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SYMPLECTIC REAL BOTT MANIFOLDS (EXTENDED ABSTRACT)

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1. Introduction

A *real Bott tower* (of height n) is a sequence of $\mathbb{R}\mathrm{P}^1$-bundles:

$$M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 = \{\text{a point}\},$$

where each $\mathbb{R}\mathrm{P}^1$-bundle $M_i \to M_{i-1}$ is the projectivization of a Whitney sum of two real line bundles on M_{i-1}. Each M_i is called a *real Bott manifold*. Clearly $M_1 = \mathbb{R}\mathrm{P}^1$ and $M_2 = (\mathbb{R}\mathrm{P}^1)^2$ or a Klein bottle. If every bundle in the tower is trivial, then $M_n = (\mathbb{R}\mathrm{P}^1)^n$. However, there are many choices of non-trivial bundles at each stage in the tower and it is known that there are many different diffeomorphism classes in real Bott manifolds ([6], [7]). A real Bott manifold is also an example of a real toric manifold which admits a flat Riemannian metric ([6]).

Although orientable ones occupy a small portion in all real Bott manifolds ([3]), the number of orientable ones of dimension n approaches infinity as n approaches infinity. Among those orientable ones, some are *symplectic*, i.e., admit a symplectic form. In this paper we give a complete characterization of symplectic real Bott manifolds (Theorem 3.1). In particular, we prove that among real Bott manifolds M the following are equivalent:

(1) M is cohomologically symplectic,
(2) M is symplectic,
(3) M admits a Kähler structure.

We remark that the implication (3) $\Rightarrow$ (2) $\Rightarrow$ (1) always holds but the reverse implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) do not hold in general as is well-known. For example, $\mathbb{C}P^2 \# \mathbb{C}P^2$ is cohomologically symplectic but not symplectic because it does not admit an almost complex structure and a certain T^2-bundle over T^2 constructed in [8] is symplectic but does not admit a Kähler structure.

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2. Quotient description of real Bott manifolds

In this section, we recall the quotient description of real Bott manifolds (see [6] and [7] for details).

Let $\mathfrak{B}(n)$ be the the set of $n \times n$ upper triangular $(0,1)$ matrices with zero diagonal entries. For a matrix $A \in \mathfrak{B}(n)$, A^i_j denotes the (i,j) entry of A and A^i (respectively, A_j) denotes the i-th row (respectively, j-th column) of A. Let S^1 be the unit circle in $\mathbb{C}$. For $z \in S^1$ and $a \in \mathbb{Z}/2 = \{0,1\}$, we set $z(a) := a$ if $a = 0$ and $\bar{z}$ if $a = 1$. We then define the involution a_i on $T^n := (S^1)^n$ by

$$a_i(z_1, \dots, z_n) := (z_1, \dots, z_{i-1}, -z_i, z_{i+1}(A^i_{i+1}), \dots, z_n(A^i_n)) \tag{2.1}$$

for $i = 1, \dots, n$. Let $G(A)$ denote the transformation group on T^n generated by a_i's. Then the quotient space $M(A) := T^n/G(A)$ is known to be a real Bott manifold and every real Bott manifold can be obtained as $M(A)$ for some $A \in \mathfrak{B}(n)$. Although A is not necessarily uniquely determined by a real Bott manifold, A contains all geometrical information on $M(A)$. For example,

$$M(A) \text{ is orientable } \Longleftrightarrow \sum_{j=1}^{n} A^i_j = 0 \text{ in } \mathbb{Z}/2 \text{ for any } i \tag{2.2}$$

(see [6]).

3. Main theorem

The following is our main theorem.

Theorem 3.1 (see [5].). *Let $A \in \mathfrak{B}(2n)$. The following conditions are equivalent:*

(1) *$M(A)$ is cohomologically symplectic, that is, there exists an $\alpha \in \mathrm{H}^2(M(A))$ such that α^n is nonzero.*
(2) *There exist n subsets $\{j_1, j_{n+1}\}, \dots, \{j_n, j_{2n}\}$ of $\{1, 2, \dots, 2n\}$ such that*
   - $\coprod_k^n \{j_k, j_{k+n}\} = \{1, 2, \dots, 2n\}$ *and*
   - $A_{j_1} = A_{j_{n+1}}, \dots, A_{j_n} = A_{j_{2n}}$.
(3) *There exists a symplectic form on $M(A)$.*
(4) *There exists a Kähler structure on $M(A)$.*

Moreover, any $\alpha \in \mathrm{H}^2(M(A))$ in (1) can be represented by a symplectic form on $M(A)$.

Example 3.2. Let $A \in \mathfrak{B}(4)$. If A is the zero matrix, then $M(A)$ is the 4-dimensional torus and symplectic. Suppose that A is non-zero and $M(A)$ is symplectic. Then it follows from Theorem 3.1 (2) that A is one of the following:

$$\begin{pmatrix} 0&1&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&1\\0&0&1&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&1\\0&0&0&0\\0&0&0&0 \end{pmatrix}.$$

Real Bott manifolds $M(A)$ for A above are diffeomorphic to each other but not diffeomorphic to the 4-dimensional torus ([6], [7]). On the other hand, T^2-bundles over T^2 which are symplectic are classified in [4]. One can easily check that our $M(A)$ is of type $\{-I, I, (0,0)\}$ in [4, Table 1] .

Finally we note that if

$$A = \begin{pmatrix} 0&1&1&0\\0&0&1&1\\0&0&0&0\\0&0&0&0 \end{pmatrix},$$

then $M(A)$ is orientable by (2.2), but not symplectic. Therefore the class of symplectic real Bott manifolds is strictly smaller than that of orientable real Bott manifolds.

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