

BRIEF NOTES ON 3-FOLD BIRATIONAL GEOMETRY

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ABSTRACT. A brief survey of 3-fold birational geometry with a special look at del Pezzo fibrations is given. The subject involves questions such as the rationality and birational rigidity problems, birational automorphisms, Mori fiber spaces, and ctr..

1. WHAT WE WOULD LIKE TO KNOW.

These notes are devoted to give a short description of some ideas and results of up-to-date birational geometry. In what follows, we will always assume the ground field k to be algebraically closed of characteristic 0, e.g., the field of complex numbers \mathbb{C} .

Given an algebraic variety X , we can naturally attach some objects, e.g., the field of functions $k(X)$, the essential object in birational geometry. So, assuming classification to be one of the most important problems in algebraic geometry, we may be asked to describe all algebraic varieties with the same field of functions, i.e., that are birationally isomorphic to X . Of course "all" is a too huge class, and usually we are restricted by projective and normal varieties (though non-proper or non-normal cases may naturally arise in some questions). Typically there are two main tasks:

- A. Given a variety V , one need to determine whether it is birational to another variety W .
- B. V and W are birational to each other, and one need to get a decomposition of the birational map between them into "elementary links", i.e., birational maps that are simple enough.

The rationality problem is the essential example of task A. Recall that a variety is said to be rational if it is birational to \mathbb{P}^n (or, which is the same, if its field of functions is $k(x_1, \dots, x_n)$). As to task B, examples will be given below, we only point the usual way to solve it: first, we prove that a variety can be mapped birationally to its "good" model (i.e., to a variety with nice properties), and then describe elementary links between models.

2000 *Mathematics Subject Classification.* 14E05, 14E06, 14E07.

Key words and phrases. birational transformation, minimal model, Mori fiber space, birational rigidity, maximal singularity.

This work was partially supported by the grants RFBR no. 99-01-01132, Grant of Leading Scientific Schools no. 96-15-96146, INTAS-OPEN 97/2072, and NWO-RFBR 047-008-005.

Received May 2, 2001.

2. CURVES AND SURFACES.

Dimension 1. Normal algebraic curves are exactly smooth ones. It is well known that a birational map between projective smooth curves is an isomorphism, so the birational and biregular classifications coincide in dimension 1. Projective spaces have no moduli, so \mathbb{P}^1 is the unique representative of rational curves.

Dimension 2. The simplest birational maps in dimension 2 are blow-up at a smooth point and contraction of a **-1-curve** (recall that **-1-curve** is a smooth rational curve with self-intersection -1). It is well known that any birational map between smooth surfaces can be decomposed into a chain of these birational maps, each time staying in the class of smooth varieties. Now it is clear that we can contract all **-1-curves** and get the so-called (relatively) minimal (smooth) model (i.e., nothing to contract). So it is very convenient to use minimal models as the class of "good" models and the indicated blow-ups and contractions as elementary links. As to the rationality problem, the famous theorem of Castelnuovo says that a smooth surface X is rational if and only if $H^1(X, \mathcal{O}_X) = H^0(X, 2K_X) = 0$. This is one of the most outstanding achievements of the classical algebraic geometry.

Summarizing results in dimensions 1 and 2, we can formulate the rationality criterion as follows: X is rational if and only if all essential differential-geometric invariants, i.e., $H^0(X, (\Omega_X^p)^{\otimes m})$, vanish (here Ω^p are sheaves of holomorphic forms).

3. THREEFOLDS.

As soon as we get to dimension 3, the situation become much harder. Now it isn't obvious at all what is "a good model". We may proceed with the way as in dimension 2, i.e., contracting everything that can be contracted. This is the viewpoint of Mori's theory. But then starting from a smooth variety, we may loose smoothness very quickly and even get a "very bad" variety with two Weil divisors intersecting by a point (this is the case of "a small contraction", i.e., birational morphism which is an isomorphism in codimension 1).

Nevertheless, Mori's theory states that there is the smallest category of varieties which is stable under divisorial contractions and flips (the last is exactly the tool which allows to "correct" a small contraction). In what follows we shall not need details of this theory, the reader can find them in many monographs (e.g., [15]). We only point that X belongs the Mori category if it is a projective normal variety with at most \mathbb{Q} -factorial terminal singularities. It means that every Weil divisor is \mathbb{Q} -Cartier, i.e., it becomes Cartier as soon as we take it with some multiplicity, and for every resolution of singularities $\varphi : Y \rightarrow X$ we have

$$K_Y = \varphi^*(K_X) + \sum a_i E_i,$$

where all a_i are positive rational numbers, E_i are exceptional divisors, and "=" means "equal as \mathbb{Q} -divisors", i.e., multiplying by a suitable integer number, we get the linear equivalency. In particular, there exists the intersection theory on such varieties, which is very similar to the usual one, the difference is mostly that we must involve rational numbers as intersection indices.

The Mori category has some nice properties. In particular, Kodaira dimension is a birational invariant under maps in this category. We recall that Kodaira dimension $kod(X)$ is the largest dimension of images under (rational) maps defined

by linear systems $|mK_X|$. We are mostly interested in studying varieties of negative Kodaira dimension, i.e., when $H^0(X, mK_X) = 0$ for all $m > 0$, because their birational geometry is very non-trivial.

From now on, we will only consider varieties of negative Kodaira dimension. So, what are "minimal models" in the Mori theory for such varieties? These are so-called Mori fiber spaces. By definition, a projective normal variety X is a Mori fiber space if there exists a surjective morphism $\pi : X \rightarrow S$ onto a normal variety S such that the relative Picard number $\rho(X/S)$ is equal to 1 and $-K_X$ is relatively ample (i.e., $m(-K_X)$ is very ample on each fiber of π for large enough m). $\rho(X/S) = 1$ means that for any two curves C_1 and C_2 lying in fibers there exist numbers n_1 and n_2 such that for any divisor D on X we have $n_1 C_1 \circ D = n_2 C_2 \circ D$ (i.e., $n_1 C_1$ and $n_2 C_2$ are numerically equivalent).

In dimension 3 (up to now the only dimension where the Mori theory is proved) we have three possible types of Mori fiber spaces (or, briefly, Mori fibrations) $\rho : X \rightarrow S$:

- 1) \mathbb{Q} -Fano, if $\dim S = 0$ (i.e., a point);
- 2) del Pezzo fibration, if $\dim S = 1$ (the fiber over the generic point of S is a del Pezzo surface of the corresponding degree);
- 3) conic bundle, if $\dim S = 2$ (the fiber over the generic point of S is a plane conic).

Factorization of birational maps between Mori fibrations is given by the Sarkisov program (it is proved in dimension 3, see [4]). There are 4 types of elementary links in this program, but unfortunately they haven't got a clear description as in dimension 2. So, up to now, the Mori fibrations theory and the Sarkisov program mostly serve as theoretical tools.

In dimension 3, the rationality problem becomes enormously hard. During a long time, many mathematicians believed that it should be possible to find the rationality criterion more or less simple as in lower dimensions. But nearly simultaneously, in early 70th, three outstanding works of different authors which gave examples of unirational but non-rational varieties, appeared. These were Iskovskikh and Manin ([12]), Clemens and Griffiths ([3]), and Artin and Mumford ([2]). Recall that an algebraic variety is said to be unirational if there exists a rational map from projective space which is finite at the generic point. The matter is that all essential differential-geometric invariants vanish on unirational varieties as well, so we can not even hope to find something like the rationality criterion for curves and surfaces, combining these invariants. The reader may find an excellent survey of the rationality problem in higher dimensions in [10].

Nevertheless, during the last 10 years, a considerable progress in the birational classification problem has been achieved, mostly due to conception of birationally rigid varieties. By definition, a Mori fibration $\rho : V \rightarrow S$ ($\dim S > 0$) is birationally rigid if any birational map $\chi : V \dashrightarrow V'$ to another Mori fibration $\rho' : V' \rightarrow S'$ defines a birational map $\psi : S \dashrightarrow S'$ by means of ρ and ρ' , i.e., we have the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\chi} & V' \\ \rho \downarrow & & \downarrow \rho' \\ S & \xrightarrow{\psi} & S' \end{array}$$

Then, a \mathbb{Q} -Fano V (i.e., $\dim S = 0$) is birationally rigid if for any birational map $\chi : V \dashrightarrow V'$ to another Mori fibrations $V' \rightarrow S'$ there exists a birational automorphism $\psi \in \text{Bir}(V)$ such that the composition $\chi \circ \psi$ is an isomorphism $V \rightarrow V'$ (i.e., V and V' are biregularly isomorphic to each other).

Here are some examples of (birationally) rigid and non-rigid varieties. But first let me note that any rigid variety is not rational. Indeed, \mathbb{P}^3 is birational to $\mathbb{P}^1 \times \mathbb{P}^2$, so we have at least three different Mori structures: \mathbb{Q} -Fano (\mathbb{P}^3 itself), del Pezzo fibration $\mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^1$, and conic bundle $\mathbb{P}^1 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$. The best example of birationally rigid \mathbb{Q} -Fano is a smooth quartic 3-fold in \mathbb{P}^4 - it is nothing but the result of Iskovskikh and Manin ([12]). Note that they proved even more: any birational automorphism of a smooth quartic is actually biregular (and in general case, there are no non-trivial automorphisms at all). As to conic bundle, there exists the following result of Sarkisov ([22], [23]): any standart conic bundle $V \rightarrow S$ over a rational surface S with the discriminant curve C is rigid if the linear system $|4K_S + C|$ is not empty. Say, if S is a plane, it is enough for C to have degree 12 or higher. Then, a smooth cubic 3-fold in \mathbb{P}^4 gives us an example of non-rigid conic bundles. Indeed, the projection from a line lying on such a cubic, realizes it as a conic bundle over a plane with a quintic as the discriminant curve. Examples of rigid and non-rigid del Pezzo fibrations will be given in the next section.

The maximal singularities method ([12], [18]) is the main tool in studying of Mori fiber spaces. Moreover, except for very unusual but a little freakish Kollar's method ([13]), it is the unique method in birational geometry that works effectively in any dimension (e.g., [19]). It is very simple to show the key idea. Suppose $\chi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a quadratic transformation of a plane, \mathcal{L} the full linear system of lines, and $\mathcal{D} = \chi_*^{-1}\mathcal{L}$ its strict transform. Note that \mathcal{L} has no base points, but \mathcal{D} is a linear system of conics with 3 base points of multiplicity 1 because of action of χ . So the key object in the maximal singularities method is singularities of linear systems. The usual way to apply this method: given a linear system with singularities (i.e., base points with multiplicities), either we prove that it is impossible, or we construct a birational map onto another variety which decreases singularities of the system.

Let me conclude this section with a little remark. Minimal models or Mori fiber spaces are not convenient models all the way. In many cases, there are more preferable classes of varieties. For examples, sometimes it is useful to consider Gorenstein terminal varieties with numerically effective anticanonical divisors ([1], [5]). In other words, we have to select each time.

4. DEL PEZZO FIBRATIONS.

The aim of this section is to light up recent results in studying the birational rigidity problem for del Pezzo fibrations. Recall that a projective surface X is said to be del Pezzo of degree d if $-K_X$ is ample and $(-K_X)^2 = d$.

So our objects are Mori fibrations $\rho : V \rightarrow S$, S is a non-singular curve, and the fiber V_η over the generic point η of S is a smooth del Pezzo surface of degree $d \geq 1$. Note that it does not make a sense to say about the rigidity problem if S is non-rational or $d > 3$. Indeed, if genus S is more than 0, there are no Mori structures lying transversally with respect to fibers, and we get an answer immediately using Iskovskikh's results in [11]. So we suppose $S = \mathbb{P}^1$. In this case, V_η is a del Pezzo surface over the field of function of one variable, and if $d \geq 5$, any such a surface is rational ([17]), so V is rational too, thus it is non-rigid. If $d = 4$, we know that V_η

(actually, for any degree) has a lot of rational points ([16]), i.e., $\rho : V \rightarrow S$ has a lot of sections, so we can blow up a general section and get (probably, after some log-flips), a structure of conic bundle.

Here I will only consider the case $d = 1$. The reader can find some results in degree 2 and 3 in [20] and [7].

So, let $\rho : V \rightarrow \mathbb{P}^1$ be a Mori fibration on del Pezzo surfaces of degree 1. We will also assume it to be smooth, in this case the rigidity problem is completely solved ([7]).

First of all, it is known that if X is a reduced Gorenstein del Pezzo surface of degree 1, then $|-K_X|$ has a unique base point, $|-2K_X|$ is free and gives a morphism of degree 2 to a quadratic cone in \mathbb{P}^3 , and $|-3K_X|$ is very ample ([9], [21]). Thus we know that our fibration has a distinguished section - exactly that intersects each fiber at the base point of the anticanonical system. We will denote it s_b . Clearly,

$$s_b = \text{Bas}|-K_X + mF|$$

for $m \gg 0$. Let us also note that all fibers are reduced because of smoothness of V . Now we can put down

$$\text{Pic}(V) = \mathbb{Z}[-K_X] \oplus \mathbb{Z}[F],$$

where F is a fiber of ρ .

Then, we can easily construct a very convenient projective model of V . The linear system $|-2K_X + mF|$ has no base points for large enough m , so it gives us a degree 2 morphism $\varphi : V \rightarrow Q$ to a variety Q . On the other hand, $\mathcal{E} = \rho_*\mathcal{O}(-2K_X + mF)$ is a vector bundle of rank 4, and we may choose m such that

$$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(n_1) \oplus \mathcal{O}(n_2) \oplus \mathcal{O}(n_3) \oplus \mathcal{O}(n_4)$$

for some $0 \leq n_1 \leq n_2 \leq n_3 \leq n_4$. Suppose $X = \mathbf{Proj} \mathcal{E}$, and let $\pi : X \rightarrow \mathbb{P}^1$ be the natural projection. Thus, X is a \mathbb{P}^3 -bundle over \mathbb{P}^1 . Then, $\rho = \pi \circ \varphi$, $Q \subset X$, and the ramification divisor R_Q of φ is cutted off by some divisor $R \subset X$. Let $t_b = \varphi(s_b) \subset X$. Clearly $t_b \cap R = \emptyset$.

Let M be the tautological bundle of X , L is the class of a fiber of X , t_0 is the "minimal section" of X (i.e., the class of an effective curve such that $t_0 \circ M = 0$), and l is the class of a line in a fiber. We have ([7], lemma 2.2):

PROPOSITION 4.1. *The only two cases are possible:*

- 1) $t_b = t_0$, $R \sim 3M$, $Q \sim 2M - 2n_2L$, $2n_2 = n_1 + n_3$, n_1 and n_2 are even.
- 2) $t_b = t_0 + n_1l$, $R \sim 3M - 3n_1L$, $Q \sim 2M - 2n_2L$, $n_3 = 2n_2$, $n_2 \geq 3n_1$, n_1 is even.

We can see that the numbers n_1 , n_2 , and n_3 define completely our fibration (of course, up to moduli). The main result is proved in loc.cit., corollary 2.10, using the maximal singularities method:

THEOREM 4.2. *All smooth Mori fibrations on del Pezzo surfaces of degree 1 are rigid except for two cases:*

- a) $n_1 = n_2 = n_3 = 2$;
- b) $n_1 = 0$, $n_2 = 1$, $n_3 = 2$.

These are both non-rigid cases.

It is possible to summarize this result as follows:

$\rho : V \rightarrow \mathbb{P}^1$ is rigid if and only if for any n the linear system $|n(-K_X) - F|$ is empty or has a base component.

I conjecture that this should be also true for degree 2 or 3 (at least, in smooth cases).

What are other structures of Mori fibrations in cases *a*) and *b*) of theorem 4.2? Let us first consider case *b*). It is easy to see that $\dim |-K_V - 2F| = 0$, the unique divisor in this linear system is the direct product of \mathbb{P}^1 and an elliptic curve, and can be contracted along \mathbb{P}^1 . We get a Fano variety with the Picard number 1, index 2, and $(-K_X)^3 = 8$, which is nothing but the so-called double cone over the Veronese surface. Conversely, given such a Fano variety, we can blow up any non-singular curve of genus 1 and degree 2 (with respect to the anticanonical divisor), and then get a del Pezzo fibration as in case *b*).

As to case *a*), let us note that $\dim |-K_V - F| = 1$, $Bas|-K_V - F| = s_b$, and a general member of this pencil is an elliptic surface obtained with blowing up at the base point of the anticanonical system of a degree 1 del Pezzo surface. Moreover, the normal bundle $\mathcal{N}_{s_b|V} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, so we can make a flop centered at s_b , and get another del Pezzo fibration (of degree 1) $\rho : V' \rightarrow \mathbb{P}^1$ with the same $n_1 = n_2 = n_3 = 2$. Obviously, both varieties have different Mori structures. Actually, we can say even more ([7]):

PROPOSITION 4.3. *Let $V \rightarrow \mathbb{P}^1$ be as in case *a*) of theorem 4.2, $V' \rightarrow \mathbb{P}^1$ constructed as before, and $\psi : V \dashrightarrow V'$ the corresponding flop. Suppose $\chi : V \dashrightarrow W$ is a birational map to a Mori fibration $W \rightarrow S$. Then either χ or $\psi^{-1} \circ \chi$ is a fiber-to-fiber map, i.e., defines an isomorphism $\mathbb{P}^1 \rightarrow S$. In other words, V (or, the same, V') has exactly 2 structures of Mori fibration.*

Finally, there is one more topic in birational geometry of del Pezzo fibrations. This is fiber-to-fiber birational maps that are isomorphisms of fibers over the generic point of the base. Here is an example. Let $[x, y, z, w]$ and $[p, q, r, s]$ be the coordinates of weights $(1, 1, 2, 3)$ in two copies of weighted projective space $\mathbb{P}(1, 1, 2, 3)$, t is a local parameter on the base (we consider a local situation). Suppose $X = \{w^2 + z^3 + x^5y + t^{24}xy^5 = 0\}$, $V = \{s^2 + r^3 + p^5q + pq^5 = 0\}$. X and V are fibrations on del Pezzo surfaces of degree 1 (say, over a local ring with parameter t), V is smooth, X has cE_8 -singularity (Gorenstein and terminal). Consider two birational transformations $\chi : X \dashrightarrow V$ and $\psi : V \dashrightarrow X$:

$$\chi : \quad p = x \quad q = t^6y \quad r = t^2z \quad s = t^3w$$

and

$$\psi : \quad x = t^6p \quad y = y \quad z = t^{10}r \quad w = t^{15}s.$$

Clearly, $\chi \circ \psi = id$, χ and ψ are isomorphisms everywhere outside the central fibers. χ and ψ give examples birational maps that are isomorphisms of the fibers over the generic points of the base.

Really, there exists the following result ([8]):

THEOREM 4.4. *Let $V \rightarrow S$ and $V' \rightarrow S'$ be Mori fibrations on del Pezzo surfaces of degree 1. Suppose we have a commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{\chi} & V' \\ \downarrow & & \downarrow \\ S & \dashrightarrow & S' \end{array}$$

If both V and V' are smooth, then χ is an isomorphism.

In particular, it follows that any smooth Mori fibration on del Pezzo surfaces of degree 1 may have only biregular automorphisms, any birational map between smooth rigid fibrations is actually an isomorphism, V and V' from case a) of theorem 4.2 are unique smooth Mori fibrations in their class of birational equivalency.

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