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ON THE FIRST HOMOLOGY OF THE AUTOMORPHISM GROUPS OF G-MANIFOLDS

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Abstract. We have been calculated the first homology group of the equivarinat diffeomorphism, the equivariant Lipschitz homeomorphism group and the equivariant homeomorphism group of G-manifolds. Those first homology groups are quite different by the category of the automorphism group and reflect the feature of the category. We shall describe the results and give the outline of the proof.

§0. Preliminaries

In this paper we shall describe the recent results on the first homology group of automorphims of smooth G-manifolds. Here the automorphism means equivariant diffeomorphism, equivariant Lipschitz homeomorphism or equivariant homeomorphism.

Let M be a connected smooth manifold and $\mathcal{D}(M)$ denote the group of diffeomorphisms of M which are isotopic to the identity through diffeomorphisms with compact support. It is known that the group structure of $\mathcal{D}(M)$ determine the smooth structure of M (Filipkiewicz [FI]). Thus $\mathcal{D}(M)$ has a rich group structure but it is a huge group. Hermann [H] and Thurston[TH] proved that $\mathcal{D}(M)$ is a perfect group. The result is able be applied to the foliation theory. There are many analogous results on the group of diffeomorphisms of M which preserve a geometric structure of M.

We treat in the case where M is a smooth G-manifold with G a compact Lie group and calculate the first homology group of the automorphism group. Then we shall see that the first homology group depends on the corresponding category of the automorphism.

The paper is organized as follows. In §1 we review the structure of the commutator subgroup of $\mathcal{D}(M)$. In §2 we describe the first homology group of the

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equivariant diffeomorphism group. In §3 and §4 we treat in the case of the equivariant Lipschitz homeomorphism group and equivariant homeomorphims group, respectively. In §5 we give some examples. In §6, §7 and §8 we show the point of the proof of the main results by using the standard example in §5.

§1. Diffeomorphism groups

Let M be a connected smooth manifold and $\mathcal{D}(M)$ denote the group of diffeomorphisms of M which are isotopic to the identity through diffeomorphisms with compact support.

A group K is said to be perfect if K coincides with the commutator subgroup $[K,K]$ of K. The first homology group of K is given by $H_1(K) = K/[K,K]$.

Theorem 1.1. (Herman [H], Thurston[TH]) *$\mathcal{D}(M)$ is perfect.*

If M is a smooth manifold with boundary, then we have.

Theorem 1.2.
(1) $H_1(\mathcal{D}([0,1])) \cong \mathbf{R} \oplus \mathbf{R}$ (Fukui [FU]).
(2) *If M is a smooth manifold with boundary of dimension greater than one, then $\mathcal{D}(M)$ is perfect* ([AF9]).

§2. Equivariant diffeomorphism groups

Let $\mathcal{D}_G(M)$ denote the group of equivariant diffeomorphisms of M which are G-isotopic to the identity through equivariant diffeomorphisms with compact support

When M is a smooth orbifold, combining Theorem 1.1 and Theorem 1.2 with the lifting theorem of the isotopies from the orbit space by Bierstone [BI] and Schwartz [S], we determine the first homology group $H_1(\mathcal{D}(M))$ of diffeomorphisms of M completely ([AF7], Theorem 4.4). Then we can apply the result to calculate $H_1(\mathcal{D}_G(M))$ when G is a finite group ([AF7], Theorem 4.6).

In this section we shall describe the results on the first homology group $\mathcal{D}_G(M)$ when G is a compact Lie group.

When M is a smooth G-manifold with a free G-action or with one orbit type, we have the following.

Theorem 2.1. (Banyaga [BA], [AF1]) *Let G be a compact Lie group and M be a smooth connected G-manifold with* $\dim M/G > 0$. *If G acts freely on M or M has one orbit type, then $\mathcal{D}_G(M)$ is perfect.*

Now we consider the case when M is a connected smooth G-manifold with codimension one orbit.

Theorem 2.2. ([AF2]) *Let V be a G-module such that G acts transitively on the unit sphere $S(V) = G/H$ of V. Then*

$$H_1(\mathcal{D}_G(V)) \cong \mathbf{R} \times H_1((N(H)/H)_0).$$

Here $N(H)$ is the normalizer of the group H in G.

If M is a connected smooth G-manifold such that M/G is homeomorphic to S^1, then M has one orbit type. Then it follows from Theorem 2.1 that $\mathcal{D}_G(M)$ is perfect. If M/G is homeomorphic to $[0,1]$, then M has two or three orbit types.

Let (H) be the principal orbit type of M and $(K_0), (K_1)$ be the singular orbit types of M. Set

$$W(M) = \left(\frac{N(H) \cap N(K_0)}{H} \times \frac{N(H) \cap N(K_1)}{H}\right)_0$$

Theorem 2.3. ([AF2]) *If M/G is homeomorphic to $[0,1]$, then*

$$H_1(\mathcal{D}_G(M)) \cong \mathbf{R}^2 \times H_1(W(M)).$$

§3. Equivariant Lipschitz homeomorphism groups

In this section we consider the Lipschitz homeomorphism groups of smooth manifolds and the equivariant Lipschitz homeomorphism groups of smooth G-manifolds.

Let M and N be smooth manifolds. A map $f : M \to N$ is called *a Lipschitz map* if for any $p \in M$ there exist local charts (U, φ) around p and (V, ψ) around $f(p)$ with $f(U) \subset V$ such that there exists a positive number L such that

$$|(\psi \circ f \circ \varphi^{-1})(x) - (\psi \circ f \circ \varphi^{-1})(y)| \leq L|x - y|, \quad (x, y \in \varphi(U)).$$

f is said to be *a Lipschitz homeomorphism* if f and f^{-1} are both Lipschitz homeomorphisms. Let $\mathcal{L}(M)$ denote the group of all Lipschitz homeomorphisms of M and $L(M)$ denote the group of Lipschitz homeomorphisms of M which are isotopic to the identity through Lipschitz homeomorphisms with compact supports.

We also consider the compact open Lipschitz topology on the group of all Lipschitz homeomorphism on M as the following. Let $f \in \mathcal{L}(M)$. Let (U, φ) and (V, ψ) be charts on M and $K \subset U$ be a compact subset such that $f(K) \subset V$.
Let $\mathcal{N}(f; (U, \varphi), (V, \psi), K, \varepsilon)$ be the set of Lipschitz homeomorphisms g of M such that $g(K) \subset V$ satisfying

(1) $|(\psi \circ f \circ \varphi^{-1})(x) - (\psi \circ g \circ \varphi^{-1})(x)| < \varepsilon$ for $x \in K$.

(2) $|((\psi\circ f\circ\varphi^{-1})(x) - (\psi\circ g\circ\varphi^{-1})(x)) - ((\psi\circ f\circ\varphi^{-1})(y) - (\psi\circ g\circ\varphi^{-1})(y))| < \varepsilon|x-y|$ for $x, y \in K$.

The topology on $\mathcal{L}(M)$ generated those set $\mathcal{N}(f; (U, \varphi), (V, \psi), K, \varepsilon)$ is called *the compact open Lipschitz topology.* Let $\mathcal{H}_{LIP}(M)$ denote the group of Lipschitz homeomorphism of M which are isotopic to the identity through Lipschitz homeomorphisms with the compact open Lipschitz topology Then we have the following.

Theorem 3.1. ([AF3]) *$L(M)$, $\mathcal{H}_{LIP}(M)$ are perfect.*

Let $L(\mathbf{R}^n, \{0\})$ $(\mathcal{H}_{LIP}(\mathbf{R}^n, \{0\}))$ be the the group Lipschitz homeomorphisms of $\mathbf{R}^n$ which are isotopic to the identity through Lipschitz homeomorphisms fixing the origin 0 in compact open topology (in compact open Lipschitz topology)

Theorem 3.2. (Tsuboi [TS])
$L(\mathbf{R}, \{0\}) = \mathcal{H}_{LIP}(\mathbf{R}, \{0\})$ is perfect.

Theorem 3.3. ([AF3],[AFM])
(1) *$\mathcal{H}_{LIP}(\mathbf{R}^n, \{0\})$ $(n \geq 1)$ is perfect.*
(2) *$H_1(L(\mathbf{R}^2, \{0\}))$ admits a continuous moduli.*

Let G be a compact Lie group and M be a smooth G-manifold. $L_G(M)$ $(\mathcal{H}_{LIP,G}(M)$: the group of equivariant Lipschitz homeomorphisms of a smooth G-manifold M which are isotopic to the identity through equivariant Lipschitz homeomorphisms with compact support in compact open topology (in compact open Lipschitz topology).

Theorem 3.4. ([AF3]) *If G acts on M freely or M has one orbit type, then $L_G(M)$, $\mathcal{H}_{LIP,G}(M)$ are perfect.*

Theorem 3.5. ([AF4]) *Let G be a finite group and let M be a smooth G-manifold. Then $\mathcal{H}_{LIP,G}(M)$ is perfect.*

In the next we consider the case where M has a codimension one orbit. Let $\mathcal{C}(\mathbf{R})$ be the set of real valued functions f on $(0,1]$ such that there exists a positive number K satisfying

$$|f(x) - f(y)| \leq \frac{K}{x}(y-x) \quad \text{for } 0 < x \leq y \leq 1.$$

Let $\mathcal{C}_0(\mathbf{R})$ be the subspace of those $f \in \mathcal{C}(\mathbf{R})$ with f bounded on $(0,1]$.

Theorem 3.6. ([AFM])

$$H_1(L_{U(n)}(\mathbf{C}^n)) \cong \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R}).$$

Theorem 3.7. ([AF8]) *Let V be a G-module such that G acts on $S(V)$ transitively. Then the group $\mathcal{H}_{LIP,G}(V)$ is perfect.*

Let M be a smooth G-manifold with codimension one orbit such that M/G is homeomorphic to $[0,1]$. Set

$$\bar{W}(M) = \left(\frac{N(H) \cap N(K_0)}{K_0 \cap N(K_0)} \times \frac{N(H) \cap N(K_1)}{K_1 \cap N(K_1)}\right)_0.$$

Analyzing the behavior of Lipschitz homeomorphisms around the singular orbits and using Theorem 3.7, we have the following.

Theorem 3.8. ([AF8]) $H_1(\mathcal{H}_{LIP,G}(M)) \cong H_1(\bar{W}(M))$.

§4. Equivariant homeomorphism groups

In this section we consider the homeomorphism groups of smooth manifolds and equivariant homeomorphism groups of smooth G-manifolds.

Let $\mathcal{H}(M)$ be the group of homeomorphisms of M which are isotopic to the identity through homeomorphisms with the compact open topology

Theorem 4.1. (Mather [MA]) *$\mathcal{H}(\mathbf{R}^n)$ is perfect.*

Using the fragmentation theorem by Edwards and Kirby [EK] we see that $\mathcal{H}(M)$ is perfect for a smooth manifold M.

Theorem 4.2. (Tsuboi [TS]) *$\mathcal{H}([0,1])$ is perfect.*

Let G be a compact Lie group and M be a smooth G-manifold. Let $\mathcal{H}_G(M)$ denote the group of equivariant homeomorphisms of a smooth G-manifold M which are isotopic to the identity through equivariant homeomorphisms with compact support with compact open topology.

Theorem 4.3. (Rybicki [RY]) *If G acts on M freely or M is a G-manifold with one orbit type, then $\mathcal{H}_G(M)$ is perfect.*

Let V be a G-module such that G acts on the unit sphere $S(V)$ transitively. Let H be the principal isotropy subgroup.

Theorem 4.4. ([AFR]) *Let V be a G-module such that G acts on the unit sphere $S(V) = G/H$ transitively. Then $\mathcal{H}_G(V)$ is perfect.*

Let M be a smooth connected G-manifold such that M/G is homeomorphic to $[0,1]$. Let (H) be the principal orbit type of M and $(K_0), (K_1)$ be the singular orbit types of M. Then we have the following.

Theorem 4.5. ([AFR])

$$H_1(\mathcal{H}_G(M)) \cong H_1(\bar{W}(M)).$$

§5. Examples

Example 5.1. *Let $G = U(1)$ and $M = \mathbf{C}$ with the canonical $U(1)$-action. Applying Theorems* 2.2, 3.6, 3.7 and 4.4 *we have.*

$$\begin{aligned} H_1(\mathcal{D}_{U(1)}(\mathbf{C})) &\cong U(1) \times \mathbf{R}, \\ H_1(\mathcal{H}_{LIP,G}(M)) &= 0, \\ H_1(L_G(M)) &\cong \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R}), \\ H_1(\mathcal{H}_G(M)) &= 0. \end{aligned}$$

Example 5.2. *Let $G = U(1) \times U(1)$. Let $M = S^3 = \{(z_1, z_2) \in \mathbf{C}^2 |\ |x_1|^2 + |z_2|^2 = 1\}$ with the canonical G-action given by*

$$(u_1, u_2) \cdot (z_1, z_2) = (u_1 z_1,\ u_2 z_2), \qquad (u_1, u_2) \in G,\ (z_1, z_2) \in S^3.$$

Then

$$\begin{aligned} &H = \{1\},\ K_0 = K_1 = U(1),\ N(H) = \{1\}, \\ &W(M) = \bar{W}(M) = U(1) \times U(1). \end{aligned}$$

Applying Theorems 2.3, 3.6, 3.8 and 4.5 *we have.*

$$\begin{aligned} H_1(\mathcal{D}_G(M)) &\cong \mathbf{R}^2 \times U(1)^4. \\ H_1(\mathcal{H}_{LIP,G}(M)) &\cong U(1) \times U(1) \\ H_1(L_G(M)) &\cong \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R}) \times \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R}) \\ H_1(\mathcal{H}_G(M)) &\cong U(1) \times U(1) \end{aligned}$$

§6. $H_1(\mathcal{D}_{U(1)}(\mathbf{C}))$

The rest of the paper we shall see the point of the proof of the main results by using Example 5.1.

In this section we shall give the outline of the proof of the following.

$$H_1(\mathcal{D}_{U(1)}(\mathbf{C})) \cong U(1) \times \mathbf{R}. \tag{6.1}$$

Note that $\mathbf{C}/U(1)$ is homeomorphic to the half line $\mathbf{R}_+$. Then the natural projection $V \to \mathbf{C}/U(1)$ is identified with

$$\pi:\ \mathbf{C} \to \mathbf{C}/U(1) \approx \mathbf{R}_+, \quad \pi(z) = |z|^2.$$

Let $P:\ \mathcal{D}_{U(1)}(\mathbf{C}) \to \mathcal{D}(\mathbf{R}_+)$ be a group homomorphism defined by

$$P(h)(x) = |h(\sqrt{x})|^2 \quad (x \in \mathbf{R}_+),$$

and let $\Psi : \mathcal{D}(\mathbf{R}_+) \to \mathcal{D}_{U(1)}(\mathbf{C})$ be a map defined by

$$\Psi(f)(z) = \begin{cases} \dfrac{\sqrt{f(|z|^2)}}{|z|} z & (z \neq 0) \\ 0 & (z = 0). \end{cases}$$

Then Ψ is a well defined group homomorphism which is a right inverse of P.

For each $h \in KerP$ let $a_h : \mathbf{R}_{>0} \to U(1)$ be the map satisfying

$$h(x) = x a_h(x^2) \quad \text{for} \quad x \in \mathbf{R}_{>0}.$$

Lemma 6.1.

(1) *a_h is a smooth map.*

(2) *We can extend the map a_h to the smooth map $\bar{a}_h : \mathbf{R}_+ \to U(1)$.*

(3) *Let $L :\ KerP \to C^\infty(\mathbf{R}_+)$ be a map given by $L(h) = a_h^{-1}$. Then L is a group isomorphism.*

Let $\Phi :\ \mathcal{D}_{U(1)}(\mathbf{C}) \to U(1)$ be a map defined by

$$\Phi(h) = \bar{a}_{(\Psi(P(h^{-1})\circ h)^{-1}}.$$

Put

$$\Theta = (\Phi, P) :\ \mathcal{D}_{U(1)}(\mathbf{C}) \to U(1) \times \mathcal{D}(\mathbf{R}_+).$$

Lemma 6.2.

Then Θ is a group epimorphism.

Let $E : \mathbf{R} \to U(1)$ denotes the exponential map. We shall prove that each $h \in Ker\,\Theta$ can be written as a composition of elements in $[Ker\,\Theta,\ \mathcal{D}_{U(1)}(\mathbf{C})]$.

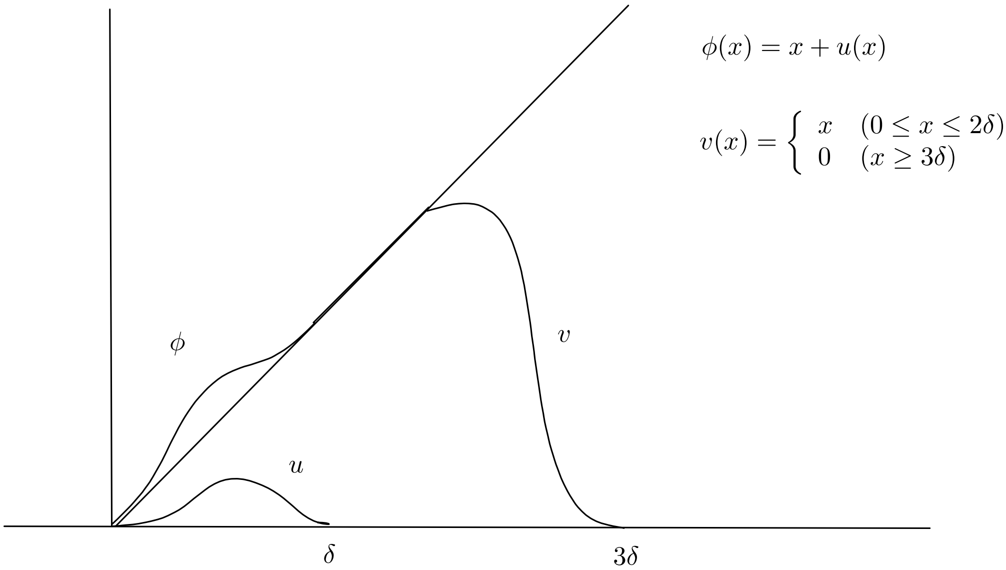
We can assume that h is C^1-close to the identity and by the fragmentation theorem *supp*(h) is contained in the δ disk in $\mathbf{C}$ for $\delta > 0$. Note that $h \in KerP$ and $\bar{a}_h(0) = 1$. Let $\alpha_h : \mathbf{R}_+ \to \mathbf{R}$ be the map such that $E \circ \alpha_h = \bar{a}_h$ with $\alpha_h(0) = 0$. Then α_h is C^1-close to zero and *supp*$(\alpha_h) \subset [0, \delta]$.

Lemma 6.3. *For $\delta > 0$, let $u :\ \mathbf{R}_+ \to \mathbf{R}$ be a C^∞ function supported in $[0, \delta)$ which is C^1-close to the zero map and $u(0) = 0$. Then there exist a C^∞ function $v : \mathbf{R}_+ \to \mathbf{R}$ and $\phi \in \mathcal{D}(\mathbf{R}_+)$ such that*

(1) *supp*$(v) \subset [0, 3\delta)$, $|v(x)| \leq \delta$,

(2) *supp*$(\phi) \subset [0, 3\delta)$ *and ϕ is isotopic to the identity through a C^∞ isotopy supported in $[0, 3\delta)$,*

(3) $u = v \circ \phi - v$



By Lemma 6.3 there exist a C^∞ function $v : \mathbf{R}_+ \to \mathbf{R}$ and $\phi \in \mathcal{D}(\mathbf{R}_+)$ such that $a_h = v \circ \phi - v$. Let $h_1 = L(E \circ v)$ and $h_2 = \Psi(\phi)$. Then we have the following.

Lemma 6.4.

$$h = h_1^{-1} \circ h_2^{-1} \circ h_1 \circ h_2 \in [Ker\,\Theta,\ \mathcal{D}_{U(1)}(\mathbf{C})].$$

There exists the following exact sequence:

$$Ker\Theta/[Ker\Theta,\ \mathcal{D}_{U(1)}(\mathbf{C})] \overset{\iota_*}{\to} H_1(\mathcal{D}_{U(1)}(\mathbf{C}))$$
$$\overset{\Theta_*}{\to} H_1(U(1) \times \mathcal{D}(\mathbf{R}_+)) \to 1$$

From Lemma 6.4 and Theorem 1.2 we have.

$$H_1(\mathcal{D}_{U(1)}(\mathbf{C})) \cong H_1(U(1) \times \mathcal{D}(\mathbf{R}_+)) \cong U(1) \times \mathbf{R}.$$

Therefore (6.1) follows.

§7. $\mathcal{H}_{LIP,U(1)}(\mathbf{C})$

In this section we shall prove that $\mathcal{H}_{LIP,U(1)}(\mathbf{C})$ is perfect.

Let $P :\ \mathcal{H}_{LIP,U(1)}(\mathbf{C}) \to \mathcal{H}_{LIP}(\mathbf{R}_+)$ be a group homomorphism defined by

$$P(h)(x) = |h(\sqrt{x})|^2 \quad (x \in \mathbf{R}_+),$$

and let $\Psi : \mathcal{H}_{LIP}(\mathbf{R}_+) \to \mathcal{H}_{LIP,U(1)}(\mathbf{C})$ be a map defined by

$$\Psi(f)(z) = \begin{cases} \dfrac{\sqrt{f(|z|^2)}}{|z|}\, z & (z \neq 0) \\ 0 & (z = 0). \end{cases}$$

Then Ψ is a well defined group homomorphism which is a right inverse of P.

For each $h \in KerP$ let $a_h : \mathbf{R}_{>0} \to U(1)$ be the map satisfying

$$h(x) = xa_h(x^2) \quad \text{for} \quad x \in \mathbf{R}_{>0}.$$

By the fragmentation theorem we can assume, for a positive number δ ($\delta < \frac{1}{6}$), that

(1) supp(h) is contained in the unit disk in $\mathbf{C}$,

(2) h is δ-close to the identity in compact open Lipschitz topology.

Then $a_h(x) = 1$ for $x \geq 1$. Let $\hat{a}_h : \mathbf{R}_{>0} \to \mathbf{R}$ be a function such that

(1) $E(\hat{a}_h(x)) = a_h(x^2)$,

(2) $\hat{a}_h(x) = 0$ for $x \geq 1$.

We say that a map $\alpha : (0,1] \to \mathbf{R}$ satisfy the *condition* (L) if there exists a positive number K such that

$$|\alpha(x) - \alpha(y)| \leq \frac{K}{x}(y-x) \quad for \quad 0 < x \leq y \leq 1.$$

Lemma 7.1. *$\hat{a}_h$ is bounded and satisfies the condition (L).*

Conversely assume that α satisfies the condition (L) and let

$$h_\alpha(xz) = \begin{cases} xzE(\alpha(x)) & 0 < x \leq 1,\ z \in U(1) \\ 0 & x = 0. \end{cases}$$

Lemma 7.2. *If $\alpha : (0,1] \to \mathbf{R}$ is a bounded map satisfying the condition (L) with $\alpha(x) = 0$ ($x \geq 1$). Then $h_\alpha \in KerP$.*

Lemma 7.3. *There exist maps β and γ from $(0,1]$ to $\mathbf{R}$ such that*

(1) *β and γ are bounded and satisfy the condition (L).*

(2) *$h_\beta \circ h_\gamma = h_{\hat{a}_h} = h$.*

(3) *$supp(\beta) \subset \bigcup_{k=1}^{\infty}[2^{-2k-1}, 2^{-2k-1}3]$.*

(4) *$supp(\gamma) \subset \bigcup_{k=1}^{\infty}[2^{-2k-2}, 2^{-2k-2}3] \cup [2^{-2}, 1]$.*

Proposition 7.4. *There exist $\xi \in \mathcal{H}_{LIP}([0,1])$ and a real valued function λ on $(0,1]$ satisfying the condition (L) such that $\beta = \lambda \circ \xi - \lambda$.*

By Proposition 7.4 we have

$$h_\beta = h_\lambda^{-1} \circ \Psi(\xi)^{-1} \circ h_\lambda \circ \Psi(\xi) \in [Ker\,P,\ \mathcal{H}_{LIP,U(1)}(\mathbf{C})].$$

Similarly we can prove that $h_\gamma \in [Ker\,P,\ \mathcal{H}_{LIP,U(1)}(\mathbf{C})]$. Thus each element of $KerP$ can be written as a composition of commutators of elements in $\mathcal{H}_{LIP,U(1)}(\mathbf{C})$. It follows from Theorem 3.2 that $\mathcal{H}_{LIP,U(1)}(\mathbf{C})$ is perfect.

§8. $H_1(L_{U(1)}(\mathbf{C}))$

We shall calculate the first homology group $H_1(L_{U(1)}(\mathbf{C}))$.

Let $P : L_{U(1)}(\mathbf{C}) \to L(\mathbf{R}_+)$ be a group homomorphism given by

$$P(h)(x) = |h(x)| \quad \text{for} \quad h \in L_{U(1)}(\mathbf{C}),\ x \in \mathbf{R}_+.$$

There exists a right inverse $\Psi : L(\mathbf{R}_+) \to L_{U(1)}(\mathbf{C})$ of P defined by

$$\Psi(f)(xz) = f(x)z \quad \text{for} \quad f \in L(\mathbf{R}_+),\ x \in \mathbf{R}_+,\ z \in U(1).$$

For $h \in KerP$, let $a_h : \mathbf{R}_{>0} \to U(1)$ be the map satisfying

$$h(xz) = xza_h(x) \quad \text{for} \quad x \in \mathbf{R},\ z \in U(1).$$

Lemma 8.1. *$\hat{a}_h$ satisfies the condition (L).*

By the fragmentation theorem and Theorem 3.4, we can assume that *supp*(h) is contained in the unit disk in $\mathbf{C}$. If α satisfies the condition (L), let

$$F_\alpha(xz) = \begin{cases} xzE(\alpha(x)) & x \in \mathbf{R}_{>0},\ z \in U(1) \\ 0 & x = 0. \end{cases}$$

Let $\hat{a}_h :\ \mathbf{R}_{>0} \to \mathbf{R}$ be a function such that
(1) $E(\hat{a}_h(x)) = a_h(x)$,
(2) $\hat{a}_h(x) = 0$ if $x \geq 1$.

Lemma 8.2. *Let $\alpha :\ \mathbf{R}_{>0} \to \mathbf{R}$ be a function satisfying the condition (L) and supp$(\alpha) \subset [0,1]$. Then $F_\alpha \in KerP$.*

Let $\mathcal{C}(\mathbf{R})$ denote the space of real valued function on $\mathbf{R}_{>0}$ satisfying the condition (L) and $\mathcal{C}_0(\mathbf{R})$ the subspace of those $f \in \mathcal{C}(\mathbf{R})$ with f bounded.
Define a homomorphism

$$T :\ KerP \to \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R}), \qquad T(h) = \hat{a}_h \mod \mathcal{C}_0(\mathbf{R}).$$

Then we have a map

$$\Theta :\ L_{U(1)}(\mathbf{C}) \to L(\mathbf{R}_+) \times \mathcal{C}/\mathcal{C}_0$$

given by

$$\Theta(h) = (P(h),\ T(\Psi(P(h))^{-1} \circ h)).$$

Lemma 8.3. *Θ is an onto group homomorphism.*

Proposition 8.4. *$Ker\ \Theta$ is contained in the commutator subgroup of $L_{U(1)}(\mathbf{C})$.*

Key to the proof of Proposition 8.4
If $h \in Ker\ \Theta$, then $h \in KerP$ and $\hat{a}_h \in \mathcal{C}_0(\mathbf{R})$.
Thus, for any positive number ε, there exists an integer n such that $\left|\dfrac{\hat{a}_h(x)}{n}\right| \leq \varepsilon$ for $0 < x$ and

$$\left|\frac{\hat{a}_h(x)}{n} - \frac{\hat{a}_h(y)}{n}\right| \leq \frac{\varepsilon}{x}(y - x) \quad \text{for} \quad 0 < x \leq y.$$

Note that $a_h = E(n\hat{a}_h) = E(\hat{a}_h)^n$. Then, for a sufficiently small positive number ε, we can assume that $|\hat{a}_h(x)| \leq \varepsilon$ for $0 < x$ and

$$|\hat{a}_h(x) - \hat{a}_h(y)| \leq \frac{\varepsilon}{x}(y - x) \quad \text{for} \quad 0 < x \leq y.$$

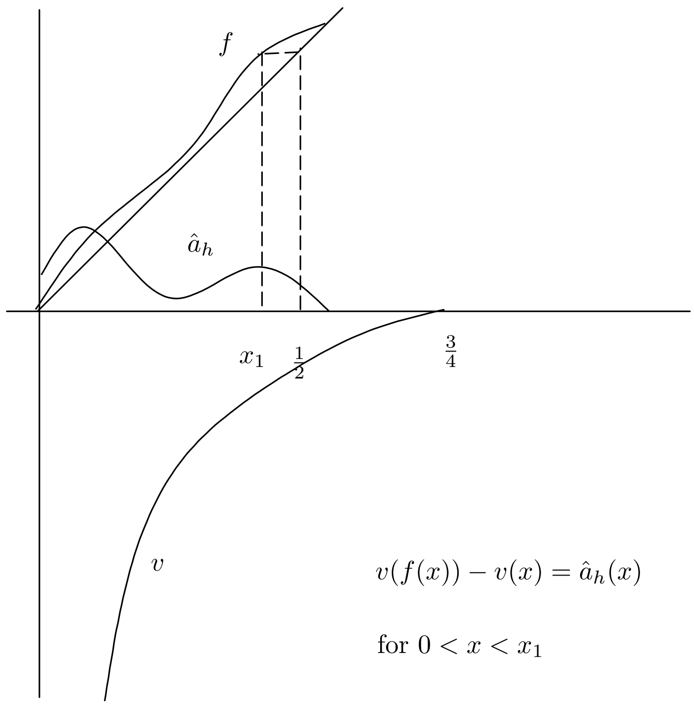
Let v be a real valued smooth monotone increasing function on $\mathbf{R}_{>0}$ such that

$$v(x) = \begin{cases} \log x & (0 < x \leq 1/2), \\ 0 & (3/4 \leq x). \end{cases}$$

Then $v \in \mathcal{C}(\mathbf{R})$.
Let f be a real valued function on $\mathbf{R}_+$ defined by

$$f(x) = \begin{cases} 0 & (x = 0) \\ xe^{\hat{a}_h(x)} & (0 < x \leq 1) \\ x & (x \geq 1). \end{cases}$$

Lemma 8.5. *If ε is sufficiently small, then $f \in L(\mathbf{R}+)$.*



Set

$$g = h \circ \Psi(f)^{-1} \circ F_{E\circ v}^{-1} \circ \Psi(f) \circ F_{E\circ v}.$$

Lemma 8.6.
If $0 < x \leq \dfrac{1}{2e^{\varepsilon}}$, *then we have*

(1) $v(f(x)) - v(x) = \hat{a}_h(x)$,
(2) $g(xz) = h(xz)$ *for* $z \in U(1)$.

By the fragmentation theorem Proposition 8.4 follows from Theorem 3.5 and Lemma 8.5.
Therefore

$$H_1(L_{U(1)}(\mathbf{C})) \cong \mathcal{C}(\mathbf{R})/\mathcal{C}_0(\mathbf{R}).$$

§9. $\mathcal{H}_{U(1)}(\mathbf{C})$

We shall prove that $\mathcal{H}_{U(1)}(\mathbf{C})$ is perfect.

Let $P: \mathcal{H}_{U(1)}(\mathbf{C}) \to \mathcal{H}(\mathbf{R}_+)$ be a group homomorphism defined by

$$P(h)(x) = |h(x)| \quad (x \in \mathbf{R}_+).$$

Let $\Psi : \mathcal{H}(\mathbf{R}_+) \to \mathcal{H}_{U(1)}(\mathbf{C})$ be a map defined by

$$\Psi(f)(xze) = f(x)ze \quad (x \in \mathbf{R}_+,\ z \in U(1)).$$

Then we have

Lemma 9.1. *Ψ is a group homomorphism which is a right inverse of P.*

By the fragmentation theorem and Theorem 4.3, we can assume that $supp(h)$ is contained in the unit disk in $\mathbf{C}$. For $h \in KerP$, let $a_h : \mathbf{R}_{>0} \to U(1)$ be the map satisfying

$$h(xz) = xza_h(x) \quad \text{for} \quad x \in \mathbf{R},\ z \in U(1).$$

Then a_h is a continuous map.

Let $\hat{a}_h : \mathbf{R}_{>0} \to \mathbf{R}$ be a continuous function such that

(1) $E(\hat{a}_h(x)) = a_h(x)$,

(2) $\hat{a}_h(x) = 0$ if $x \geq 1$.

If α is a real valued continuous function on $\mathbf{R}_{>0}$, let

$$F_\alpha(xz) = \begin{cases} xzE(\alpha(x)) & x \in \mathbf{R}_{>0},\ z \in U(1) \\ 0 & x = 0. \end{cases}$$

Lemma 9.2. *If α is a real valued continuous function on $\mathbf{R}_{>0}$, then $F_\alpha \in \mathcal{H}_{U(1)}(\mathbf{C})$.*

By the fragmentation theorem we can assume that $supp(h)$ is contained in the unit disk in $\mathbf{C}$.

Lemma 9.3. *There exist real valued functions β and γ on $\mathbf{R}_{>0}$ such that*

(1) $F_\beta \circ F_\gamma = F_{\hat{a}_h} = h$,

(2) $supp(\beta) \subset \bigcup_{k=1}^{\infty} [2^{-2k-1}, 2^{-2k-1}3]$,

(3) $supp(\gamma) \subset \bigcup_{k=1}^{\infty} [2^{-2k-2}, 2^{-2k-2}3] \cup [2^{-2}, 1]$.

Proposition 9.4. *There exist a continuous map $\hat{\beta} : (0,1] \to \mathbf{R}$ with $\hat{\beta}(1) = 0$ and piecewise homeomorphism ξ such that $\beta = \hat{\beta} - \hat{\beta} \circ \xi$.*

Proof. We shall define the pieswise linear homeomorphism ξ as follows. Let a_n be a sequence of numbers such that

$$a_1 = 1, \quad a_{2n} = a_n, \quad a_{2n+1} = n + 1.$$

Let

$$t(n,k) = 2^{k-1}(2n-1) \quad \text{for} \quad n, k = 1, 2, \cdots.$$

Then we have $a_{t(n,k)} = n$. Put

$$p_n = 2^{-2n-1}, \quad q_n = 2^{-2n-1}3, \quad r_n = 2^{-2n-5}13, \quad s_n = 2^{-2n-5}15.$$

Then $q_{n+1} < r_n < s_n < p_n < q_n$.

Let $\xi : (0,1] \to (0,1]$ be a piecewise linear map such that

(1) $\xi(p_n) = r_{2n-1}, \quad \xi(q_n) = s_{2n-1}$

(2) $\xi(r_{t(n,k)}) = r_{t(n,k+1)}, \quad \xi(s_{t(n,k)}) = s_{t(n,k+1)}$

(3) ξ is linear on each interval $[q_{n+1}, r_n], [r_n, s_n], [s_n, p_n]$ or $[p_n, q_n]$, where $n, k = 1, 2, \cdots$.

Define

$$\hat{\beta}(x) = \beta(\xi^{-k}(x)) \quad \text{if } x \in [\xi^k(p_n),\ \xi^k(q_n)] \quad (n \geq 1,\ k \geq 0).$$

Then $\hat{\beta} - \hat{\beta} \circ \xi = \beta$.

Using Proposition 9.4 we have that

$$h_\beta = h_{\hat{\beta}} \circ \Psi(\xi)^{-1} \circ h_{\hat{\beta}}^{-1} \circ \Psi(\xi).$$

Then $h_\beta \in [KerP,\ \mathcal{H}_G(V)]$.

Similarly we can prove that $h_\gamma \in [KerP,\ \mathcal{H}_G(V)]$.

Therefore $\mathcal{H}_{U(1)}(\mathbf{C})$ is perfect.

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