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HYPERBOLICITY OF C^1-STABLY SHADOWING HOMOCLINIC CLASSES

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Abstract. Let f be a diffeomorphism on a closed manifold M, and let $p \in M$ be a hyperbolic periodic point of f. Denote $H_f(p)$ the homoclinic class of f associated to p. We say that $H_f(p)$ is C^1-stably shadowing if $H_f(p)$ is locally maximal (in $U \subseteq M$) and there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the shadowing property on Λ_g, where $\Lambda_g = \cap_{n\in\mathbb{Z}} g^n(U)$ and which is called the *continuation* of $H_f(p) = \cap_{n\in\mathbb{Z}} f^n(U)$. We show in this paper that $H_f(p)$ is C^1-stably shadowing if and only if $H_f(p)$ is hyperbolic.

The notion of pseudo orbit very often appears in several branches of modern theory of dynamical systems([10]), and the pseudo orbit shadowing property usually plays an important role in the investigation of stability theory and ergodic theory.

Let X be a compact metric space with metric d, and let f be a homeomorphism of X. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b}$ in X $(-\infty \le a < b \le \infty)$ is called a *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$. Let $\Lambda \subseteq X$ be a closed f-invariant set. We say that f has the *shadowing property* on Λ if for any $\epsilon > 0$ there is a $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=a}^{b} \subseteq \Lambda$ of $f(-\infty \le a < b \le \infty)$, there is $y \in X$ that ϵ-shadowes the pseudo orbit; that is, $d(f^i(y), x_i) < \epsilon$ for all $a \le i \le b-1$. Notice that, in this definition, only δ-pseudo-orbits of f "contained in Λ" can be ϵ-shadowed, but the shadowing point $y \in X$ is "not necessarily" contained in Λ. This property does not depend on the metric used and is preserved

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under topological conjugacy. It is easy to see that f has the shadowing property on Λ if and only if f^n has the shadowing property on Λ for each $n \in \mathbb{Z}$.

Let M be a closed C^∞ manifold, and let $\text{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. It is well known that if p is a hyperbolic periodic point of f with period k then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\} \text{ and}$$
$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M. A point $x \in W^s(p) \cap W^u(p)$ is called a *homoclinic point* of f associated to p, and it is said to be a *transversal homoclinic point* of f if the above intersection is transversal at x; i.e., $x \in W^s(p) \pitchfork W^u(p)$. The closure of the homoclinic points of f associated to p is called the *homoclinic class* of f associated to p, and it is denoted by $H_f(p)$. The closure of the transversal homoclinic points of f associated to p is called the *transversal homoclinic class* of f associated to p, and it is denoted by $H_f^T(p)$. It is clear that both $H_f(p)$ and $H_f^T(p)$ are compact f-invariant sets. Homoclinic classes are the natural candidates to replace hyperbolic basic sets in nonhyperbolic theory.

Let q be a hyperbolic periodic point of f. We say that p and q are *homoclinic related*, and write $p \sim q$ if

$$W^s(p) \pitchfork W^u(q) \neq \phi \text{ and } W^u(p) \pitchfork W^s(q) \neq \phi.$$

It is clear that if $p \sim q$ then index(p) = index(q); i.e., $\dim W^s(p) = \dim W^u(q)$. By the Smale's transeverse homoclinic point theorem, $H_f^T(p)$ coincides with the closure of the set of hyperbolic periodic points q of f such that $p \sim q$. Note that if p is a hyperbolic periodic point of f then there is a neighborhood U of p and a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$ there exists a unique hyperbolic periodic point p_g of g in U with the same period as p and index(p_g) = index(p). Such that point p_g is called the *continuation* of $p = p_f$.

Note that,when a transverse homoclinic class is not hyperbolic, it may contain periodic points having different indices. Actually, there are examples of diffeomorphisms with transverse homoclinic classes containing hyperbolic periodic points with different indices in a robust way. These systems do not have the shadowing property in general. One of the main obstructions for shadowing is the existence of hetrodymensional cycles. Such dynamical phenomena easily give pseudo orbit which can not be shadowable.

For given $x, y \in X$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo-orbit $\{x_i\}_{i=a_\delta}^{b_\delta}$ $(a_\delta < b_\delta)$ of f such that $x_{a_\delta} = x$ and $x_{b_\delta} = y$. Write $x \leftrightsquigarrow y$

if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The set of points $\{x \in X : x \leftrightsquigarrow x\}$ is called the *chain recurrent set* of f and which is denoted by $\mathcal{R}(f)$. It is easy to see that the set is closed and $f(\mathcal{R}(f)) = \mathcal{R}(f)$. If we denote the set of periodic points of f by $P(f)$, then $P(f) \subset \Omega(f) \subset \mathcal{R}(f)$. Here $\Omega(f)$ is the non-wandering set of f. The relation $\leftrightsquigarrow$ induces on $\mathcal{R}(f)$ an equivalence relation, whose classes are called *chain components* of f. These are compact invariant sets and can not be decomposed into two disjoint compact invariant sets and so serve as "elementary pieces" of the dynamical systems. Denote $C_f(p)$ the chain component of f that contains a hyperbolic periodic point of f. We say $C_f(p)$ is C^1-*stably shadowable* if there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for every $g \in \mathcal{U}(f)$, g has the shadowing property on $C_g(p_g)$, where p_g is the continuation of p and $C_g(p_g)$ is the chain component of g containing p_g.

Very recently, Sakai and Wen *et al.* [11-12] proved that if $C_f(p)$ is C^1-stably shadowable, then $C_f(p)$ is the hyperbolic homoclinic class of f associated to p. Also Lee *et al.* [8] showed that $C_f(p)$ is C^1-stably shadowing if and only if $C_f(p)$ is hyperbolic.

In this paper, we introduce the notion of C^1-stably shadowing for a closed f-invariant subset of M (see also [8]), and show the homoclinic class $H_f(p)$ of f associated to p is hyperbolic if and only if $H_f(p)$ is C^1-stably shadowing.

Definition 1. We say that a closed f-invariant set Λ is C^1*-stably shadowing* if Λ is locally maximal (in U) and there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the shadowing property on Λ_g. Here $\Lambda_g = \cap_{n\in\mathbb{Z}} g^n(U)$ and which is called the *continuation* of $\Lambda_f = \Lambda = \cap_{n\in\mathbb{Z}} f^n(U)$.

Definition 2. Let p be a hyperbolic periodic point of f. We say that the homoclinic class $H_f(p)$ is C^1*-stably shadowable* if there exists a C^1-neighborhood $\mathcal{U}(f)$ of f such that for every $g \in \mathcal{U}(f)$, g has the shadowing property on $H_g(p_g)$, where p_g is the continuation of p and $H_g(p_g)$ is the homoclinic class of g associated to p_g. The same definition can be applied to $H_f^T(p)$.

Recall that a compact invariant set Λ is called *hyperbolic* for f if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist constants $C > 0, 0 < \lambda < 1$ such that

$$\|Df^n|_{E(x)}\| \leq C\lambda^n$$

and

$$\|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

Our main theorems are the following.

Theorem A. *Let p be a hyperbolic periodic point of f. Then the homoclinic class $H_f(p)$ is C^1-stably shadowing if and only if $H_f(p)$ is hyperbolic.*

Theorem B. *Let p be a hyperbolic periodic point of f. If the homoclinic class $H_f(p)$ is C^1-stably shadowable then $H_f(p)$ is hyperbolic.*

References

[1] F. Abdenur and L.J.Diaz, Pseudo-orbit shadowing in the C^1topology, Discrete and continuous Dynam. Sys., **17** (2007), 223-245.

[2] C. Bonatti and S. Crovisier, Recurrence et genericite, Invent. Math, **158** (2004) 33-104.

[3] C. Bonatti and L.J. Diaz, Persistence of transitive diffeomorphisms, Annal of Math., **143** (1995), 367-396.

[4] C. Bonatti, L.J. Diaz and G. Turcat, Pas de " shadowinh lemma " pour les dynamiques partiellment hyperboliques, R. Acad. Sci. Paris, Ser. I, **330** (2000), 587-592.

[5] C. Bonatti, L.J. Diaz and M. Viana, "dynamics Beyond Uniform Hyperbolicity," Encyclopedia of mathematical sciences (mathematical Physics), **102**(Springer- Verlag, Berlin, 2005).

[6] L. J. Diaz and J.Rocha, Partially hyperbolic and transitive dynamics generated by hetroclinic cycles, Ergodic. Th. Dynam. Sys., **21** (2001), 25-76

[7] J. Franks, *Necessary conditions for stability of diffeomorphisms*, Trans. Amer. Math. Soc. **158** (1971), 301-308.

[8] K. Lee, K. Moriyasu and K. Sakai, *C^1-stable shadowing diffeomorphisms*, Contin. Dyn. Syst. **22** (2008)

[9] R. Mañé, *An ergodic closing lemma*, Ann of Math. **116** (1982), 503-540.

[10] S. Yu. Pilyugin, " Shadowing In dynamical Systems, " Lecture Note In math.**1706** (springer-Verlage, Berlin, 1999)

[11] K. Sakai, *C^1-stably shadowable chain components*, Ergod. Th. & Dynam. Sys.**28**(2008) 987-1029.

[12] X. Wen, S. Gan and L. Wen, C^1-stably shadowable chain components are hyperbolic, J. Differential Equations, in press.

[13] G. C. Yuan and J.A. Yorke, an open set of maps for wich every points is absolutely non shadowable, Proc. Amer. Math. Soc., **128** (2000),909-918

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