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# RECURRENCE PROPERTIES OF INTERVAL EXCHANGE MAPS

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Abstract. If an ergodic system has positive entropy, then the Shannon-McMillan-Breiman theorem provides a relationship between the entropy and the size of an atom of the iterated partition. The system also has Ornstein-Weiss' first return time property, which offers a method of computing the entropy via an orbit. We consider irrational rotations and interval exchange maps which are the typical model of zero entropy. For almost every interval exchange map we show that the logarithm of the recurrence time and hitting time to $r$-neighborhood normalized by $-\log r$ goes to 1.

## 1. Introduction

Let $(X, \mu)$ be a probability space and $T : X \to X$ be a $\mu$-preserving transformation. Celebrated Poincaré's recurrence theorem says that under suitable assumptions a typical trajectory of the system comes back infinitely many times in any neighborhood of its starting point. It is natural to ask that how many iterations of an orbit is necessary to come back within a distance $r$ from the starting point. The quantitative recurrence theory investigates this kind of questions.

Define $\tau_r(x)$ to be the first return time of $x$ into the ball $B(x,r)$ centered in $x$ with radius $r$ and $\tau_r(x,y)$ to be the hitting time of $x$ into the ball $B(y,r)$ centered in $y$ with radius $r$. Then for many "hyperbolic" systems we have

$$\lim_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = d_\mu(x), \quad \lim_{r\to 0} \frac{\log \tau_r(x,y)}{-\log r} = d_\mu(y) \quad \text{a.e.}$$

Another interesting sequence of neighborhoods can be defined, using the symbolic dynamics generated by a partition $\mathcal{P}$. The symbolic name $u = u_1 u_2 u_3 \dots$ of $x$ is

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given by the rule

$$u_i = k \Longleftrightarrow T^i(x) \in P_k, \quad k = 0, \ldots, \ell - 1.$$

Let $\{\mathcal{P}_n\}$ be the sequence of partitions of $X$ according to the first $n$ $\mathcal{P}$-names. Let $P_n(x)$ be the element in $\mathcal{P}_n$ containing $x$. Define $R_n(u)$ to be the smallest $k$ such that $T^k(x) \in P_n(x)$.

**Theorem 1** (Wyner-Ziv([10]), Ornstein-Weiss([8])). *For ergodic processes with entropy $h$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log R_n(u) = h \quad \textit{almost surely.}$$

The Shannon-McMillan-Brieman theorem states that

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu(P_n(x)) = h(T, \mathcal{P}) \qquad \text{a.e.}$$

Therefore, if the entropy $h(T, \mathcal{P}) > 0$, then we have

$$\lim_{n\to\infty} \frac{\log R_n(u)}{-\log \mu(P_n(x))} = 1 \qquad \text{a.e.} \tag{1}$$

But when the entropy is zero, this approach does not applicable. In this talk, we consider entropy zero systems - irrational rotations and interval exchange maps. Not only for almost every irrational rotations, but for almost every interval exchange maps it's shown that (1) holds.

## 2. Irrational rotations

Let $T : [0, 1) \to [0, 1)$ be the rotation by an irrational $\theta$, i.e., $T(x) = x + \theta \pmod 1$. Then $T$ preserves the Lebesgue measure $\mu$. The recurrence property of the irrational rotation is closely related to the Diophantine approximation.

An irrational number $\theta$, $0 < \theta < 1$, is said to be of type $\eta$ if

$$\eta = \sup\{\beta : \varliminf_{j\to\infty} j^\beta \|j\theta\| = 0\},$$

where $\|\cdot\|$ is the distance to the nearest integer ($\|t\| = \min_{n\in\mathbb{Z}} |t - n|$). Every irrational number is of type $\eta \geq 1$. The set of irrational numbers of type 1 (Roth type) has measure 1. A number with bounded partial quotients is of type 1. There exist numbers of type $\infty$, called the Liouville numbers. The relation between the recurrence property of the irrational rotation and the type of the irrational is as follows.

**Theorem 2** (Choe-Seo[1]). *For every $x$, we have*

$$\varliminf_{r\to 0^+} \frac{\log \tau_r(x)}{-\log r} = \frac{1}{\eta}, \qquad \varlimsup_{r\to 0^+} \frac{\log \tau_r(x)}{-\log r} = 1.$$

**Theorem 3** (Kim-Seo[4])**.** *For almost every $x$, we have*

$$\varliminf_{r\to 0^+} \frac{\log \tau_r(x,y)}{-\log r} = 1, \qquad \varlimsup_{r\to 0^+} \frac{\log \tau_r(x,y)}{-\log r} = \eta.$$

Let $\mathcal{P} = \{[0, 1-\theta), [1-\theta, 1)\}$ and $\mathcal{P}_n = \vee_{i=0}^{n-1} T^{-i}\mathcal{P}$ be the partition of $[0,1)$ obtained by the orbit $\{-k\theta\}$, $0 \le k \le n$. Denote the partition of $[0,1)$ by the orbit $\{k\theta\}, 0 \le k \le n$ by $\mathcal{Q}_n$, that is, $\mathcal{Q}_n = \vee_{i=1}^{n} T^{i}\mathcal{P}$.

**Theorem 4** (Kim-Park[5])**.** *We have*

$$\varliminf_{n\to\infty} \frac{-\log \mu(P_n(x))}{\log n} = \frac{1}{\eta}, \quad \varlimsup_{n\to\infty} \frac{-\log \mu(P_n(x))}{\log n} = 1 \qquad a.e$$

*and*

$$\varliminf_{n\to\infty} \frac{\log R_n(u)}{\log n} = \frac{1}{\eta}, \ \varlimsup_{n\to\infty} \frac{\log R_n(u)}{\log n} = 1, \ a.e.$$

*But we have the limit of the ratio as*

$$\lim_{n\to\infty} \frac{\log R_{Q_n(x)}(x)}{-\log \mu(Q_n(x))} = 1 \ a.e. \ x.$$

$$\varliminf_{n\to\infty} \frac{\log R_{P_n(x)}(x)}{-\log \mu(P_n(x))} = 1, \quad \varlimsup_{n\to\infty} \frac{\log R_{P_n(x)}(x)}{-\log \mu(P_n(x))} = \eta \ a.e. \ x,$$

*where $R_E$ is the first return time to a subset $E$.*

## 3. Interval exchange maps

The interval exchange map is a (finite) piecewise isometry on the unit interval, which preserves the orientation. An interval exchange map (i.e.m.) is determined by the combinatorial data: two bijections $(\pi_0, \pi_1)$ from $\mathcal{A}$ (names for the intervals) onto $\{1, \dots, d\}$. ($d =$ card $(\mathcal{A})$) and the length data $(\lambda_\alpha)_{\alpha\in\mathcal{A}}$.

We set

$$I_\alpha := [0, \lambda_\alpha) \times \{\alpha\}, \ 1 = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha, \ I := [0,1).$$

Define, for $\varepsilon = 0, 1$, a bijection $j_\varepsilon$ from $\sqcup_{\alpha\in\mathcal{A}} I_\alpha$ onto $I$:

$$j_\varepsilon(x, \alpha) = x + \sum_{\pi_\varepsilon(\beta)<\pi_\varepsilon(\alpha)} \lambda_\beta \ .$$

The i.e.m. $T$ associated to these data is the bijection $T = j_1 \circ j_0^{-1}$ of $I$ and

$$T(x) = x + \sum_{\pi_1(\alpha)>\pi_1(\beta)} \lambda_\beta - \sum_{\pi_0(\alpha)>\pi_0(\beta)} \lambda_\beta, \ \text{for } x \in I_\alpha$$

Keane[6] showed that if the length data are rationally independent, then the interval exchange map is minimal (i.e., all orbits are dense). Veech and Masur showed that almost every interval exchange map is uniquely ergodic. We refer to [11] and references therein for the general introduction to the interval exchange map.

Let $\alpha_0, \alpha_1 \in \mathcal{A}$, $\pi_0(\alpha_0) = \pi_1(\alpha_1) = d$. Define $\mathcal{R}_0(\pi_0, \pi_1)$, $\mathcal{R}_1(\pi_0, \pi_1)$ as follows:

$$\mathcal{R}_0(\pi_0, \pi_1) = (\pi_0, \hat{\pi}_1), \quad \mathcal{R}_1(\pi_0, \pi_1) = (\hat{\pi}_0, \pi_1),$$

$$\hat{\pi}_1(\alpha) \quad = \quad \begin{cases} \pi_1(\alpha) & \text{if } \pi_1(\alpha) \leq \pi_1(\alpha_0), \\ \pi_1(\alpha) + 1 & \text{if } \pi_1(\alpha_0) < \pi_1(\alpha) < d, \\ \pi_1(\alpha_0) + 1 & \text{if } \alpha = \alpha_1, \ (\pi_1(\alpha_1) = d); \end{cases}$$

$$\hat{\pi}_0(\alpha) \quad = \quad \begin{cases} \pi_0(\alpha) & \text{if } \pi_0(\alpha) \leq \pi_0(\alpha_1), \\ \pi_0(\alpha) + 1 & \text{if } \pi_0(\alpha_1) < \pi_0(\alpha) < d, \\ \pi_0(\alpha_1) + 1 & \text{if } \alpha = \alpha_0, \ (\pi_0(\alpha_0) = d). \end{cases}$$

Each vertex $(\pi_0, \pi_1)$ being the origin of two arrows joining $(\pi_0, \pi_1)$ to $\mathcal{R}_0(\pi_0, \pi_1)$, $\mathcal{R}_1(\pi_0, \pi_1)$. The name of an arrow joining $(\pi_0, \pi_1)$ to $\mathcal{R}_\varepsilon(\pi_0, \pi_1)$ ($\varepsilon \in \{0, 1\}$) is the element $\alpha_\varepsilon \in \mathcal{A}$ such that $\pi_\varepsilon(\alpha_\varepsilon) = 1$.

Let $T$ be an i.e.m. by $(\pi_0, \pi_1)$, $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$. $T$ is of type $\varepsilon$ if one has $\lambda_{\alpha_\varepsilon} \geq \lambda_{\alpha_{1-\varepsilon}}$; ($\pi_\varepsilon(\alpha_\varepsilon) = d$ as above ) Define a new i.e.m. $\mathcal{V}(T)$ by the admissible pair $\mathcal{R}_\varepsilon(\pi_0, \pi_1)$ and the lengths $(\hat{\lambda}_\alpha)_{\alpha \in \mathcal{A}}$ given by

$$\begin{cases} \hat{\lambda}_\alpha = \lambda_\alpha & \text{if } \alpha \neq \alpha_\varepsilon, \\ \hat{\lambda}_{\alpha_\varepsilon} = \lambda_{\alpha_\varepsilon} - \lambda_{\alpha_{1-\varepsilon}} & \text{otherwise,} \end{cases}$$

i.e. the length data of $T$ are obtained from those of $\mathcal{V}(T)$ as

$$\lambda = V(T)\hat{\lambda}$$

where the matrix $V(T)$ has all diagonal entries equal to 1 and all off-diagonal entries equal to 0 except the one corresponding to $(\alpha_\varepsilon, \alpha_{1-\varepsilon})$ which is also equal to 1. The i.e.m. $\mathcal{V}(T)$ is the first return map of $T$ on $\left[0, \sum_{\alpha \neq \alpha_{1-\varepsilon}} \lambda_\alpha\right)$. We also associate to $T$ the arrow in the Rauzy diagram joining $(\pi_0, \pi_1)$ to $\mathcal{R}_\varepsilon(\pi_0, \pi_1)$. Iterating this process, we obtain a sequence of i.e.m $(\mathcal{V}^k(T))_{k \geq 0}$ and an infinite path in the Rauzy diagram starting from $(\pi_0, \pi_1)$.

Let $T = T(0)$, define $T(n) = \mathcal{V}^{k(n)}(T)$ by the following property: $k(n+1)$ is the largest integer $k > k(n)$ such that not all names in $\mathcal{A}$ are taken by arrows associated to iterations of $\mathcal{V}$ from $T(n)$ to $\mathcal{V}^k(T)$. Let $(T(n))_{n \geq 0}$ be the sequence of i.e.m. obtained by the accelerated Zorich algorithm, with associated lengths $(\lambda_\alpha(n))_{\alpha \in \mathcal{A}}$. A matrix $Z(n) \in \mathrm{SL}\,(d, \mathbb{Z})$ with non negative entries such that

$$\lambda(n-1) = Z(n)\lambda(n).$$

We write, for $m < n$

$$Q(m,n) = Z(m+1)\cdots Z(n), \quad Q(n) = Q(0,n).$$

Marmi, Moussa and Yoccoz showed that almost every interval exchange map the following "diophantine" condition holds:
*(A) for any $\varepsilon > 0$ there exist $C_\varepsilon > 0$ such that for all $n \geq 1$ we have*

$$\|Z(n+1)\| \leq C_\varepsilon \|Q(n)\|^\varepsilon.$$

Note that there are interval exchange maps with condition (A) which are not uniquely ergodic. Using Condition (A) we have the following theorem:

**Theorem 5** (Kim-Marm[3])**.** *For almost every interval exchange maps we have*

$$\lim_{r\to 0^+} \frac{\log \tau_r(x)}{-\log r} = 1, \qquad \lim_{r\to 0^+} \frac{\log \tau_r(x,y)}{-\log r} = 1, \qquad a.e.\ x$$

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