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ITERATION, ITERATIVE ROOTS AND ITERATIVE EQUATIONS

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Abstract. Iteration exists extensively in the nature. Iteration of a homeomorphism generates a dynamical system. To embed such a homeomorphism into a flow we need to define fractional iteration and find iterative roots. The problem of iterative roots is to discuss iterative equations. This is a way to lead us to explore the mathematics of iteration.

Iteration is a basic concept in the theory of dynamical systems. Over a hundred years ago E. Schröder, N. H. Abel, and C. Babbage studied iteration and obtained many nice results [20, 1, 4]. With iteration we trace where an object will go under the action of a dynamical system. On the other hand, we are also concerned with the course between two succesive states of iteration, inserting data to preserve iteration and embedding a discrete dynamical system into a flow. In section 1 we introduce some basic knowledges about iteration. Then we give some concepts, ideas, results and problems for iterative roots and iterative equations respectively in section 2 and 3.

1. Iteration

Iteration is a repeated action of the same operation. It exists extensively in mathematics, in science, in the nature and in the world. The population x_n of a kind of insect in the n-th year depends on the population of last year, i.e., $x_n = x_{n-1}(a - bx_{n-1})$ in the famous Logistic model, where $0 \leq x_{n-1} \leq a/b$. This is iteration of the function $f(x) = x(a - bx)$. Usually a principal P placed at a rate of interest r for n years accumulates to an amount of loan $A_n = P(1 + nr)$ at simple interest or $A_n = P(1 + r)^n$ at interest compounded annually. A_n is just the n-th iterate of the function $a(x) = x + rP$ or $a(x) = x(1 + r)$. In computer science iteration is a basic terminology and cyclic computation is very important. In general, consider a self-mapping f on a topological space X, i.e., f maps X into itself. f^n defined by

$$f^n(x) = f \circ f^{n-1}(x), \quad f^0(x) = x, \tag{1.1}$$

is called the n-th *iterate* of f. n is called the *index* of iteration.

In astronautics we consider a spaceship B being launched at time $t = 0$ from the place A, where an observer stands, and returns to A at $t = \xi \leq \infty$. Suppose the observer at A sends a signal at time t to B and let $g(t)$ denote the time when the observer receives the signal reflected by B. It is easy to see that $g \in C(I, I),\ I = [0, \xi],\ g(0) = 0,\ g(\xi) = \xi,\ g(t) > t\ \forall t \in (0, \xi)$. When ($B$ time) does the astronaut

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in B receive the signal sent from A at t (A time)? Suppose the B time when B receives the signal is $f(t)$. Since reflection is in the same principle, we see

$$f(f(t)) = g(t), \tag{1.2}$$

that is, g is the second iterate of the unknown function f. However, it is not easy to solve this equation even for some simple functions $g(x) = \sqrt{x}$ and $g(x) = \log(x+1)$.

In engineering, the nonlinear oscillaton of a rigid rotor is governed by the differential equation

$$\frac{d^2\theta}{dt^2} + g(\theta) = A\cos(\omega t).$$

Its dynamics is decribed by iteration of its Poincaré mapping $T(\frac{2\pi}{\omega}, 0)$, where $T(t, s)$ is the fundamental solution matrix defined by $T(t, s)x_0 = x(t; s, x_0)$. Here $x = \mathrm{col}(\theta, \dot{\theta})$ and $x(t; s, x_0)$ denotes the solution with the initial data $x(s) = x_0$.

For continuous mapping $f : X \to X$, the sequence $\{f^n : n \in \mathbf{Z}_+\}$ satisfies
(i) $f^0 = \mathrm{id}$ (identity), and
(ii) $f^m \circ f^n = f^{m+n}$,
that is, it becomes a semigroup. In this case we say that f generates a *(discrete) semi dynamical system.* If $f : X \to X$ is a homeomorphism then we say that f generates a *(discrete) dynamical system.* See [18, 28].

Given $x_0 \in X$, we are concerned with the set $\mathrm{Orb}_f(x_0) = \{f^n(x_0) : n \in \mathrm{Z}\}$, called the *orbit* of f at x_0, or the sets $\mathrm{Orb}_f^{\pm}(x_0) = \{f^n(x_0) : n \in \mathrm{Z}_{\pm}\}$, called the *positive half-orbit* and *negative half-orbit* of f at x_0 respectively. Especially we are concerned with the long-term behaviors of iteration, that is, the topological structure of *limit sets* $\omega_f(x_0)$ and $\alpha_f(x_0)$, which consist of the accumulating points of $\mathrm{Orb}_f^+(x_0)$ and $\mathrm{Orb}_f^-(x_0)$ respectively. There are many important accumulating points such as fixed points, periodic points and nonwandering points. Let $F(f), P(f)$ and $\Omega(f)$ consist of all fixed points, all periodic points and all nonwandering points respectively. They are all *invariant sets* of f.

Two systems $f : X \to X$ and $g : Y \to Y$ are regarded to be *topologically equivalent* or *topologically conjugate* if $g \circ h = h \circ f$ for a homeomorphism $h : X \to Y$. In this case f and g have the same topological structure of orbits. A system f is said to be *structural stable* if all small perturbations in the C^1-topology are topologically conjugate to f itself. A system near its hyperbolic fixed point is structural stable. More generally, a Morse-Smale system, defined by the following conditions that
(i) there are only finitely many fixed points and periodic points,
(ii) all fixed points and periodic points are hyperbolic, and
(iii) all stable manifolds intersect transversally with unstable manifolds,
is also structural stable. Can we find other systems which are structural stable?

If a system $f_\mu : X \to X$ parametrized by real μ is not structural stable at $\mu = \mu_0$, that is, the system for $\mu < \mu_0$ are "essentially" different from the system for $\mu > \mu_0$, or precisely, the system before μ_0 and the system after μ_0 are not topologically conjugate to each other, then we say the system bifurcates at $\mu = \mu_0$. It is easy to check with the Logistic mapping $f_\mu(x) = \mu x(1 - x)$ on $I = [0, 1]$ how many periodic points arise from bifurcation for different $\mu \in [0, 4]$. It is interesting to analyze bifurcations for various mappings.

2. Iterative Roots

An autonomous differential equation defines a *continuous dynamical system.* In fact, let $x(t, x_0)$ be the solution with the initial data $x(0) = x_0 \in X$ and $T(t)x_0 = x(t, x_0)$. Clearly, $T(t) : X \to X$ are continuous and satisfies that $T(0) = \mathrm{id}$ and $T(t+s) = T(t)T(s)$. Sometimes $T(t)$ is also called a *flow.* A flow $T(t)$ can be discretized easily into a discrete dynamical system (or a homeomorphism) F by taking

$$F(x) = T(1)x, \quad x \in X. \tag{2.3}$$

Conversely, we also want to embed a homeomorphism $F : X \to X$ into a flow, so as to append those "lost" data and repair the whole process of iteration. It is the first step to define the fractional iterate $F^{1/n}$. If it is possible then we can extend easily the index n of iteration for F to rational and hopefully to real. Then F^t is well defined for all real t. Let $f = F^{1/n}$. It requires to solve the iterative equation

$$f^n(x) = F(x), \quad \forall x \in X. \tag{2.4}$$

That is just the problem encountered in (1.2). A solutions of equation (2.4) is called an n-th *iterative root* of F. See [4, 9, 10, 19].

It is not easy to calculate the n-th iterate of a function even for some simple functions $f(x) = \frac{ax+b}{cx+d}$ and $f(x) = \sin x$. Solving iterative roots will be more complicated. Let us only consider the one-dimensional case and start with the simplest Babbage's equation

$$f^n(x) = x, \quad \forall x \in \mathbf{R}, \tag{2.5}$$

the solutions of which are also called *unit iterative roots.* Clearly (2.5) has a trivial solution $f(x) \equiv x$. Is there any nontrivial solution?

A function f satisfying $f^2 = \mathrm{id}$ is called a *convolution.* A function f such as $1/x$ and $1 - x$, whose graph $y = f(x)$ is symmetric to the line $y = x$, is a convolution. For equation (2.5) we have the following result [10].

Theorem 1. *Continuous real unit n-th iterative roots f are of the forms:*
(i) $f(x) \equiv x$, or
(ii) f is a strictly decreasing convolution when n is even.

There are given many results [9, 10, 35, 36] for iterative roots in the monotone cases. In the following table a comparison with algebraic equations is given roughly.

$z^n = a$	$z = a^{1/n}$	$f^n(x) = F(x)$	$f(x) = F^{1/n}(x)$
n odd, $a > 0$	$\exists$, unique, $+$	n odd, $F \nearrow$	$\exists$, ∞, $\nearrow$
n even,$a > 0$	$\exists$, 2, $\pm$	n even, $F \nearrow$	$\exists$, ∞, $\nearrow$, $\searrow$
n odd, $a < 0$	$\exists$, unique, $-$	n odd, $F \searrow$	$\exists$, ∞, $\searrow$
n even,$a < 0$	no real roots	n even, $F \searrow$	no solutions

where $\exists$, $\nearrow$ and $\searrow$ mean existence, increasing and decreasing respectively and $2, \infty$ mean the number of solutions.

Especially, if both F and f are orientation-preserving homeomorphisms from $I = [a, b]$ onto itself and if f is an iterative root of F, we can prove that f is topologically conjugate to F by an orientation-preserving homeomorphism.

In the cases without monotonicity the following result [35] may disappointing us.

Theorem 2. *Continuous mapping $F : I = [a, b] \to I$ is surjective with an extreme point $c \in I$ for which F is strictly monotone both on $[a, c]$ and on $[c, b]$. Then F has no C^0 n-th iterative roots for any integer $n \geq 2$.*

However, for a large class of mappings without monotonicity there are plentiful results. Usually $x_0 \in (a, b)$ is referred to as a *monotonic point* of $F : I = [a, b] \to I$ if F is strictly monotone on a neighborhood of x_0. Otherwise, x_0 is called a *non-monotonic point* or a *fort* simply. A continuous function $F : I \to I$ is called an *S-function* or *Strictly Piecewise Monotone Continuous Function* if it has only finitely many forts. Let $S(I, I)$ consist of all S-functions in $C(I, I)$. Clearly all extreme points are forts but a fort may not be an extreme point. End-points are not forts. Let $N(F)$ denote the number of forts of F. It is not hard to prove that

$$0 = N(F^0) \leq N(F) \leq N(F^2) \leq ... \leq N(F^m) \leq ...$$

Let $H(F)$ be the minimum of the integer $m > 0$ such that $N(f^m) = N(F^{m+1})$. We can prove that

$$N(F^m) = N(F^{m+k}), \quad \forall k \in \mathbf{N},$$
$$H(F^k) = [\frac{m}{k}] + \mathrm{sgn}(\frac{m}{k} - [\frac{m}{k}]), \quad \forall k \in \mathbf{N},$$

if $H(F) = m < \infty$. As a generalization to Theorem 2, we have the following [43].

Theorem 3. *$F \in S(I, I)$, $H(F) > 1$. Then F has no C^0 n-th iterative roots for any integer $n > N(F)$.*

There are two cases to be discussed further, in one of which $H(F) \leq 1$ and in the other $n \leq N(F) < N(F^2)$.

In the case of $H(F) \leq 1$ we only need to consider $N(F) = N(F^2)$ for non-monotonic functions. In this case we can prove that F is strictly monotone on $[m, M]$ where $m := \min\{F(x) : x \in I\}$ and $M := \max\{F(x) : x \in I\}$, so we can extend the interval for F to be monotone larger to two consecutive forts or end-points. The interval $[a', b']$ is called the *characteristic interval* of F if a', b' are two consecutive forts of F on $I = [a, b]$ or end-points of I and $[m, M] \subset [a', b'] \subset I$. We can prove that every $F \in S(I, I)$ with $H(F) \leq 1$, has a unique characteristic interval $[a', b']$, on which F is a continuous and monotone self-mapping and generates a semi dynamical system.

Theorem 4. *([43]). $F \in S(I, I)$, $H(F) \leq 1$. Suppose that*
(i) F is strictly increasing on $[a', b']$ and
(ii) $F(x) \neq a', b'$ for all $x \in I$ if $F(a') \neq a'$ and $F(b') \neq b'$. Then F has a continuous n-th iterative roots for any integer $n \geq 2$. Condition (i) and (ii) are also necessary for $n > N(F) + 1$.

Theorem 5. *([43]). $F \in S(I, I)$, $H(F) \leq 1$. Suppose that*
(i) F is strictly decreasing on $[a', b']$ and
(ii) either $F(a') = b'$ and $F(b') = a'$ or $a' < F(x) < b'$ for all $x \in I$. Then F has a continuous n-th iterative roots for any odd $n > 0$.

When $n \leq N(F) < N(F^2)$, *expansions* and *piecewise expansions* are discussed in [33, 34] but the existence of iterative roots for more general mappings is still an open problem.

Concerning smoothness of iterative roots, a classic result was given by Böedewadt in the case of no fixed points, that is, if $F \in C^\infty(I, I)$, $I = (a, b)$, $F(x) > x$ and

$F'(x) > 0$ then F has n-th iterative roots in $C^\infty(I, I)$ for any integer $n \geq 2$. It is also a well-known result in the case of unique fixed point that if (i) $F \in C^1(I, I)$, $F'(x) > 0$, $\forall x \in I$, (ii) F has a unique fixed point $x_0 \in I$ such that $F'(x_0) \neq 1$, and (iii) $F'(x_0)$ exists, then F has a unique n-th iterative roots in $C^1(I, I)$ for any integer $n \geq 2$, which is increasing. This result is also known as *local smoothness* of iterative roots. It was conjectured that there are seldom iterative roots with global smoothness, namely, C^1-smooth simultaneously at two or more fixed points. This was proved in [42] in 1995.

Theorem 6. *Let $I = [0, 1]$ and*

$$\begin{aligned} A^1 &:= \{F \in C^1(I, I) : \ F'(x) > 0, \ F(x) \neq x \text{ on } (0, 1), \\ &\quad F(0) = 0, F(1) = 1, F'(0) \neq 1 \neq F'(1), \ F''(0) \text{ and } F''(1) \text{ exist}\}, \\ E^1 &:= \{f \in C^1(I, I) : \ f(0) = 1, f(1) = 1, \ 0 < m \leq f'(x) \leq M, \\ &\quad |f'(x) - f'(y)| \leq M|x - y| \ \forall x, y \in I\}. \end{aligned}$$

For given integer $n \geq 2$, it is generic for F in $A^1(I)$ that there is no n-th iterative roots in E^1. More precisely, all functions F which do not have n-th iterative roots in E^1 become a dense open subset in $A^1(I)$.

If F has more than two fixed points, we can obtain a similar result. Uniqueness of local smooth itertive roots gives us the "best" choice in the infinitely many continuous iterative roots. Some more results are also obtained for uniqueness in [41]. It is also interesting to give uniqueness or generic non-existence for decreasing functions and under weaker conditions.

3. Iterative Equations

An equation including iteration as the main operation is called an *iterative equation.* The problem of iterative roots, namely (2.4), is a special iterative equation, seen in [4, 9, 10, 19, 37]. A general problem is to discuss the equation

$$\lambda_1 f(x) + \lambda_2 f^2(x) + ... + \lambda_n f^n(x) = F(x), \tag{3.6}$$

where the constants λ_i are real and satisfy the normalization condition $\sum_{i=1}^n \lambda_i = 1$. Equation(3.6) is called *polynomial-like iterative equation* or *linear iterative equation* as being a linear combination of iterates. In some aspects it is processed more difficultly than linear differential equations because iteration is a nonlinear operation but differentiation is linear. Such a linear combination of iterates often appears in problems of invariant curves of dynamical systems [7, 17].

Another further generalization is the equation

$$G(f^{n_1}(x), f^{n_2}(x), ..., f^{n_k}(x)) = F(x), \tag{3.7}$$

where $G : X^k := \overbrace{X \times \cdots \times X}^{k} \to X$ and $F : X \to X$ are given mappings. Sometimes equation(3.6) is considered as a *nonlinear iterative equation.*

Equation (3.6) has been studied extensively in one dimensional case, seen in [2, 35, 36]. In 1970's S.Nabeya [16] and J.G.Dhombres [3] investigated the equations

$$f(p + qx + rf(x)) = a + bx + cf(x)$$

and

$$f^2(x) = af(x) + (1 - a)x.$$

In the early 1980's L.Zhao [47] and A.Mukherjea and J.Ratti [15] considered (3.6) for $n = 2$ and for special F respectively. However, it remained an open problem how to give existence and structure of solutions for the general (3.6), as remarked by J.Matkowski [12].

In 1986 a result of existence for the general (3.6) was given in [38] with fixed point theory and a method of construction for an operator, called "structural operator" sometimes for convenience since it describes the basic relation of iteration in this equation.

Theorem 7. *If $X = I := [a, b]$, $F : I \to I$ is an increasing function with fixed points at a and b and Lipschitz constant $M > 1$ and coefficients $\lambda_1 > 0, \lambda_i \geq 0,\ i = 2, ..., n$, satisfy the normalization condition $\sum_{i=1}^{n} \lambda_i = 1$, then (3.6) has a continuous solution of the same type with Lipschitz constant M/λ_1.*

With this method uniqueness and continuous dependence are also obtained in [38, 39]. Later C^1 and C^2 smoothness are given in [40, 21] respectively. Furthermore, some similar results are also given for the nonlinear equation (3.7) in [23, 24, 26]. A rigorous proof for the general C^k smoothness, as a problem proposed in [35], is also given for (3.6) in [11].

In 2000 W.Zhang and J.A.Baker [44] discussed the problem of variate coefficients, i.e., the equation

$$\lambda_1(x)f(x) + \lambda_2(x)f^2(x) + ... + \lambda_n(x)f^n(x) = F(x), \tag{3.8}$$

where all $\lambda_i(x)$ are continuous functions. They modified the method of construction for the "structural operator" and gave results on existence, uniqueness and continuous dependence for its solutions. Such a method is applied to study (3.7) in [25, 29].

It is interesting to study the symmetry of solutions, which is described by equivariance of a mapping under the action of a Lie group. Suppose that Γ is a Lie group of linear transformations of X. As in [5, 6], say that $f : X \to X$ is Γ-*equivariant* if

$$f(\gamma x) = \gamma f(x) \quad \forall x \in X,\ \gamma \in \Gamma.$$

Sometimes we also say that $f : A \subset X \to X$ is of Γ-*equivariance* if f is a restriction of a Γ-equivariant mapping on the subset A. An odd function as a mapping on $\mathbf{R}$ is $\mathbf{Z}_2$-equivariant, where $\mathbf{Z}_2 = \{1, -1\}$ is a group acting on $\mathbf{R}$. In the case of $\mathbf{R}^1$, invertible linear transformations of $\mathbf{R}$ take the form $x \mapsto \gamma x$ where $0 \neq \gamma \in \mathbf{R}$. Any Lie group acting linearly on $\mathbf{R}$ can therefore be identified with a subgroup of $GL(\mathbf{R})$, the multiplicative topological group of nonzero reals. All such groups are abelian. Let $V \subset \mathbf{R}_0$ be a set of generators for a subgroup $\Gamma \subset GL(\mathbf{R})$ and write $\Gamma = \langle V \rangle$, which consists of all finite products $v_1 v_2 \cdots v_m$ where $v_1, v_2, \cdots, v_m \in V \cup V^{-1}$. We say that Γ is *finitely generated* if it has a finite set of generators, and *topologically finitely generated* if it is the closure in $GL(\mathbf{R})$ of a finitely generated group. Clearly, if $V = (1, c]$ for $c > 1$ then $\Gamma = (0, \infty)$; if V contains an element γ with $|\gamma| \neq 1$ then Γ is infinite; if $V = \{-1\}$ then $\Gamma = Z_2 = \{-1, 1\}$, the unique nontrivial compact Lie group contained in $GL(\mathbf{R})$.

Let $I=[-1,1], M\geq 1\geq m\geq 0$ and $\Gamma\subset GL(\mathbf{R})$. Define

$$\begin{aligned}\mathcal{F}_\Gamma(I) &= \{f\in\mathcal{C}(I)|f(\gamma x)=\gamma f(x),\ \forall\gamma\in\Gamma \text{ and } \forall x,\gamma x\in I\},\\ \mathcal{F}(I;m,M) &= \{f\in\mathcal{C}(I)|f(-1)=-1, f(1)=1, \text{and}\\ &\qquad m(y-x)\leq f(y)-f(x)\leq M(y-x),\forall y>x\in I\},\\ \mathcal{F}_\Gamma(I;m,M) &= \mathcal{F}(I;m,M)\cap\mathcal{F}_\Gamma(I).\end{aligned}$$

where $\mathcal{C}(I)$ is a real Banach space consisting of all continuous real-valued functions on I with respect to the uniform norm $\|f\|=\max\{|f(t)|:t\in I\}$ for $f\in\mathcal{C}(I)$. We can prove that $\mathcal{F}_\Gamma(I;m,M)$ is a compact convex subset of $\mathcal{C}(I)$ if $\Gamma\subset GL(\mathbf{R})$ is topologically finitely generated. In [45] the following result is given.

Theorem 8. *Suppose that Γ is a topologically finitely generated Lie group acting on $\mathbf{R}$, and that $M>1$. If $F\in\mathcal{F}_\Gamma(I;0,M)$ and coefficients $\lambda_1>0,\lambda_i\geq 0,\ i=2,...,n$, satisfy the normalization condition $\sum_{i=1}^n\lambda_i=1$, then equation (3.6) has a continuous solution $f\in\mathcal{F}_\Gamma(I;0,\frac{M}{\lambda_1})$ of Γ-equivariance.*

With symmetry it is possible to generalize the theory to $\mathbf{R}^N$ for $N>1$. Let $\Gamma=O(N)$ be the orthogonal group in its standard representation on $\mathbf{R}^N$, i.e., $\mathbf{O}(N)=\{A\in GL(N):\ AA^T=I_N\}$, where A^T denotes the transpose of A. Let $B=B^N$ be the unit ball $B^N=\{x\in\mathbf{R}^N|\|x\|\leq 1\}$. Obviously B is a Γ-invariant subset of $\mathbf{R}^N$. Let $\langle\cdot,\cdot\rangle$ denote the usual inner product on $\mathbf{R}^N$, and define

$$\begin{aligned}&\mathcal{F}_\Gamma(B)=\{f\in\mathcal{C}(B)|f(\gamma x)=\gamma f(x),\ \forall\gamma\in\Gamma \text{ and } \forall x,\gamma x\in B\},\\ &\mathcal{F}(B;m,M)=\{f\in\mathcal{C}(B)|f \text{ fixes } \partial B \text{ pointwise, and for any } v\in B,\\ &\qquad\qquad t_2\geq t_1 \text{ we have}\\ &\qquad\qquad m(t_2-t_1)\|v\|^2\leq\langle f(t_2v)-f(t_1v),v\rangle\leq M(t_2-t_1)\|v\|^2\\ &\qquad\qquad \text{when both } t_1v \text{ and } t_2v\in B\},\\ &\mathcal{F}_\Gamma(B;m,M)=\mathcal{F}(B;m,M)\cap\mathcal{F}_\Gamma(B),\end{aligned}$$

where $M\geq 1\geq m\geq 0$. The following result is also given in [45].

Theorem 9. *Let $F\in\mathcal{F}_{O(N)}(B;0,M)$ where $M>1$. Then equation(3.6) where $\lambda_1>0,\lambda_i\geq 0$ for $i=2,...,n$, and $\sum_{i=1}^n\lambda_i=1$, has an continuous solution $f\in\mathcal{F}_{O(N)}(B;0,\frac{M}{\lambda_1})$ of $O(N)$-equivariance.*

We can also give uniqueness and continuous dependence for those symmetric solutions. Furthermore, we can generalize our results to the action of another group $\Gamma=\mathbf{O}(N)\times\langle cI_N\rangle$ on $\mathbf{R}^N$, where I_N is the identity on $\mathbf{R}^N$ and $0<c\in\mathbf{R}$. C^1 smoothness of symmetric solutions is given in [46]. Symmetric solutions for nonlinear equation (3.7) and the iterative differential equation

$$\frac{d}{dt}x(t)=G(x^{n_1}(t),x^{n_2}(t),...,x^{n_k}(t)) \tag{3.9}$$

are discussed in [31, 32].

A special form of (3.6) is in the case of linear $F(x)=-\lambda_0 x$, which is called *homogeneous linear iterative equation* and can be represented equivalently by

$$f^n(x)=a_{n-1}f^{n-1}(x)+a_{n-2}f^{n-2}(x)+...+a_1f(x)+a_0x,\quad x\in\mathbf{R}. \tag{3.10}$$

It is also attracts attention of many mathematicians [3, 8, 13, 14, 15, 16, 27].

We hope to have a theory of characteristic for (3.10) but it does not have so simple algebraic structure as homogeneous linear differential equations, for which all solutions can be spanned linearly by finitely many characteristic solutions (or eigensolutions). Enlightened by Euler's idea of characteristic for differential equations, for equation (3.10) we formally consider a linear solution

$$f(x) = rx, \quad x \in \mathbf{R}, \tag{3.11}$$

where $r \in \mathbf{C}$ is indeterminate. It reduces easily to the algebraic equation

$$r^n - a_{n-1}r^{n-1} - ... - a_1 r - a_0 = 0, \tag{3.12}$$

where the polynomial on the left-hand-side is denoted simply by $P_n(r)$. We call $P_n(r)$ the *characteristic polynomial* of equation (3.6) and call (3.12) the *characteristic equation* of equation (3.6). Its roots are called the *characteristic roots* of equation (3.6) and the corresponding solutions (3.11) are called the *characteristic solutions* of equation (3.6). Of course there is one-to-one correspondence between the characteristic roots and the characteristic solutions of Eq.(3.6), and there are at most n characteristic solutions. We are mainly interested in the real solutions.

By the relations between roots and coefficients of polynomials, equation (3.6) is equivalent to

$$f^n(x) - (\sum_{i=1}^{n} r_i) f^{n-1}(x) + (\sum_{i<j}^{n} r_i r_j) f^{n-2}(x) + ... + (-1)^n r_1 r_2 \cdots r_n x, \tag{3.13}$$

where $r_1, ..., r_n$ are n complex roots of the polynomial $P_n(r)$. If $a_0 \neq 0$, it is easy to prove that equation (3.6) is also equivalent to

$$f^{-n}(x) - (\sum_{i=1}^{n} s_i) f^{-(n-1)}(x) + (\sum_{i<j}^{n} s_i s_j) f^{-(n-2)}(x) + ... + (-1)^n s_1 s_2 \cdots s_n x, \tag{3.14}$$

where f^{-k} denotes the k-th iterate of f^{-1} and $s_i = r_i^{-1}$, $i = 1, 2, ..., n$. Sometimes we call (3.14) is the *dual equation* of (3.13).

When $n = 2$ a complete description of solutions with characteristic is given in [13]. For the general n, we have the following basic result, which was proposed first in [12] and proved in [14].

Theorem 10. . *Let $m, k \in \mathbf{N}$ be such that $m \geq k \geq 1$. Suppose*

$$P(x) = x^m - a_{m-1}x^{m-1} - ... - a_0; \quad Q(x) = x^k - b_{k-1}x^{k-1} - ... - b_0$$

are polynomials such that $Q|P$ (i.e. Q divides P). If $f : \mathbf{R} \to \mathbf{R}$ satisfies the iterative equation

$$f^k(x) = b_{k-1}f^{k-1}(x) + ... + b_1 f(x) + b_0 x, \quad x \in \mathbf{R},$$

then f satisfies the iterative equation

$$f^m(x) = a_{m-1}f^{m-1}(x) + ... + a_1 f(x) + a_0 x, \quad x \in \mathbf{R}.$$

With *linear difference forms*, in [30] the general iterate f^m of a solution f is formulated in a linear combination of $f^0, f, ..., f^{n-1}$. Thus structure of solutions are given in some cases where $r_n > ... > r_2 > r_1 > 1$, $0 < r_1 < r_2 < ... < r_n < 1$, $r_1 < r_2 < ... < r_n < -1$ or $-1 < r_1 < r_2 < ... < r_n < 0$. Some results are also obtained in the case of $r_1 = r_2 = ... = r_n = r$. It is proved in [30] that equation (3.10) has no continuous solutions on $\mathbf{R}$ if it has no real characteristic roots. However, we do not have a perfect result on structure of solutions in the cases of multiple characteristic roots or even in the cases where it has only simple characteristic roots but some of them are positive and the others are negative.

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