *be the discrete valuation of* $K = k(X)$ *associated with a monic irreducible polynomial* π_v *of degree* d*. Denote by* α *the image of* X *in the residue field* $k_v = k[X]/(\pi_v)$*. Then the group* $K_l^M(k_v)$ *is generated by the symbols of the form* $\{a_1, a_2, \ldots, a_{l-r}, f_1(\alpha), f_2(\alpha), \ldots, f_r(\alpha)\}$*, where* $a_1, a_2, \ldots, a_{l-r}$ *are elements of* $k^\times$ *and* $f_1, \ldots, f_r$ *are monic irreducible polynomials in* $k[X]$ *with* $0 < \deg f_1 < \deg f_2 < \cdots < \deg f_r < d$.

Proof. Any element $w \in K_l^M(k_v)$ can be written as $w = \{g_1(\alpha), \ldots, g_l(\alpha)\} = \partial_v(\{g_1, \ldots, g_l, \pi_v\})$ for some polynomials $g_1, \ldots, g_l \in k[X]$ of degree less than d. So, it suffices to prove that if $\{g_1, \ldots, g_l\}$ is a symbol in $K_l^M(K)$ where $g_1, \ldots, g_l \in k[X]$ are of degree less than d, then it can be written as a sum of symbols of the form $\{a_1, a_2, \ldots, a_{l-r}, f_1, f_2, \ldots, f_r\}$ where $a_1, a_2, \ldots, a_{l-r}$ are elements of $k^\times$ and $f_1, \ldots, f_r$ are monic irreducible polynomials in $k[X]$ with $0 < \deg f_1 < \deg f_2 < \cdots < \deg f_r < d$.

First of all, by multilinearity of Milnor symbols, we may suppose that $g_1, \dots, g_l$ are monic irreducible polynomials in $k[X]$.

Now let us assume that f and g are monic irreducible polynomials of the same degree, say, $m > 0$. Write $f = g + h$ where $\deg h$ is less than m. In case $f = g$, $\{f, g\} = \{-f, f\} + \{-1, f\} = \{-1, f\}$ in $K_2^M(K)$. In other cases, $h \neq 0$ and we have $\frac{g}{f} + \frac{h}{f} = 1$. Hence $\{\frac{h}{f}, \frac{g}{f}\} = 0$. If we expand out the symbol using the multilinearity, then we get $\{f, g\} = \{h, g\} - \{h, f\} + \{f, f\} = \{h, g\} - \{h, f\} + \{-1, f\}$. In both cases, we can always rewrite $\{f, g\}$ as a sum of symbols of the form $\{\phi, \psi\}$ where ϕ and ψ are polynomials of $k[X]$ with $\deg \phi < \deg \psi$.

The proof follows inductively from this observation and we're done. □

Lemma 3.6. *(Inductive formula for $N_{k_v/k}$) For a generator $x = \{a_1, a_2, \dots, a_{l-r}, f_1(\alpha), f_2(\alpha), \dots, f_r(\alpha)\} \in K_l^M(k_v)$ in Lemma 3.5, we may express $N_{k_v/k}(x)$ as a sum of $\pm\{-1, -1, \dots, -1\}$ and $N_{v_i}(x_i)$ in $K_l^M(k)$, for $i = 1, 2, \dots, r$, where $\deg \pi_{v_i} < \deg \pi_v$ and $x_i \in K_l^M(k_{v_i})$ for each i.*

Proof. Let d be the degree of the monic irreducible polynomial π_v in $k[X]$. By Lemma 3.5, $K_l^M(k_v)$ is generated by symbols of the form $x = \{a_1, a_2, \dots, a_{l-r}, f_1(\alpha), f_2(\alpha), \dots, f_r(\alpha)\}$, where $a_1, a_2, \dots, a_{l-r}$ are elements of $k^\times$ and $f_1, \dots, f_r$ are monic irreducible polynomials in $k[X]$ with $0 < \deg f_1 < \deg f_2 < \cdots < \deg f_r < d$. Let $v_1, v_2, \dots, v_r$ be the discrete valuation of $K = k(X)$ associated with $f_1, f_2, \dots, f_r$, respectively. Write $y = \{a_1, a_2, \dots, a_{l-r}, f_1, f_2, \dots, f_r, \pi_v\} \in K_{l+1}^M(K)$ so that $\partial_v(y) = x$. Then $N_{k_v/k}(x)$ appears as a term in the Weil reciprocity law $\sum_w N_w(\partial_w(x)) = 0$. But, we have $\partial_w(y) = 0$ unless w is equal to either v, v_i for some $i \in \{1, 2, \dots, r\}$ or v_∞. Note that $\partial_{v_i}(y) = (-1)^{r-i} x_i$, where $x_i = \{a_1, a_2, \dots, a_{l-r}, f_1(\alpha_i), \dots, f_{i-1}(\alpha_i), f_{i+1}(\alpha_i), \dots, f_r(\alpha_i), \pi_v(\alpha_i)\}$ and α_i is the image of X in k_v under the identification $k_v = k[X]/(f_i)$. Also, we have $\partial_{v_\infty}(y) = (-1)^{r+1} \deg(f_1) \dots \deg(f_r) \deg(\pi_v)\{-1, -1, \dots, -1\}$. Therefore, we have

$$N_{k_v/k}(x) = (-1)^r \deg(f_1) \dots \deg(f_r) \deg(\pi_v)\{-1, -1, \dots, -1\} - \sum_{i=1}^{r} (-1)^r N_{v_i}(x_i).$$

Note that each x_i may be written explicitly once x is known. □

4. Introduction to the Goodwillie groups

We define the first Goodwillie group $GW_1(k)$ as follows when k is a field.

Definition 4.1. *$GW_1(k)$ is the abelian group generated by elements of $GL_n(k)$ for various $n \geq 1$, subject to the following 4 kinds of relations. For each $A \in GL_n(k)$, we denote by (A) the corresponding element of $GW_1(k)$.*

(i) (Identity Matrices) $(I) = 0$ *for an identity matrix* I.
(ii) (Similar Matrices) $(A) = (SAS^{-1})$ *for* $A, S \in GL_n(k)$.
(iii) (Direct Sum) $(A) + (B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ *for* $A \in GL_n(k)$ *and* $B \in GL_m(k)$.
(iv) (Polynomial Homotopy) $(A(0)) = (A(1))$ *for* $A(t) \in GL_n(k[t])$, *where* $k[t]$ *is the polynomial ring over* k *with the indeterminate* t.

It is immediate from the definition that, for a field extension $k \subset L$, we have a natural homomorphism $GW_l(k) \to GW_l(L)$ of groups.

Let us denote by $GL(k) = \bigcup_{n \geq 1} GL_n(k)$ the set of invertible matrices of finite ranks over k, where two matrices A and B are considered equal if $A = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}$ for some identity matrix I, or the other way around. Then $GW_1(k)$ can be thought of as generated by elements of $GL(k)$, by using the relations (i) and (iii).

The following lemma is easily proved using the definition of $GW_1(k)$.

Lemma 4.2. *(i) Every elementary matrix represents 0 in* $GW_1(k)$.
(ii) $(A) + (B) = (AB)$ *for* $A, B \in GL_n(k)$ *in* $GW_1(k)$

Proof. (i) An immediate consequence of the relation (iv) is that the element of $GW_1(k)$ represented by a 2×2 block matrix $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is equal to the one represented by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. To see this, simply take the invertible polynomial matrix $\begin{pmatrix} A & Ct \\ 0 & B \end{pmatrix}$ which have entries in $k[t]$ and put $t = 0$ and $t = 1$. Similarly, we see that an elementary matrix $E_{ij} \in GL_n(k)$, whose diagonal entries and (i, j)-th term are 1 and the rest entries are all 0, represents 0 in $GW_1(k)$.

(ii) In $GW_1(k)$, we have

$$\begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} B & I \\ 0 & A \end{pmatrix} = \begin{pmatrix} I & 0 \\ B & I \end{pmatrix} \begin{pmatrix} B & I \\ 0 & A \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ -AB & A+B \end{pmatrix}.$$

On the other hand, we also have

$$(AB) = \begin{pmatrix} I & 0 \\ 0 & AB \end{pmatrix} = \begin{pmatrix} I & I \\ 0 & AB \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \begin{pmatrix} I & I \\ 0 & AB \end{pmatrix} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ -AB & I+AB \end{pmatrix}.$$

But, by letting $t = 0$ and $t = 1$ in the polynomial homotopy

$$\begin{pmatrix} 0 & I \\ -AB & t(A+B) + (1-t)(I+AB) \end{pmatrix} \in GL_{2n}(k[t]),$$

we see that

$$\begin{pmatrix} 0 & I \\ -AB & A+B \end{pmatrix} = \begin{pmatrix} 0 & I \\ -AB & I+AB \end{pmatrix} \text{ in } GW_1(k)$$

and we're done. □

Corollary 4.3. *Every element of $GW_1(k)$ can be written as (A) for some single invertible matrix $A \in GL(k)$.*

Proof. This follows from the simple observation that $(A) - (B) = (A) + (B^{-1}) = \begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix}$ in $GW_1(k)$. □

Proposition 4.4. *We have $GW_1(k) \simeq k^\times$, the multiplicative group of units in k.*

Proof. Let us define the map $\phi : GW_1(k) \to k^\times$ by $\phi\big(\sum_i n_i(A_i)\big) = \prod_i (\det A_i)^{n_i}$. ϕ is well-defined by Definition 4.1 and is clearly surjective.

On the other hand, let (A), where A is an invertible matrix in $GL(k)$, be an arbitrary element in the kernel of ϕ (c.f. Corollary 4.3). When the determinant of A is 1, it is well-known (c.f. 2.2.6 & 2.3.2 in [7]) that A is a product of elementary matrices, which are trivial in $GW_1(k)$ by Lemma 4.2 (i). □

This construction can be easily generalized to l commuting invertible matrices instead of a single invertible matrix. So, we define the l-th Goodwillie group $GW_l(k)$ for $l \geq 1$ as follows.

Definition 4.5. *$GW_l(k)$ is the abelian group generated by l-tuples of commuting matrices $(A_1, \ldots, A_l)$ ($A_1, \ldots, A_l \in GL_n(k)$ for some $n \geq 1$), subject to the following 4 kinds of relations.*
(i) (Identity Matrices) $(A_1, \ldots, A_l) = 0$ when A_i for some i is equal to the identity matrix $I_n \in GL_n(k)$.
(ii) (Similar Matrices) $(A_1, \ldots, A_l) = (SA_1S^{-1}, \ldots, SA_lS^{-1})$ for any $S \in GL_n(k)$.
(iii) (Direct Sum) $(A_1, \ldots, A_l) + (B_1, \ldots, B_l) = \left(\begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}, \ldots, \begin{pmatrix} A_l & 0 \\ 0 & B_l \end{pmatrix} \right)$ for commuting $A_1, \ldots, A_l \in GL_n(k)$ and commuting $B_1, \ldots, B_l \in GL_m(k)$.
(iv) (Polynomial Homotopy) $(A_1(0), \ldots, A_l(0)) = (A_1(1), \ldots, A_l(1))$ if $A_1(t), \ldots, A_l(t)$ are commuting matrices in $GL_n(k[t])$, where $k[t]$ is the polynomial ring over k with the indeterminate t.

The motivic cohomology, which appear in various literatures and defined in many different ways although most of them turned out to be isomorphic, $H^d_{\mathcal{M}}\big(\mathrm{Spec}k, \mathbb{Z}(l)\big)$, when the degree d is equal to the weight l, is isomorphic to the Goodwillie group GW_l (See [9]). Therefore, in the present article, the Goodwillie group $GW_l(k)$ will be used to represent the motivic cohomology when the degree is equal to the weight.

To understand the Goodwillie group $GW_l(k)$ as a homology group of a complex, we introduce the following notation.

Definition 4.6. *$C(k[t_1,\dots,k_d],l)(d \geq 0, l \geq 1)$ is defined to be the abelian group generated by l-tuples $(A_1,\dots,A_l) = (A_1(t_1,\dots,t_d),\dots,A_l(t_1,\dots,t_d))$ where $A_1 = A_1(t_1,\dots,t_d),\dots,A_l = A_l(t_1,\dots,t_d)$ are commuting matrices in $GL_n(k[t_1,\dots,t_d])$ for various $n \geq 1$, subject to the following 3 kinds of relations.*
(i) (Identity Matrices) $(A_1,\dots,A_l) = 0$ when A_i for some i is equal to the identity matrix $I \in GL_n(k[t_1,\dots,t_d])$.
(ii) (Similar Matrices) $(A_1,\dots,A_l) = (SA_1S^{-1},\dots,SA_lS^{-1})$ for any $S \in GL_n(k[t_1,\dots,t_d])$.
(iii) (Direct Sum) $(A_1,\dots,A_l)+(B_1,\dots,B_l) = \left(\begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix},\dots,\begin{pmatrix} A_l & 0 \\ 0 & B_l \end{pmatrix}\right)$ for commuting $A_1,\dots,A_l \in GL_n(k[t_1,\dots,t_d])$ and commuting $B_1,\dots,B_l \in GL_m(k[t_1,\dots,t_d])$.

Our main concern in this article is when $d = 0$ and $d = 1$. When $d = 1$, we set $t = t_1$ and we define the boundary map $\partial : C(k[t],l) \to C(k,l)$ by sending $(A_1(t),\dots,A_l(t))$ in $C(k[t],l)$ to $(A_1(1),\dots,A_l(1)) - (A_1(0),\dots,A_l(0))$ in $C(k,l)$. Then, the Goodwillie group $GW_l(k)$ is nothing but the cokernel of the map $\partial : C(k[t],l) \to C(k,l)$. We will denote by the same notation $(A_1,\dots,A_l)$ the element in $C(k,l)/\partial C(k[t],l) = GW_l(k)$ represented by $(A_1,\dots,A_l)$, by abuse of notation, whenever $A_1,\dots,A_l$ are commuting matrices in $GL_n(k)$.

Before we proceed to the next section, we define products in the Goodwillie group. Recall that, for $A = (a_{ij}) \in GL_m(k)$ and $B = (b_{ij}) \in GL_n(k)$, the Kronecker product $A \otimes B$ is defined to a block matrix in $GL_{mn}(k)$ whose (i,j)-th block $(1 \leq i,j \leq m$ is given by $a_{ij}B$, i.e.,

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mm}B \end{pmatrix}.$$

One of the basic properties of the Kronecker product of matrices is that $(A \otimes B)(C \otimes D) = AC \otimes BD$ when $A, C \in GL_m(k)$ and $B, D \in GL_n(k)$ and thus $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Definition 4.7. *A product $\cdot : GW_p(k) \times GW_q(k) \to GW_{p+q}(k)$ for $p, q \geq 1$ is defined as follows. For commuting matrices $A_1,\dots,A_p \in GL_n(k)$ and commuting* *matrices* $B_1, \ldots, B_q \in GL_m(k)$, $(A_1, \ldots, A_p) \cdot (B_1, \ldots, B_q) \in GW_{p+q}(k)$ *is represented by the symbol* $(A_1 \otimes I_n, \ldots, A_p \otimes I_n, I_m \otimes B_1, \ldots, I_m \otimes B_q)$ *where* I_m *and* I_n *are identity matrices of size* m *and* n*, respectively.*

Let us verify that the product $\cdot : GW_p(k) \times GW_q(k) \to GW_{p+q}(k)$ is well-defined. First of all, $A_i \otimes I_n$ and $I_m \otimes B_j$ ($1 \le i \le p$, $1 \le j \le q$) are easily seen to be commuting. If either $(A_1, \ldots, A_p)$ or $(B_1, \ldots, B_q)$ is in one of the relations in Definition 4.5, let us check that $(A_1 \otimes I_n, \ldots, A_p \otimes I_n, I_m \otimes B_1, \ldots, I_m \otimes B_q)$ is also in the relations. We will check it only when $(A_1, \ldots, A_p)$ is in the relations.
(i) If, say, A_i is the identity matrix, then $A_i \otimes I_n$ is the identity matrix.
(ii) If S is in $GL_m(k)$, then $\big((S^{-1}A_1S) \otimes I_n, \ldots, (S^{-1}A_pS) \otimes I_n, I_m \otimes B_1, \ldots, I_m \otimes B_q\big)$ $= \big(T^{-1}(A_1 \otimes I_n)T \otimes I_n, \ldots, T^{-1}(A_p \otimes I_n)T, T^{-1}(I_m \otimes B_1)T, \ldots, T^{-1}(I_m \otimes B_q)T\big)$ where $T = S \otimes I_n$.
(iii) For commuting $A_1, \ldots, A_p \in GL_m(k)$, commuting $C_1, \ldots, C_p \in GL_r(k)$, and commuting $B_1, \ldots, B_q \in GL_n(k)$, we have

$$\left(\begin{pmatrix} A_1 & 0 \\ 0 & C_1 \end{pmatrix} \otimes I_n, \ldots, \begin{pmatrix} A_p & 0 \\ 0 & C_p \end{pmatrix} \otimes I_n, I_{m+r} \otimes B_1, \ldots, I_{m+r} \otimes B_q\right)$$
$$= \left(\begin{pmatrix} A_1 \otimes I_n & 0 \\ 0 & C_1 \otimes I_n \end{pmatrix}, \ldots, \begin{pmatrix} A_p \otimes I_n & 0 \\ 0 & C_p \otimes I_n \end{pmatrix}, \begin{pmatrix} I_m \otimes B_1 & 0 \\ 0 & I_r \otimes B_1 \end{pmatrix}, \right.$$
$$\left. \ldots, \begin{pmatrix} I_m \otimes B_q & 0 \\ 0 & I_r \otimes B_q \end{pmatrix}\right).$$

(iv) If $A_1(t), \ldots, A_p(t)$ are commuting matrices in $GL_m(k[t])$, then $A_1(t) \otimes I_n, \ldots, A_p(t) \otimes I_n, I_m \otimes B_1, \ldots, I_m \otimes B_q$ are commuting matrices in $GL_{mn}(k[t])$ and it is equal to $(A_1(0) \otimes I_n, \ldots, A_p(0) \otimes I_n, I_m \otimes B_1, \ldots, I_m \otimes B_q)$ and $(A_1(1) \otimes I_n, \ldots, A_p(1) \otimes I_n, I_m \otimes B_1, \ldots, I_m \otimes B_q)$ when $t = 0$ and $t = 1$, respectively.

We remark that, for commuting matrices $A, B \in GL_m(k)$, it is not necessarily true that $(A) \cdot (B) = (A, B)$ in $GW_2(k)$ unless A and B are 1×1 matrices.

For the next lemma, we set $GW_0(k) = \mathbb{Z}$, the ring of integers. Then the products $\cdot : GW_0(k) \times GW_l(k) \to GW_l(k)$ and $\cdot : GW_l(k) \times GW_0(k) \to GW_l(k)$, for $l \ge 0$ can be naturally defined by considering each $GW_l(k)$ as a $\mathbb{Z}$-module which arises from its abelian group structure.

Lemma 4.8. *The product* $\cdot : GW_p(k) \times GW_q(k) \to GW_{p+q}(k)$ *makes* $\bigoplus_{l \ge 0} GW_l(k)$ *into a graded ring.*

Proof. To show that it's a graded ring, we need to check that the product is associative and distributive with respect to the addition. Associativity is easily verified using the property $(A\otimes B)\otimes C = A\otimes(B\otimes C)$ of the Kronecker product of matrices. Distributive law may be also easily proved. □

5. Some fundamental properties of the Goodwillie group

The basic reference for this section and the subsequent section is [4]. We begin by the following useful computation.

Lemma 5.1. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be elements in $\bar{k}$ (an algebraic closure of k) not equal to either 0 or 1. Suppose also that $a_1a_2\cdots a_n = b_1b_2\cdots b_n$ and $(1-a_1)(1-a_2)\cdots(1-a_n) = (1-b_1)(1-b_2)\cdots(1-b_n)$. If all the elementary symmetric functions evaluated at $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are in k, then there is a matrix $A(t)$ in $GL_n(k[t])$ such that $I_n - A(t)$ is also invertible and the eigenvalues of $A(0)$ and $A(1)$ are $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, respectively.*

Proof. Let

$$p(\lambda) = (1-t)\prod_{i=1}^{n}(\lambda - a_i) + t\prod_{i=1}^{n}(\lambda - b_i)$$

be a polynomial in λ with coefficients in $k[t]$. It is a monic polynomial with the constant term equal to $(-1)^n a_1a_2\cdots a_n$. It has roots $b_1, b_2, \ldots, b_n$ and $a_1, a_2, \ldots, a_n$ when $t = 1$ and $t = 0$, respectively.

Now let $A(t)$ be its companion matrix in $GL_n(k[t])$. Then $\det(I_n - A(t)) = p(1)$ since $\det(\lambda I_n - A(t)) = p(\lambda)$. But $p(1) = (1-a_1)(1-a_2)\cdots(1-a_n) = (1-b_1)(1-b_2)\cdots(1-b_n)$ is in $k^\times$, and so $I_n - A(t)$ is invertible. It is clear that the eigenvalues of $A(t)$ are $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ when $t = 0$ and $t = 1$, respectively. □

Lemma 5.2. *We have the following equalities in $GW_l(k)$.*

(i) $(BC, A_2, \ldots, A_l) = (B, A_2, \ldots, A_l) + (C, A_2, \ldots, A_l)$, for all commuting $A, C, A_2, \ldots, A_l \in GL_n(k)$;

Similarly, $(A_1, \ldots, A_{i-1}, BC, A_{i+1}, \ldots, A_l) = (A_1, \ldots, A_{i-1}, B, A_{i+1}, \ldots, A_l) + (A_1, \ldots, A_{i-1}, C, A_{i+1}, \ldots, A_l)$ for all commuting $B, C, A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_l \in GL_n(k)$;

(ii) $(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_l) + (A_1, \ldots, A_j, \ldots, A_i, \ldots, A_l) = 0$, for all commuting $A_1, \ldots, A_l \in GL_n(k)$;

(iii) $(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_l) = 0$, when $A_i = -A_j$ for commuting $A_1, \ldots, A_l \in GL_n(k)$;

(iv) $(c_1, \ldots, b, \ldots, 1-b, \ldots, c_l) = (c_1, \ldots, a, \ldots, 1-a, \ldots, c_l)$, for $a, b \in k - \{0, 1\}$ and $c_i \in k^\times$ for each appropriate i.

Proof. (i) Let $\Theta(t)$ be the $2n \times 2n$ matrix

$$\begin{pmatrix} 0 & I_n \\ -BC & t(I_n + BC) + (1-t)(B+C) \end{pmatrix}.$$

Then, $\Theta(t)$ is in $GL_{2n}(k[t])$, and the boundary of $\big(\Theta(t),\, A_2 \oplus A_2, \ldots, A_l \oplus A_l\big)$ $\big(\Theta(t),\, A_2 \oplus A_2, \ldots, A_l \oplus A_l\big)$ under $\partial : C(k[t], l) \to C(k, l)$ is $(I \oplus BC, A_2 \oplus A_2, \ldots, A_l \oplus A_l) - (B \oplus C, A_2 \oplus A_2, \ldots, A_l \oplus A_l) = (BC, A_2, \ldots, A_l) - (B, A_2, \ldots, A_l) - (C, A_2, \ldots, A_l)$.

The proof is similar for other cases.

(ii) We let $\Theta(t)$ be the matrix

$$\begin{pmatrix} 0 & I_n \\ -A_iA_j & t(I_n + A_iA_j) + (1-t)(A_i + A_j) \end{pmatrix}.$$

Then the boundary of $\big(A_1 \oplus A_1, \ldots, \Theta(t), \ldots, \Theta(t), \ldots, A_l \oplus A_l\big)$ is

$$\begin{aligned}
&(A_1, \ldots, A_iA_j, \ldots, A_iA_j, \ldots, \ldots, \ldots, A_l) - (A_1, \ldots, A_i, \ldots, A_i, \ldots, A_l) \\
&- (A_1, \ldots, A_j, \ldots, A_j, \ldots, A_l) = \big((A_1, \ldots, A_i, \ldots, A_i, \ldots, A_l) + ((A_1, \ldots, A_i, \ldots, A_j, \ldots, A_l) \\
&\quad + (A_1, \ldots, A_j, \ldots, A_i, \ldots, A_l) + (A_1, \ldots, A_j, \ldots, A_j, \ldots, A_l)\big) \\
&\qquad - (A_1, \ldots, A_i, \ldots, A_i, \ldots, A_l) - (A_1, \ldots, A_j, \ldots, A_j, \ldots, A_l) \\
&= (A_1, \ldots, A_j, \ldots, A_i, \ldots, A_l) + (A_1, \ldots, A_j, \ldots, A_i, \ldots, A_l) \quad \text{in } GW_l(k) \text{ by } (i).
\end{aligned}$$

(iii) The boundary of $\left(\begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}, \ldots, \begin{pmatrix} -A & 0 \\ 0 & -A \end{pmatrix}, \ldots, \begin{pmatrix} 0 & I_n \\ -A & t(A+I_n) \end{pmatrix} \ldots, \begin{pmatrix} A_l & 0 \\ 0 & A_l \end{pmatrix}\right)$ is equal to

$$\begin{aligned}
&\left(\begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}, \ldots, \begin{pmatrix} -A & 0 \\ 0 & -A \end{pmatrix}, \ldots, \begin{pmatrix} 0 & I_n \\ -A & A+I_n \end{pmatrix}, \ldots, \begin{pmatrix} A_l & 0 \\ 0 & A_l \end{pmatrix}\right) \\
&\quad - \left(\begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}, \ldots, \begin{pmatrix} -A & 0 \\ 0 & -A \end{pmatrix}, \ldots, \begin{pmatrix} 0 & I_n \\ -A & 0 \end{pmatrix}, \ldots, \begin{pmatrix} A_l & 0 \\ 0 & A_l \end{pmatrix}\right) \\
&= \left(\begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}, \ldots, \begin{pmatrix} -A & 0 \\ 0 & -A \end{pmatrix}, \ldots, \begin{pmatrix} A & I_n \\ 0 & I_n \end{pmatrix}, \ldots, \begin{pmatrix} A_l & 0 \\ 0 & A_l \end{pmatrix}\right) \\
&\quad - \left(\begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}, \ldots, \begin{pmatrix} -A & 0 \\ 0 & -A \end{pmatrix}, \ldots, \begin{pmatrix} 0 & I_n \\ -A & 0 \end{pmatrix}, \ldots, \begin{pmatrix} A_l & 0 \\ 0 & A_l \end{pmatrix}\right) \\
&= (A_1, \ldots, -A, \ldots, A, \ldots, A_l) - \left(\begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}, \ldots, \begin{pmatrix} -A & 0 \\ 0 & -A \end{pmatrix}, \ldots, \begin{pmatrix} 0 & I_n \\ -A & 0 \end{pmatrix}\right. \\
&\left. \quad, \ldots, \begin{pmatrix} A_l & 0 \\ 0 & A_l \end{pmatrix}\right).
\end{aligned}$$

Thus it suffices to prove that $\left(\begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}, \dots, \begin{pmatrix} -A & 0 \\ 0 & -A \end{pmatrix}, \dots, \begin{pmatrix} 0 & I_n \\ -A & 0 \end{pmatrix}, \dots, \begin{pmatrix} A_l & 0 \\ 0 & A_l \end{pmatrix}\right)$ vanishes in $GW_l(k)$. But it is equal to

$$\left(\begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 & I_n \\ -A & 0 \end{pmatrix}^2, \dots, \begin{pmatrix} 0 & I_n \\ -A & 0 \end{pmatrix}, \dots, \begin{pmatrix} A_l & 0 \\ 0 & A_l \end{pmatrix}\right)$$
$$= 2\left(\begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 & I_n \\ -A & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & I_n \\ -A & 0 \end{pmatrix}, \dots, \begin{pmatrix} A_l & 0 \\ 0 & A_l \end{pmatrix}\right),$$

which vanishes in $GW_l(k)$ by (ii) above.

(iv) Apply Lemma 5.1 to $a_1 = a,\ a_2 = \sqrt{b},\ a_3 = -\sqrt{b},\ b_1 = -\sqrt{a},\ b_2 = \sqrt{a},\ b_3 = b$ to get $A(t) \in GL_3(k[t])$ with the properties stated in the lemma. Take $z = 2\big(c_1^{\oplus 3}, \dots, A(t), \dots, I_3 - A(t), \dots, c_l^{\oplus 3}\big)$. By the theory of rational canonical form, we have

$$\partial z = 2\left((c_1, \dots, b, \dots, 1-b, \dots, c_l) + \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, \dots, \begin{pmatrix} 1 & -1 \\ -a & 1 \end{pmatrix}\right.\right.$$
$$\left.\left., \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix}\right)\right) - 2\left((c_1, \dots, a, \dots, 1-a, \dots, c_l) + \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix},\right.\right.$$
$$\left.\left.\dots, \begin{pmatrix} 1 & -1 \\ -b & 1 \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix}\right)\right)$$
$$= -2(c_1, \dots, a, \dots, 1-a, \dots, c_l) + 2(c_1, \dots, b, \dots, 1-b, \dots, c_l)$$
$$- \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}^2, \dots, \begin{pmatrix} 1 & -1 \\ -b & 1 \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix}\right)$$
$$+ \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}^2, \dots, \begin{pmatrix} 1 & -1 \\ -a & 1 \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix}\right)$$
$$= \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \dots, \begin{pmatrix} 1-b & 0 \\ 0 & 1-b \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix}\right)$$
$$- \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \dots, \begin{pmatrix} 1 & -1 \\ -b & 1 \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix}\right)$$
$$- \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \dots, \begin{pmatrix} 1-a & 0 \\ 0 & 1-a \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix}\right)$$
$$+ \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \dots, \begin{pmatrix} 1 & -1 \\ -a & 1 \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \dots, \begin{pmatrix} 1-b & 0 \\ 0 & 1-b \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -b & 1 \end{pmatrix}^{-1}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$
$$- \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \dots, \begin{pmatrix} 1-a & 0 \\ 0 & 1-a \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -a & 1 \end{pmatrix}^{-1}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$
$$= \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \dots \begin{pmatrix} 1 & 1 \\ b & 1 \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$
$$- \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \dots, \begin{pmatrix} 1 & 1 \\ a & 1 \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$
$$= \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \dots, \begin{pmatrix} \frac{-b}{1-b} & \frac{1}{1-b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ b & 1 \end{pmatrix} \begin{pmatrix} \frac{-b}{1-b} & \frac{1}{1-b} \\ 0 & 1 \end{pmatrix}^{-1}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$
$$- \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \dots, \begin{pmatrix} \frac{-a}{1-a} & \frac{1}{1-a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ a & 1 \end{pmatrix} \begin{pmatrix} \frac{-a}{1-a} & \frac{1}{1-a} \\ 0 & 1 \end{pmatrix}^{-1}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$
$$= \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ b-1 & 2 \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$
$$- \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ a-1 & 2 \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right).$$

By taking the boundary of the element

$$\left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ b-1 & (2-b)t + 2(1-t) \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$
$$- \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ a-1 & (2-a)t + 2(1-t) \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right),$$

we see that

$$\partial z = \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ b-1 & 2-b \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$
$$- \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ a-1 & 2-a \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$
$$= \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \dots, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ b-1 & 2-b \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$
$$- \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \dots, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ a-1 & 2-a \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$
$$= \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}, \dots, \begin{pmatrix} 1-b & 0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$

$$- \left(\begin{pmatrix} c_1 & 0 \\ 0 & c_1 \end{pmatrix}, \dots, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \dots, \begin{pmatrix} 1-a & 0 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} c_l & 0 \\ 0 & c_l \end{pmatrix} \right)$$
$$= (c_1, \dots, b, \dots, 1-b, \dots, c_l) - (c_1, \dots, a, \dots, 1-a, \dots, c_l)$$

in $GW_l(k)$. Therefore, (iv) vanishes in $GW_l(k)$. □

Corollary 5.3. *(Multilinearity and Skew-symmetry for $GW_l(k)$)*

(i) $(A_1, \dots, A_{i-1}, BC, A_{i+1}, \dots, A_l) = (A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_l) + (A_1, \dots, A_{i-1}, C, A_{i+1}, \dots, A_l)$ *in* $GW_l(k)$*, for all commuting* $B, C, A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_l \in GL_n(k)$

(ii) $(A_1, \dots, A_i, \dots, A_j, \dots, A_l) = -(A_1, \dots, A_j, \dots, A_i, \dots, A_l)$ *in* $GW_l(k)$ *for all commuting* $A_1, \dots, A_l \in GL_n(k)$

If $A_1, \dots, A_l$ and $A'_1, \dots, A'_l$ are commuting matrices in $GL_n(k)$ and $GL_m(k)$, respectively, then $(A_1, \dots, A_l) + (A'_1, \dots, A'_l) = (A_1 \oplus A'_1, \dots, A_l \oplus A'_l)$ in $GW_l(k)$. Therefore, we obtain the following result from Corollary 5.3, which can be viewed as a generalization of Corollary 4.3.

Corollary 5.4. *Every element in* $GW_l(k)$ *can be written as a single symbol* $(A_1, \dots, A_l)$*, where* $A_1, \dots, A_l$ *are commuting matrices in* $GL_n(k)$*.*

Corollary 5.5. *The graded ring* $\bigoplus_{l \geq 0} GW_l(k)$ *in Lemma 4.8 is anti-commutative.*

Proof. Note that, when $A \in GL_m(k)$ and $B \in GL_n(k)$, then the matrix $A \otimes B$ is similar to $B \otimes A$ by the similarity matrix S which is given by a base change which sends the (i, j)-th canonical vector to (j, i)-th canonical vector for each $i = 1, \dots, m$ and $j = 1, \dots, n$. Therefore, for commuting $A_1, \dots, A_p \in GL_m(k)$ and commuting $B_1, \dots, B_q \in GL_n(k)$, we have $(A_1, \dots, A_p) \cdot (B_1, \dots, B_q) = (A_1 \otimes I_n, \dots, A_p \otimes I_n, I_m \otimes B_1, \dots, I_m \otimes B_q) = (I_n \otimes A_1, \dots, I_n \otimes A_p, B_1 \otimes I_m, \dots, B_q \otimes I_m)$ by Definition 4.5 (ii). But, by Corollary 5.3, it is equal to $(-1)^{pq}(B_1 \otimes I_m, \dots, B_q \otimes I_m, I_n \otimes A_1, \dots, I_n \otimes A_p) = (-1)^{pq}(B_1, \dots, B_q) \cdot (A_1, \dots, A_p)$. □

In fact, this corollary can be also deduced from Lemma 2.2 (ii) and Theorem 6.7, which is to be proved later. Thanks to Lemma 5.2, we may now construct a map from Milnor's K-groups to the Goodwillie groups.

Proposition 5.6. *For any field k, the assignment* $\{a_1, a_2, \dots, a_l\} \mapsto (a_1, a_2, \dots, a_l)$ *for each Steinberg symbol* $\{a_1, a_2, \dots, a_l\}$ *gives a well-defined homomorphism* ρ_l *from the Milnor's* K*-group* $K^M_l(k)$ *to* $GW_l(k)$*. In fact, it gives rise to a graded ring homomorphism from* $\bigoplus_{l \geq 0} K^M_l(k)$ *to* $\bigoplus_{l \geq 0} GW_l(k)$*.*

Proof. This proposition turns out to be straightforward when $l = 1$ (see Proposition 4.4). So we assume that $l \geq 2$. By Corollary 5.3 (i), the multilinearity is satisfied by our symbol $(\ ,\ldots,\ )$. Therefore all we need to show is that for every $\alpha \in K - \{0,1\}$ and $c_r \in K^\times$ for $1 \leq r \leq l$, $r \neq i, j$, $(c_1, \ldots, \alpha, \ldots, 1-\alpha, \ldots, c_l)$ is in $\partial C(k[t], l)$.

The proposition is immediate for a Galois field $\mathbb{F}_p$ because $K_l^M(\mathbb{F}_p) = 0$ for $l \geq 2$. So we may assume that there exists an element $e \in k$ such that $e^3 - e \neq 0$. By Lemma 5.2 (iv) with $a = e$, $b = 1-e$, we have $(c_1, \ldots, e, \ldots, 1-e, \ldots, c_l) - (c_1, \ldots, 1-e, \ldots, e, \ldots, c_l) = 2(c_1, \ldots, e, \ldots, 1-e, \ldots, c_l) = 0$ in $GW(k)$. With $a = -e$, $b = 1+e$, we have $2(c_1, \ldots, e, \ldots, 1+e, \ldots, c_l) = 2(c_1, \ldots, -e, \ldots, 1+e, \ldots, c_l) = 0$. Hence, $(c_1, \ldots, e^2, \ldots, 1-e^2, \ldots, c_l) = 2(c_1, \ldots, e, \ldots, 1-e, \ldots, c_l) + 2(c_1, \ldots, e, \ldots, 1+e, \ldots, c_l) = 0$.

On the other hand, by Lemma 5.2 (iv) with $a = e^2$, $b = \alpha$, we see that $-(c_1, \ldots, e^2, \ldots, 1-e^2, \ldots, c_l) + (c_1, \ldots, \alpha, \ldots, 1-\alpha, \ldots, c_l) = 0$ in $GW(k)$ and we're done.

The fact that the ρ_l gives rise to a graded ring homomorphism follows from the definition of products. □

6. The transfer maps for the Goodwillie group and the Nesterenko-Suslin theorem

In case of the Goodwillie groups, there is a natural functorial definition of the transfer map for any finite extension $k \subset L$.

Definition 6.1. *If $A_1, \ldots, A_l$ are commuting invertible matrices of rank n with coefficients in $L[t]$ (respectively, L), then by identifying $L[t]$ (respectively L) as a free $k[t]$-module (respectively k-module) of rank $d = [L : k]$, we may consider $A_1, \ldots, A_l$ as commuting invertible linear maps on $L[t]^n$ (respectively L^n), i.e., commuting invertible linear maps on $k[t]^{nd}$ (respectively, k^{nd}). So, they are associated with certain commuting invertible matrices $A'_1, \ldots, A'_l$ of rank nd with coefficients in $k[t]$ (respectively, k). This gives a map $C(L[t], l) \to C(k[t], l)$ (see Definition 4.6) (respectively, a map $C(L, l) \to C(k, l)$. These maps are compatible with the boundary map ∂ and thus they induce a homomorphism $N_{L/k} : GW_l(L) \to GW_l(k)$, which is called the transfer map for the Goodwillie group.*

This definition of transfer can be easily generalized for the motivic cohomology group of an arbitrary regular noetherian local ring, using Goodwillie-Lichtenbaum complex. We refer [4] for interested readers.

Let us describe the transfer map $N_{L/k}$ when $L = k(\alpha)$ is a simple extension. We let $f(X) = X^d + c_1 X^{d-1} + \cdots + c_d \in k[X]$ be the monic irreducible polynomial of α

over k, which is of degree $d = [L : k]$, and let $C = C_\alpha \in GL_d(k)$ be its companion matrix.

Now for $b_1, \dots, b_l \in L^\times$, take $A_r = g_r(C) = a_{r1} + a_{r2}C + \cdots + a_{rd}C^{d-1} \in GL_d(k)$ where g_r is a polynomial with coefficients in k such that $b_r = g_r(\alpha) = a_{r1} + a_{r2}\alpha + \cdots + a_{rd}\alpha^{d-1}$ $(r = 1, \dots, l)$. Then the corresponding symbol $(b_1, \dots, b_l) \in GW_l(L)$ is sent to the symbol $(A_1, A_2, \dots, A_l)$ of $GW_l(k)$ under the transfer map.

For convenience, we define $N_{L/k} : GW_0(L) = \mathbb{Z} \to GW_0(k) = \mathbb{Z}$ to be a multiplication by the degree $[L : k]$ of the field extension. It is immediate from the definition that $N_{L'/L} \circ N_{L/k} = N_{L'/k}$ whenever we have a tower of finite field extensions $k \subset L \subset L'$.

If $d = [L : k]$, L is isomorhpic to $k^{\oplus d}$ as k-vector space. So, a multiplication by a matrix A of rank n with entries in k induces a k-linear map on L, which is associated with the matrix $A \otimes I_d$ of rank nd whose diagonal blocks are equal to A. Therefore, the composition

$$GW_l(k) \xrightarrow{i_{L/k}} GW_l(L) \xrightarrow{N_{L/k}} GW_l(k) \ ,$$

where the first map $i_{L/k}$ is induced by the inclusion of the fields $k \subset L$, is just a multiplication by d.

More generally, we have the following projection formula.

Lemma 6.2. *(Projection formula) For $z \in GW_p(k)$ and $w \in GW_q(L)$, we have $N_{L/k}(i_{L/k}(z) \cdot w) = z \cdot N_{L/k}(w)$, where the product $\cdot$ is defined as in Definition 4.7.*

Proof. We may assume that $z = (A_1, \dots, A_p)$ and $w = (B_1, \dots, B_q)$ for some commuting matrices $A_1, \dots, A_p \in GL_m(k)$ and commuting matrices $B_1, \dots, B_q \in GL_n(L)$.

For a fixed basis of L as a k-vector space, let $B'_1, \dots, B'_q$ be the commuting matrices in $GL_{nd}(k)$ which are associated with the commuting invertible linear maps induced by $B_1, \dots, B_q$ on $L^n \simeq k^{nd}$. Then we have

$$\begin{aligned}
&N_{L/k}\left((A_1, \dots, A_p) \cdot (B_1, \dots, B_q)\right) \\
&= N_{L/k}\left(A_1 \otimes I_n, \dots, A_p \otimes I_n, I_m \otimes B_1, \dots, I_m \otimes B_q\right) \\
&= \big((A_1 \otimes I_n) \otimes I_d, \dots, (A_p \otimes I_n) \otimes I_d, I_m \otimes B'_1, \dots, I_m \otimes B'_q\big) \\
&= (A_1, \dots, A_p) \cdot N_{L/k}\left((B_1, \dots, B_q)\right).
\end{aligned}$$

The case when $p = 0$ (or $q = 0$) is already shown above already and the proof is complete. $\square$

Note that we also have $N_{L/k}(w \cdot i_{L/k}(z)) = N_{L/k}(w) \cdot z$ by Corollary 5.5. By Proposition 4.4, every element in $GW_1(k)$ can be identified with a symbol represented by its determinant, so we have the following corollary from the projection formula.

Corollary 6.3. *Suppose that* $\alpha_1, \dots, \alpha_l \in k^\times$ *and* $\beta \in L^\times$. *Then* $N_{L/k}(\alpha_1, \dots, \alpha_l, \beta) = \big(\alpha_1, \dots, \alpha_l, N_{L/k}(\beta)\big)$, *where* $N_{L/k}(\beta) \in k^\times$ *is the image of* β *under the usual norm map* $N_{L/k} : L^\times \to k^\times$. *By Corollary 5.3 (ii), even if* β *is not located in the last coordinate, a similar equality holds.*

On the other hand, we have seen that the transfer maps $N_{L/k} : K_l^M(L) \to K_l^M(k)$ for the Milnor's K-groups are defined whenever L/k is a finite field extension in Section 3.

We will need the following lemma to prove the compatibility between the two transfer maps.

Lemma 6.4. *For monic irreducible polynomials* $f_0(X), \dots, f_l(X)$ *in* $k[X]$, *which are relatively prime, we have* $\sum_v N_{k_v/k}\left(\rho_l \partial_v \{f_0(X), \dots, f_l(X)\}\right) = 0$, *where the sum is taken over all discrete valuations, including* v_∞ *on* $k(X)$, *which vanish on* k.

Proof. We have $\partial_{v_\infty}\{f_0(X), \dots, f_l(X)\} = (-1)^{l+1} \deg f_0(X) \dots \deg f_l(X) \{-1, -1, \dots, -1\} \in K_l^M(k)$ since $f_0(X), \dots, f_l(X)$ are monic, so we need to show that

$$\sum_{v \neq v_\infty} N_{k_v/k}\left(\rho_l \partial_v \{f_0(X), \dots, f_l(X)\}\right) = (-1)^l \deg f_0(X) \dots \deg f_l(X) \left(-1, -1, \dots, -1\right).$$

We first illustrate the case $l = 2$. Let $f(X), g(X)$, and $h(X)$ be monic irreducible polynomials in $k[X]$. Then $\partial_v\{f(X), g(X), h(X)\}$, when $v \neq v_\infty$, vanishes unless v is one of the discrete valuations v_f, v_g, and v_h associated with the prime ideals $(f(X))$, $(g(X))$, and $(h(X))$, respectively.

Write $f(X) = \prod_{i=1}^{m}(X - \alpha_i)$, $g(X) = \prod_{i=1}^{n}(X - \beta_i)$, and $h(X) = \prod_{i=1}^{q}(X - \gamma_i)$ with $\alpha = \alpha_1, \beta = \beta_1, \gamma = \gamma_1$. Then $\partial_{v_f}\{f(X), g(X), h(X)\} = \{\overline{g(X)}, \overline{h(X)}\}$ in $K_2^M(L)$, where $L = k[X]/(f(X)) \simeq k(\alpha)$. Hence, $\partial_{v_f}\{f(X), g(X), h(X)\} = \{g(\alpha), h(\alpha)\}$.

If $k(\alpha)$, $k(\beta)$ and $k(\gamma)$ are linearly disjoint over k, then we may apply Corollary 6.3 twice to obtain

$$\begin{aligned} N_{k(\alpha,\beta,\gamma)/k}(\alpha-\beta,\alpha-\gamma) &= N_{k(\alpha,\beta)/k}\left(N_{k(\alpha,\beta,\gamma)/k(\alpha,\beta)}(\alpha-\beta,\alpha-\gamma)\right) \\ &= N_{k(\alpha)/k}\Big(N_{k(\alpha,\beta)/k(\alpha)}\big(\alpha-\beta,\prod_{i=1}^{q}(\alpha-\gamma_i)\big)\Big) \\ &= N_{k(\alpha)/k}\big(\prod_{i=1}^{n}(\alpha-\beta_i),\prod_{i=1}^{q}(\alpha-\gamma_i)\big) = N_{k_{v_f}/k}(g(\alpha),h(\alpha)) \end{aligned}$$

in $GW_2(k)$. In this case, we have

$$N_{k_{v_f}/k}(g(\alpha),h(\alpha)) = \frac{\deg f(X)\deg g(X)\deg h(X)}{[k(\alpha,\beta,\gamma):k]}N_{k(\alpha,\beta,\gamma)/k}(\alpha-\beta,\alpha-\gamma) \tag{4}$$

in $GW_2(k)$. Now let us show that the equality (4) is true without the assumption of linear disjointness. We first observe that if K/k is a finite field extension and $\phi : K \to K'$ is a k-linear field isomorphism, then $N_{K/k}(a,b) = N_{K'/k}(\phi(a),\phi(b))$ since it is true when K is a simple extension by definition. Next, we factor $g(X) = g_1(X)\dots g_s(X)$ and $h(X) = h_1(X)\dots h_t(X)$ where $g_1,\dots,g_s$ are monic irreducible in $k(\alpha)[X]$ and $h_1,\dots,h_t$ are monic irreducible in $k(\alpha,\beta)[X]$ so that $t = \deg(g)/[k(\alpha,\beta),k(\alpha)]$ and $s = \deg(h)/[k(\alpha,\beta,\gamma),k(\alpha,\beta)]$. By rearranging β_i and γ_j if necessary, let $\beta_1 = \beta,\beta_2,\dots,\beta_s$ and $\gamma_1 = \gamma,\gamma_2,\dots,\gamma_t$ be roots of $g_1,g_2,\dots,g_s$ and $h_1,h_2,\dots,h_t$, respectively. These elements reside in an algebraic closure of k. By the above observation, we see that $N_{k(\alpha,\beta,\gamma)/k}(\alpha-\beta,\alpha-\gamma) = N_{k(\alpha,\beta_i,\gamma_j)/k}(\alpha-\beta_i,\alpha-\gamma_j)$ for each $i = 1,\dots,s$ and $j = 1,\dots,t$. Hence, by Corollary 6.3 and Corollary 5.3 (i),

$$\begin{aligned} stN_{k(\alpha,\beta,\gamma)/k}(\alpha-\beta,\alpha-\gamma) &= \sum_{i=1,\dots,s}\sum_{j=1,\dots,t} N_{k(\alpha,\beta_i,\gamma_j)/k}(\alpha-\beta_i,\alpha-\gamma_j) \\ &= \sum_{i=1,\dots,s}\sum_{j=1,\dots,t} N_{k(\alpha,\beta_i)/k}\Big(N_{k(\alpha,\beta_i,\gamma_j)/k}\big(\alpha-\beta_i,h_j(\alpha)\big)\Big) \\ &= \sum_{i=1,\dots,s}\sum_{j=1,\dots,t} N_{k(\alpha)/k}\Big(N_{k(\alpha,\beta_i)/k(\alpha)}\big(\alpha-\beta_i,h_j(\alpha)\big)\Big) \\ &= \sum_{i=1,\dots,s}\sum_{j=1,\dots,t} N_{k(\alpha)/k}\big(g_i(\alpha),h_j(\alpha)\big) = N_{k(\alpha)/k}\big(g(\alpha),h(\alpha)\big). \end{aligned}$$

Note that $st = \dfrac{\deg(h)}{[k(\alpha,\beta,\gamma),k(\alpha,\beta)]}\dfrac{\deg(g)}{[k(\alpha,\beta),k(\alpha)]} = \dfrac{\deg f(X)\deg g(X)\deg h(X)}{[k(\alpha,\beta,\gamma):k]}$ and thus we verified the equality (4). Similarly, we have

$$N_{k_{v_g}/k}\rho_2\partial_{v_g}\{f(X),g(X),h(X)\} = \frac{\deg f(X)\deg g(X)\deg h(X)}{[k(\alpha,\beta,\gamma):k]}N_{k(\alpha,\beta,\gamma)/k}(\beta-\gamma,\beta-\alpha)$$

and $\quad \mathrm{N}_{\mathrm{k}_{\mathrm{v_h}}/\mathrm{k}}\rho_2\partial_{\mathrm{v_h}}\{\mathrm{f(X)},\mathrm{g(X)},\mathrm{h(X)}\} = \dfrac{\deg \mathrm{f(X)}\deg \mathrm{g(X)}\deg \mathrm{h(X)}}{[\mathrm{k}(\alpha,\beta,\gamma):\mathrm{k}]}\mathrm{N}_{\mathrm{k}(\alpha,\beta,\gamma)/\mathrm{k}}(\gamma-\alpha,\gamma-\beta).$

Therefore, by Proposition 5.6 and by the fact that the transfer map is just a multiplication by the degree $[k(\alpha,\beta,\gamma):k]$ of the field extension for the elements contained in the base field, it suffices to show that $\{\alpha-\beta,\alpha-\gamma\}+\{\beta-\gamma,\beta-\alpha\}+\{\gamma-\alpha,\gamma-\beta\}=\{-1,-1\}$ in $K_2^M(k(\alpha,\beta,\gamma))$. But, since $\dfrac{\alpha-\beta}{\gamma-\beta}+\dfrac{\alpha-\gamma}{\beta-\gamma}=1$, we have $\{\dfrac{\alpha-\beta}{\gamma-\beta},\dfrac{\alpha-\gamma}{\beta-\gamma}\}=0$. Hence, $\{\alpha-\beta,\alpha-\gamma\}-\{\alpha-\beta,\beta-\gamma\}-\{\gamma-\beta,\alpha-\gamma\}=0$ and we're done since $-\{\alpha-\beta,\beta-\gamma\}=\{\beta-\gamma,\alpha-\beta\}=\{\beta-\gamma,\beta-\alpha\}+\{\beta-\gamma,-1\}$, $-\{\gamma-\beta,\alpha-\gamma\}=\{\alpha-\gamma,\gamma-\beta\}=\{\gamma-\alpha,\gamma-\beta\}+\{-1,\gamma-\beta\}$, and $\{\beta-\gamma,-1\}+\{-1,\gamma-\beta\}=\{\beta-\gamma,-1\}-\{\gamma-\beta,-1\}=\{-1,-1\}$.

For $l\geq 3$, if we go through a similar argument which is only notationally more complicated, the proof boils down to the computation of the following element in the Milnor's K-group: $(-1)^l\{\vartheta_0-\vartheta_1,\vartheta_0-\vartheta_2,\ldots,\vartheta_0-\vartheta_{l-1},\vartheta_0-\vartheta_l\}+\{\vartheta_1-\vartheta_2,\vartheta_1-\vartheta_3,\ldots,\vartheta_1-\vartheta_l,\vartheta_1-\vartheta_0\}+(-1)^l\{\vartheta_2-\vartheta_3,\vartheta_2-\vartheta_4,\ldots,\vartheta_2-\vartheta_0,\vartheta_2-\vartheta_1\}+\cdots+\{\vartheta_l-\vartheta_0,\vartheta_l-\vartheta_1,\ldots,\vartheta_l-\vartheta_{l-2},\vartheta_l-\vartheta_{l-1}\}$, where none of ϑ_i $(i=0,\ldots,l)$ and their differences are 0. Note that the signs for the $(l+1)$-terms are all plus if l is even and alternating if l is odd. We claim that this expression is equal to $\{-1,\ldots,-1\}$ in $K_l^M(L)$, where $L=k(\vartheta_0,\ldots,\vartheta_l)$.

We regard the indices modulo $l+1$ and write $x_i=\vartheta_0-\vartheta_i$. Then the i-th term $(i=0,1,\ldots,l)$ in the above expression, if we disregard signs, becomes $\{x_{i+1}-x_i,x_{i+2}-x_i,x_{i+3}-x_i,\ldots,x_{i+l}-x_i\}$.

Therefore, the proof is complete by Proposition 2.7. □

The following key result shows the compatibility between these two types of transfer maps.

Proposition 6.5. *For every finite field extension $k\subset L$, we have the following commutative diagram, where the vertical maps are the transfer maps and the horizontal maps are the homomorphisms in Proposition 5.6:*

$$\begin{array}{ccc} K_l^M(L) & \xrightarrow{\rho_l} & GW_l(L) \\ \big\downarrow {\scriptstyle N_{L/k}} & & \big\downarrow {\scriptstyle N_{L/k}} \\ K_l^M(k) & \xrightarrow{\rho_l} & GW_l(k) \end{array}$$

Proof. Because of the functoriality properties of the transfer maps for both Milnor's K-groups and Goodwillie groups, we may assume that $L=k(\alpha)$ is a simple extension of k.

We need to prove that, for each generator $x=\partial_v(y)\in K_l^M(k_v)$ as in Lemma 3.5, we have an equality $N_{k_v/k}(\rho_l(x))=\rho_l(N_{k_v/k}(x))$ in $GW_l(k)$. For such a generator $x\in K_l^M(k_v)$, by Lemma 3.6, we may express $N_{k_v/k}(x)=N_{k_v/k}(\partial_v(y))$ as a sum of

$\pm\{-1,-1,\dots,-1\}$ and $N_{k(v_i)/k}(x_i)$ in $K_l^M(k)$, for $i=1,2,\dots,r$, where $\deg\pi_{v_i} < \deg\pi_v$ and $x_i \in K_l^M(k_{v_i})$ for each i. Note also that, for such a generator $x=\partial_v(y)$, $\partial_w(y)$ vanishes if $\deg(w) \geq \deg(v)$ unless w is equal to v or v_∞. Thanks to Definition 3.3 and Lemma 6.4, we have $\sum_v \rho_l\Big(N_{k_v/k}\big(\partial_v(y)\big)\Big) = 0$ and $\sum_v N_{k_v/k}\Big(\rho_l\big(\partial_v(y)\big)\Big) = 0$. So, the proof of our proposition is complete by induction on the degree of L/k as the proposition holds trivially when $L=k$. □

The proof of the following lemma essentially gives a concrete method of computing the transfer map of particular elements of a Goodwillie group.

Lemma 6.6. *For any field k and commuting matrices $A_1,\dots,A_l \in GL_n(k)$, there exist finite field extensions $L_1,\dots,L_r$ of k and $\alpha_{ij} \in GL_1(L_j) = L_j^\times$ $(1 \leq i \leq l,\ 1 \leq j \leq r)$ such that $\sum_{j=1}^{r} N_{L_i/k}(w_j) = (A_1,\dots,A_l)$ in $GW_l(k)$, where $w_j = (\alpha_{1j},\dots,\alpha_{lj}) \in GW_l(L_j)$ $(j=1,\dots,r)$. Moreover, the choices of the fields L_i $(i=1,\dots,r)$ and the elements $\alpha_{ij} \in L_j^\times$ can be made canonically so that it depends only on given $A_1,\dots,A_l$.*

Proof. Let us write $z=(A_1,A_2,\dots,A_l)$, where $A_1,A_2,\dots,A_l$ are commuting matrices in $GL_n(k)$. We then consider the vector space $E=k^n$ as a $k[t_1,\dots,t_l]$-module, on which t_i acts as A_i. Since E is of finite rank over k, it has a composition series $0=E_0 \subset E_1 \subset \cdots \subset E_r = E$ with simple factors $L_j = E_j/E_{j-1}$ $(j=1,\dots,r)$.

Then, there exists a maximal ideal $\mathfrak{m}_j$ of $k[t_1,\dots,t_l]$ such that $L_j \simeq k[t_1,\dots,t_l]/\mathfrak{m}_j$. So we see that L_j is a finite extension of k, and $z = \sum_{j=1}^{r}(A_1|L_j,\dots,A_l|L_j)$, where $A_i|L_j$ is the invertible linear map from L_j onto itself induced by A_i.

Let us denote by α_{ij} the element of $L_j^\times$ which corresponds to $t_i \pmod{\mathfrak{m}_j}$ for $i=1,\dots,l$, then $(A_1|L_j,\dots,A_l|L_j) = N_{L_j/k}\left((\alpha_{1j},\dots,\alpha_{lj})\right)$. Take $w_j=(\alpha_{1j},\dots,\alpha_{lj})$ and we are done.

Finally, note that the list of the fields L_i $(i=1,\dots,r)$ and the elements $\alpha_{ij} \in L_j^\times$ do not vary, except for the indices, if we choose a different composition series, e.g., by Jordan Hölder theorem. □

An isomorphism between Milnor's K-theory and motivic cohomology of fields was first given by Nesterenko and Suslin ([6]) for Bloch's higher Chow groups. Here, we present a similar result for the Goodwillie group.

Theorem 6.7. *For any field k and $l \geq 1$, the assignment $\{a_1, a_2, \ldots, a_l\} \mapsto (a_1, a_2, \ldots, a_l)$ for each Steinberg symbol $\{a_1, a_2, \ldots, a_l\}$ gives rise to an isomorphism $K_l^M(k) \simeq GW_l(k)$. In fact, it is an graded ring isomorphism between $\bigoplus_{l\geq 0} K_l^M(k)$ and $\bigoplus_{l\geq 0} GW_l(k)$*

Proof. The case $l = 1$ is done in Proposition 4.4. By Proposition 5.6, the assignment $\{a_1, a_2, \ldots, a_l\} \mapsto (a_1, a_2, \ldots, a_l)$ gives rise to a homomorphism ρ_l from the Milnor's K-group $K_l^M(k)$ to the Goodwillie group $GW_l(k)$. We will construct the inverse map $\phi_l : GW_l(k) \to K_l^M(k)$ for $l \geq 2$ as follows.

For each commuting matrices $A_1, \ldots, A_l$ in $GL_n(k)$, by Lemma 6.6, we may canonically find finite field extensions $L_1, \ldots, L_r$ of k and $\alpha_{ij} \in L_j^{\times}$ $(1 \leq i \leq l$, $1 \leq j \leq r)$ such that $(A_1, \ldots, A_l) = \sum_{j=1}^{r} N_{L_j/k}\big((\alpha_{1j}, \ldots, \alpha_{lj})\big)$ in $GW_l(k)$. We set $\phi_l(A_1, \ldots, A_l) = \sum_j N_{L_j/k}(\{\alpha_{1j}, \ldots, \alpha_{lj}\})$, where $\{\alpha_{1j}, \ldots, \alpha_{lj}\}$ is a Steinberg symbol and $N_{L_j/k} : K_l^M(L_j) \to K_l^M(k)$ is the transfer map for the Milnor's K-groups. By Lemma 6.6,ϕ_l gives a map from the set of l-tuples of commuting matrices into $K_l^M(k)$. Then, it is immediate that $\rho_l(\phi_l(A_1, \ldots, A_l)) = (A_1, \ldots, A_l)$ in $GW_l(k)$ by Proposition 6.5. In particular, ρ_l is surjective.

To show that ϕ_l actually gives a map from $GW_l(k) \to K_l^M(k)$, it remains to show that ϕ_l vanishes on the relations (i),(ii),(iii) and (iv) in Definition 4.5. But, ϕ_l clearly vanishes on the relations of type (i), (ii) and (iii) and thus ϕ_l gives rise to a homomorphism from $C(k, l)$ onto $K_l^M(k)$. To verify it for the relation (iv), let $A_1(X), \ldots, A_l(X)$ be commuting matrices in $GL_n(k[X])$, where X is an indeterminate. Then $M = k(X)^n$ can be considered as $k(X)[t_1, \ldots, t_l]$-module, on which t_i acts as $A_i(X)$. Then find a composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ with simple factors $Q_j = M_j/M_{j-1}$ $(j = 1, \ldots, r)$ and maximal ideals $\mathfrak{m}_j$ of $k(X)[t_1, \ldots, t_l]$ such that $Q_j \simeq k(X)[t_1, \ldots, t_l]/\mathfrak{m}_j$. We also denote by β_{ij} the element of $Q_j^{\times}$ which corresponds to $t_i \pmod{\mathfrak{m}_j}$ for $i = 1, \ldots, l$ and $j = 1, \ldots, r$. Now each Q_j is a finite extension field of $k(X)$ and let $x = \sum_{j=1}^{r} N_{Q_j/k(X)}(\{\beta_{1j}, \ldots, \beta_{lj}\}) \in K_l^M(k(X))$. Now consider the element $y = \{x, (X-1)/X\}$ in $K_{l+1}^M(k(X))$, where the symbol $\{x, (X-1)/X\}$ denotes $\sum_u \{x_{1u}, \ldots, x_{lu}, (X-1)/X\}$ if $x = \sum_u \{x_{1u}, \ldots, x_{lu}\}$ in $K_l^M(k(X))$. Then $\partial_v(y) = \phi_l\big((A_1(0), \ldots, A_l(0))\big)$ if $\pi_v = X$ and $\partial_v(y) = \phi_l\big((A_1(1), \ldots, A_l(1))\big)$ if $\pi_v = X - 1$. Also, the image $\partial_v(y)$ is zero unless v is the valuation associated with either $\pi_v = X - 1$ or $\pi_v = X$.

Let me illustrate the proof of these facts when $l = 2$ because it is only notationally more complicated in case $l > 2$. Denote by f and g be the characteristic polynomials of $A_1(X)$ and $A_2(X)$ over the field $k(X)$, respectively. Then coefficients of f and g are in the polynomial ring $k[X]$ and the constant terms are actually in k as $A_1(X)$ and $A_2(X)$ are in $GL_n(k[X])$. For each $j = 1, \dots, r$, β_{1j} and β_{2j} are roots of the characteristic polynomials f_j and g_j over $k(X)$ of automorphisms of Q_j induced by $A_1(X)$ and $A_2(X)$, respectively. We note that $f = \prod_{j=1}^r f_j$ and $g = \prod_{j=1}^r g_j$. Also, let f_j' and g_j' be the monic irreducible polynomials of β_{1j} and β_{2j}, respectively, over $k(X)$. Now recall from Lemma 3.4 that a projection formula for Milnor's K-groups is valid. Thus a similar argument we used in verifying (4) during the proof of Lemma 6.4 shows that $N_{Q_j/k(X)}(\{\beta_{1j}, \mu_{j1}\})$ is equal to an integer multiple of $\{\pm f_j'(0), \pm g_j'(0)\}$, whose coordinates are in $k \subset k(X)$ because f_j' and g_j' are factors of f and g in the polynomial ring over $k(X)$, hence in the polynomial ring over $k[X]$ by a Gauss' Lemma, and thus the constants terms of f_j' and g_j' are in $k \subset k(X)$. Therefore, we may write x as a sum of Milnor symbols with coordinates in $k \subset k(X)$ and the assertion is now clear.

By Definition 3.3, we see that $\phi_l\big((A_1(0), \dots, A_l(0))\big) = \phi_l\big((A_1(1), \dots, A_l(1))\big)$ since $N_v = Id$ if π_v corresponds to a linear polynomial in X. Therefore, ϕ_l gives a well-defined map from $GW_l(k)$ onto $K_l^M(k)$.

Finally, we note that $\phi_l \circ \rho_l$ is the identity map on $K_l^M(k)$ since each Steinberg symbol is fixed by it and the proof is complete. □

Acknowledgment. I thank Prof. Hahn, Sang Geun and Prof. Bae, Sung Han of KAIST for an invitation to give a talk at East Asia Number Theory Conference in Jan. 2008. The present article is an expanded and detailed version of my talk at the conference.

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