Trends in Mathematics
Information Center for Mathematical Sciences
Volume 4, Number 2, December 2001, Pages 81–86

GLOBAL EXISTENCE OF SOLUTIONS TO THE EINSTEIN-MAXWELL-KLEIN-GORDON EQUATIONS IN THE SPHERICAL SYMMETRY

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Abstract. We present the result of the global unique existence of classical solutions to the Einstein equations coupled with Maxwell-Klein-Gordon fields for small initial data under the spherical symmetry. We also obtain the decay estimates of the solutions.

Introduction

The Cauchy problem of the Einstein equations is initiated by Choquet-Bruhat[4], and the general type of local unique existence of solution is established using the harmonic coordinates(See [8],[10],[15] for updated surveys.). Compared to the local existence the global existence results are not general in terms of the assumptions on the initial data, and symmteries of the Lorentzian geomteries to construct. The most remarkable one of them is the global existence results for the vacuum Einstein equations for initial data close to the trivial one due to Christodoulou and Klainerman[6], the argument of which is simplified in [7],[14]. We also refer the corresponding result for the spatially closed space-time due to Andersson and Moncrief[1]. For the Einstein-matter system there is a semi global result for the Einstein-Maxwell-Yang-Mills system for small data due to Friedrich[9]. For the small data global existence of the Einstein equations coupled with the other matters we note the self-graviting scalar system by Christodoulou[5], and the self-graviting Vlasov-Poisson system by Rein and Rendall[16] both of which under the assumption of the spherical symmetry of the space-time.

In this paper we are concerned on the global existence problem for the Einstein equations coupled with the massless Maxwell-Klein-Gordon fields in the spherical symmetry. For the Maxwell-Klein-Gordon equations in the Minkowski space-time we have global existence of solutions with arbitrary size finite energy due to Klainerman and Machedon[13]. For the Maxwell-Klein-Gordon equations coupled with the Einstein equations in the spherical symmetry there are numerical studies due to Hod and Piran([11],[12]) regarding to the critical behaviour during collapse. We will show in this paper that under the assumption of spherical symmetry there exists a unique global classical solution to the Einstein-Maxwell-Klein-Gordon system for small initial data. Our study is motivated mainly by the work of Christodoulou

2000 *Mathematics Subject Classification.* 35Q75, 35L15.
Key words and phrases. Einstein's equations, Cauchy problem, global existence.
Received May 8, 2001.

in [5].

We consider the space and time oriented Lorentzian manifold diffeomorphic to $\mathbb{R}^4$, on which the group $SO(3)$ acts as an isometry, and the group orbits are the metric spacelike 2-spheres. The invariants of the group form a time-like curve in the space-time, which is the world line of the center of the spheres. In this spherically symmetric space-time it is convenient to introduce the function r, defined by

$$r = \sqrt{\frac{A}{4\pi}},$$

where A is the area of the 2-sphere. Thus the metric on the 2-sphere is given by

$$ds^2 = r^2 d\Sigma^2 = r^2(d\theta^2 + sin^2\theta d\phi^2).$$

The quotient, Q, of the space-time by $SO(3)$ is the Lorentzian 2-manifold with signature 0. We will define a coordinate u, which is a constant on every future null cones with vertices on the centers of the 2-spheres. With this coordinate system we can represent the metric in the form

$$ds^2 = -g(u,r)\tilde{g}(u,r)du^2 - 2g(u,r)dudr + r^2 d\Sigma^2, \tag{0.1}$$

where g and $\tilde{g}$ tend to 1 as r goes to infinity. We will use the notations $e_0 = \partial_u$, $e_1 = \partial_r$, $e_2 = \partial_\theta$, $e_3 = \partial_\varphi$, and write $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ in the following sections.

1. The Einstein-Maxwell-Klein-Gordon Equations

The Lagrangian of the Maxwell-Klein-Gordon fields in a given metric $g_{\mu\nu}$ is

$$\mathcal{L}_{MKG} = -\frac{1}{8\pi}F_{\mu\alpha}F_{\nu\beta}g^{\mu\nu}g^{\alpha\beta} - g^{\mu\nu}D_\mu\phi(D_\nu\phi)^*, \tag{1.1}$$

where $\phi = \phi_1 + i\phi_2, \quad i = \sqrt{-1}$ is a complex scalar field, and $D_\mu\phi = \partial_\mu\phi + iA_\mu$ is the (gauge) covariant derivative, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The corresponding energy-momentum-stress tensor $T_{\mu\nu}$ is

$$T_{\mu\nu} = \frac{1}{2}Re\{D_\mu\phi(D_\nu\phi)^*\} + \frac{1}{4\pi}F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} + \frac{1}{2}g_{\mu\nu}\mathcal{L}_{KGM}. \tag{1.2}$$

The Einstein-Maxwell-Klein-Gordon equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}. \tag{1.3}$$

This equation is coupled with the matter equations

$$\nabla_\mu T^{\mu\nu} = 0, \tag{1.4}$$

which corresponds to

$$g^{\mu\nu}D_\mu D_\nu\phi = 0 \tag{1.5}$$

with its complex conjugate, and

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu, \qquad J^\mu = Im\{\phi(D^\mu\phi)^*\}. \tag{1.6}$$

Following [5], we introduce the new function

$$h = \frac{\partial(r\phi)}{\partial r}.$$

Then,

$$\phi = \bar{h} = \frac{1}{r}\int_0^r h(s)ds.$$

Analogously to the flat Maxwell-Klein-Gordon system we introduce the local charge function

$$Q(u,r) = \int_{B(0,r)} J^0 dv = 4\pi \int_0^r J^0 \sqrt{-\gamma} r^2 dr,$$

which, physically, represents the total charge inside $B(0,r)$, the ball of radius r, where we used the notation $\gamma = det(g_{\mu\nu})$. In our choice of metric in (0.1) we can represent the charge function in terms of h as

$$Q(u,r) = 4\pi i \int_0^r s(\bar{h}^* h - \bar{h} h^*)ds. \tag{1.7}$$

In the spherical symmetry we assume $A_2 = A_3 = 0$ as usual. Then, we can choose our gauge so that $A_1 = 0$. Thus, among the four gauge field components we are left only with $A_0 = A_0(u,r)$ as a nontrivial unknown. Then, the radial component of (1.6) can be integrated to give

$$A_0 = \int_0^r \frac{Q}{s^2} g ds. \tag{1.8}$$

The (11)− component of (1.3) is

$$\frac{2}{r}\frac{1}{g}\frac{\partial g}{\partial r} = 8\pi \left|\frac{\partial \phi}{\partial r}\right|^2. \tag{1.9}$$

This, combined with the formula

$$\frac{\partial \phi}{\partial r} = \frac{h - \bar{h}}{r}$$

gives

$$g = \exp\left[-4\pi \int_r^\infty \frac{|h-\bar{h}|^2}{s} ds\right]. \tag{1.10}$$

The (01)−component of (1.3) is

$$\frac{1}{r^2}\tilde{g}\left[\frac{g}{\tilde{g}} + f\frac{\tilde{g}}{g}\frac{\partial}{\partial r}\left(\frac{g}{\tilde{g}}\right) - 1\right] = 8\pi\left[\frac{1}{2}\tilde{g}\left|\frac{\partial\phi}{\partial r}\right|^2 + \frac{1}{8\pi}g\frac{Q^2}{r^4}\right],$$

which implies

$$\begin{aligned}\tilde{g} &= \frac{1}{r}\int_0^r \left(1 - \frac{Q^2}{s^2}\right) g ds \\ &= \bar{g} - \frac{1}{r}\int_0^r \frac{Q^2}{s^2} g ds,\end{aligned} \tag{1.11}$$

where we set $\bar{g} = \frac{1}{r}\int_0^r g ds$. By the expression in (1.10) g is a monotone nondecreasing function of r, and thus we have

$$0 < \tilde{g} \leq \bar{g} \leq g \leq 1.$$

Using (1.9),(1.10) and (1.12), we can write the equation (1.5) as

$$Dh = \frac{1}{2r}(g - \tilde{g})(h - \bar{h}) - \frac{Q^2}{2r^3}(h - \bar{h})g - \frac{iQ}{2r}g\bar{h} - ihA_0, \tag{1.12}$$

where

$$D = \frac{\partial}{\partial u} - \frac{\tilde{g}}{2}\frac{\partial}{\partial r}.$$

The local mass function is given by

$$m(u,r) = \frac{r}{2}\left(1 - \frac{\tilde{g}}{g} + \frac{Q^2}{r^2}\right), \tag{1.13}$$

which represents the total mass inside a ball of radius r at the retarded time u. The fact that $\tilde{g} \le g$ implies $m(u,r) \ge 0$. We compute

$$\begin{aligned} \frac{\partial}{\partial r}\left(m - \frac{Q^2}{2r}\right) &= \frac{\partial}{\partial r}\left[\frac{r}{2}\left(1 - \frac{\tilde{g}}{g}\right)\right] \\ &= 2\pi\frac{\tilde{g}}{g}|h - \bar{h}|^2 + \frac{Q^2}{2r^2}, \end{aligned} \tag{1.14}$$

which shows that the function $m - \frac{Q^2}{2r}$ is a monotone nondecreasing function of r at u. On the other hand, (1.14) can be integrated to give the following another representation of the mass

$$m(u,r) = \int_0^r \left[2\pi\frac{\tilde{g}}{g}|h - \bar{h}|^2 + \frac{Q^2}{2s^2}\right] ds + \frac{Q^2}{2r}, \tag{1.15}$$

where we assume $m - Q^2/2r = 0$ at $r = 0$.

2. Global Existence of Solutions

In this section we consider the initial value problem of (1.10)-(1.12) with the initial data $h(0,r)$. Let us introduce the function space X defined by

$$X = \{\ h(\cdot,\cdot) \in C^1([0,\infty)\times[0,\infty)) \quad | \quad \|h\|_X < \infty\ \},$$

where

$$\|h\|_X := \sup_{u\ge 0}\sup_{r\ge 0}\left\{\left(1 + r + \frac{u}{2}\right)^2 |h(u,r)| + \left(1 + r + \frac{u}{2}\right)^3 \left|\frac{\partial h}{\partial r}(u,r)\right|\right\}.$$

We also introduce

$$X_0 = \{\ h(\cdot) \in C^1([0,\infty)) \quad | \quad \|h\|_{X_0} < \infty\ \},$$

where

$$\|h\|_{X_0} := \sup_{r\ge 0}\left\{(1+r)^2|h(r)| + (1+r)^3\left|\frac{\partial h}{\partial r}(r)\right|\right\}.$$

and denote $\|h_0\|_{X_0} = d$. Below we will also use the space Y containing X, and defined by

$$Y = \{h(\cdot,\cdot) \in C^1([0,\infty)\times[0,\infty)) \quad | \quad h(0,r) = h_0(r), \quad \|h\|_Y < \infty\},$$

where

$$\|h\|_Y = \sup_{u\ge 0}\sup_{r\ge 0}\left\{\left(1 + r + \frac{u}{2}\right)^2 |h(u,r)|\right\}.$$

The purpose of this section is the proof of the following theorem.

Theorem 2.1. *Suppose we have initial data* $h(0,r) \in C^1[0,\infty)$ *such that* $h(0,r) = O(r^{-2})$ *and* $\frac{\partial h}{\partial r}(0,r) = O(r^{-3})$. *Let us put* $d = \|h(0,\cdot)\|_{X_0}$. *Then, there exists* $\delta > 0$ *such that if* $d < \delta$, *then there exists a unique global classical solution* $h \in C^1([0,\infty) \times [0,\infty))$ *of (1.10)-(1.12) with* $h(0,r)$ *as the initial data. This solution has the decay property:*

$$|h(u,r)| \le C(1+u+r)^{-2}, \qquad \left|\frac{\partial h}{\partial r}(u,r)\right| \le C(1+u+r)^{-3}. \tag{2.1}$$

The proof uses contraction argument. The details of the proof of the above theorem, and its generalizations to the Einstein equations coupled with the nonlinear Klein-Gordon equation, and the Maxwell-Higgs system, are found in [2], [3].

Acknowledgements

This research is supported partially by the grant no.2000-2-10200-002-5 from the basic research program of the KOSEF and the SNU Research fund..

References

[1] L.Andersson and V.Moncrief, *"On the global evolution problem in 3+1 gravity"*, J.Geom.Phys., under preparation.

[2] D. Chae, *"Global Existence of Solutions to the Coupled Einstein and Maxwell-Higgs System in the Spherical Symmetry"*, submitted.

[3] D. Chae, *Global Existence of Spherically Symmetric Solutions to the Coupled Einstein and Nonlinear Klein-Gordon System"*, submitted.

[4] Y.Choquet-Bruhat, *"Solutions* C^∞ *d'équations hyperboliques non linéaires"*, C.R.Acad.Sci.Paris **272**, (1968), 386-388.

[5] D.Christodoulou, *"The problem of a self-graviting scalar field"*, Comm.Math.Phys.,**105**, (1986), 337-361.

[6] D.Christodoulou and S.Klainerman, *"The global non linear stability of the Minkowski space"*, Princeton Mathematical series, **41**, (1993).

[7] D.Christodoulou and S.Klainerman, F.Nicolò, *"On a null hypersurface approach to the proof of stability of the Minkowski space"*. under preparation.

[8] P.T.Chrúsciel, *"On the uniqueness in the large of solutions to the Einstein equations ("strong cosmic censorship")"*, Mathematical aspects of classical field theory (Seattle,WA,1991),Amer.Math.Soc., Providence,RI,(1992), 235-273.

[9] H.Friedrich, *"On the global existence and the asymptotic behavior of solutions to the Einstein-Maxwell-Yang-Mills equations"*, J.Differential Geom. **34**, no.2, (1991), 275-345.

[10] H.Friedrich and A. Rendall, *" The Cauchy problem for the Einstein equations"*, Einstein's Field Equations and their Physical Interpretation (ed. B. G. Schmidt) Springer, Berlin, 2000, URL: http://xxx.lanl.gov/gr-qc/0002074.

[11] S.Hod and T. Piran, *"Mass inflation in dynamical gravitational collapse of a charged scalar field"*, Phys. Rev. Lett. **81**, no.8, (1998), 1554-1557.

[12] S.Hod and T. Piran, *"Critical behaviour and universality in gravitational collapse of a charged scalar field"*, Phys.Rev. **D55**,(1997), 3485-3496.

[13] S.Klainerman and M. Machedon, *"On the Maxwell-Klein-Gordon equation with finite energy"*, Duke Math. J., **74**, no.1, (1994), 19-44.

[14] S.Klainerman and F.Nicolò,"On local and global aspects of the Cauchy problem in General Relativity", *Class. Q. Grav.*, **16**, (1999), R73-R157.

[15] A.D.Rendall, *"Local and global existence theorems for the 'Einstein equations"*, URL: http://xxx.lanl.gov/gr-qc/0001008.

[16] G.Rein and A.D.Rendall, *"Global existence of solutions of the spherically symmteric Vlasov-Einstein system with small initial data"*, Comm. Math. Phys., **150**, (1992), 561-583.

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