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FREE SURFACE WAVES OVER AN OBSTRUCTION

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Abstract. Surface waves at the free surface of an incompressible fluid passing over an obstruction are considered. We review numerical and theoritical results of various types waves and some new solitary wave types of solutions of forced KdV equation and extended KdV equation with forcing term.

We consider surface waves on a two dimensional, incompressible, inviscid fluid flow over a bump on a flat bottom. This problem generate many interesting wave patterns and require new mathematical methods to find solutions. They have been investigated numerically in [1] to [4] and [11] to [13], and the mathematical existence of the interfacial solitary waves has been proved in [5] to [9], among others. Discussion of solitary waves in continuously stratified fluids can be found in the book on stratified flow by Yih [10] and the references cited there. They have also studied asymptotically and found possible many interesting wave patterns in [14] to [19].

Numerical computations of steady solutions to exact equations for a semicircular bump ([11] - [13]) indicate the following results. For $1 < F_+ < F$ there are two branches of supercritical solutions and no solution exists below F_+. Each supercritical solution behaves like a solitary wave. Here, F is so called Froude number defined by $F = c/(gH_0)^{1/2}$ where H_0 is the constant depth of the fluid flow as the size of the bump becomes zero and c is the constant speed of the fluid flow far upstream. As the size of the bump tends to zero, one branch of approaches the uniform state far upstream and the other branch approaches a solitary wave. As F increases, the branch of larger solutions approaches a limiting configuration with a 120 degrees of angle at the crest. For $F < F_- < 1$, only one branch of subcritical solutions is found and no solution exists above F_-. They exhibit a quiescent region upstream and a Stokes wave train down stream. In $F_- < F < F_+$ even if no steady solution exists unsteady waves can appear. A solution which behaves like a hydraulic fall with $F < 1$ and the down stream Froude number has been found [13]. The solution remains almost constant up to the obstacle, then decreases monotonically to a constant value far down stream.

Next, we review the asymptotic study for the similar fluid flow as was in [11] to [13]. If $F = 1 + \epsilon F_1$ is close to unity where ϵ is a small positive parameter, an inhomogeneous nonlinear ordinary differential equation can be derived as a model equation for the study of the surface waves over a bump. For this purpose the following Euler equations with boundary conditions and asymptotic expansions are considered,

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$$u^*_{x^*} + v^*_{y^*} = 0 \tag{1}$$

$$u^* u^*_{x^*} + v^* u^*_{y^*} = -p^*_{x^*}/\rho^* \tag{2}$$

$$u^* v^*_{x^*} + v^* v^*_{y^*} = -p^*_{y^*}/\rho^* - g \tag{3}$$

at the free surface, $y^* = \eta^*(x^*)$,

$$u^* \eta^*_{x^*} - v^* = 0, \;\; p^* = 0; \tag{4}$$

at the rigid lower boundaries, $y^* = h^*(x^*)$,

$$v^* - u^* h^*_{x^*} = 0, \tag{5}$$

where (u^*, v^*) are velocities, p^* is a pressure, ρ^* is the constant density of the fluid, g is the gravitational acceleration constant, and $h^*(x^*) = -H^* + b^*(x^*)$, where H^* is the constant depth of the fluid at equilibrium state, and $b^*(x^*)$ stands for the obstruction with finite support on the rigid bottom.

The following nondimensional variables are required to remove physical units in Euler equations:

$$\epsilon = (H^*/L)^{1/2}, \;\; \eta = \epsilon^{-1}\eta^*/H^*, \;\; x = \epsilon^{1/2}x^*/H^*, \;\; y = y^*/H^* \;\;\; p = p^*/gH^*\rho^*,$$

$$(u, v) = (gH^*)^{-1/2}(u^*, \epsilon^{-1}v^*), \;\; h(x) = h^*(x)/H^*, \;\; b(x) = b^*(x)(H^*\epsilon^2)^{-1},$$

where L is the horizontal length scale, which is used to be much larger than vertical length scale. This is so-called long wave assumption.

In terms of the above nondimensional variables and assuming that u, v, p and η possess asymptotic expansions of the form

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots$$

with a physically suitable conditions for u_0, v_0, p_0 and the upstream Froude number $F = C/(gH^*)^{1/2} = 1 + \epsilon\lambda$, a system of differential equations and boundary conditions for successive approximations are obtained according to the order of ϵ. Then, by solving the resulting equations the following FKdV equation is derived,

$$-\frac{1}{3}\eta_{1xxx} - 3\eta_1\eta_{1x} + 2\lambda\eta_{1x} = b_x. \tag{6}$$

Integrating (6) from $-\infty$ to x yields

$$\eta_{1xx} = (-9/2)\eta_1^2 + 6\lambda\eta_1 - 3b(x). \tag{7}$$

Existence theorems of (7) are given as follows, **Theorem 1** (S.P. Shen) [15] (7) has a positive C^2-solution which decays exponentially at $|x| = \infty$ for large λ if $b(x)$ has a compact support. **Theorem 2** (S.P. Shen) [15] There exits $\lambda_c > 0$ such that (7) has at least two positive C^2-solution for $\lambda > \lambda_c$ and no solutions for $\lambda < \lambda_c$ if $b(x)$ has a positive compact support. **Theorem 3** (Jeongwhan Choi) [20] (7) has a negative continuous solution which decays exponentially at $|x| = \infty$ for negative λ if $b(x)$ is negative and is continuous. Numerical studies for equation (7) gives interesting wave patterns. When $b(x)$ in (7) is positive two symmetric solitary wave-like solutions appear when F_1 is positive and Hydraulic fall solution appears when F_1 is a certain negative number, below which typical periodic wave solutions appear([14],[15]).When $b(x)$ is negative, many interesting new solutions appear [20]. Two critical values of F_1, say λ_1 and λ_2, exist so that, for $F_1 > \lambda_1$, two positive symmetric solitary-wave-like solutions appear and, for $F_1 > \lambda_2$, two positive unsymmetric solitary-wave-like solutions appear. In this case of negative $b(x)$, a negative symmetric solitary-wave-like solution appears for any positive value of λ. We also note that negative symmetric solitary- wave-like solution and the cut-off point for the appearance of positive unsymmetric solitary-wave-like solutions do not occur if $b(x)$ is of the positive semicircular form [14], [15]. Some other types of symmetric or unsymmetric wave solutions of [7] were found in [20] which decays to zero at $\pm\infty$.

Two-layer immiscible inviscid and incompressible fluid have been considered and exhibited many interesting wave phenomena. Numerical studies of steady flow of a two layer fluid over a bump or a step bounded by a free or rigid upper boudary were carried out by Fforbes[21], Belward and Forbes[22], Sha and Vanden-Broeck[23], and Moni and King[24], among others and an asymptotic approach for the case of rigid upper boundary was developed without surface tension by Shen on the basis of the FKdV theory and with surface tension by Choi, Sun, and Shen ([16],[17]). The KdV theory fails when the coefficient of th nonlinear term or that of the third derivative in th FKdV vanishes. In the case considered [17], when the wave speed is near the smaller critical speed of

$$u_0 = [\frac{1+h \pm [(1-h)^2 + 4\rho h]^{1/2}}{2}]^{1/2}$$

for internal wave, the amplitude of which is larger at the interface than at the free surface, the coefficient of the nonlinear term in th FKdV may vanish and the following Extended KdV with forcing is derived and studied for the wave motion of the fluid,

$$F_1\eta_{xxx} + (F_2\eta^2 + F_3\eta F_4)\eta_x = F_5 b_x.$$

Three types of solution waves have been found. The first type solution consists of symmetric solitary wave like solutions, the second type solution is one that is a part of a free solitary wave behind the bump and a periodic wave solution ahead of the bump. The free solitary wave is a solitary wave solution of the extended KdV equation without forcing. By a third type solution, we mean a solution that is constant behind the bump and a periodic wave ahead of the bump. In many cases both second and third type solutions do satisfy the conservation of mass, even if they do not tend to zero far up stream. It is found that for branches of first type solutions can appear in the supercritical case and there are no first and second type solutions in the subcritical case. The third type solutions appear in both supercritical and subcritical cases. In both cases symmetric solutions without a periodic part are embedded in the third type solutions at discrete values of a parameter, and a hydraulic jump wave solution appears as a limiting case of third type solutions in the subcritical case.

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