Trends in Mathematics - New Series
Information Center for Mathematical Sciences
Volume 10, Number 2, 2008, pages 57–64

2008 International Workshop
on Dynamical Systems and
Related Topics

ON SOME USEFUL CONDITIONS FOR HYPERBOLICITY

MARCIN MAZUR

Abstract. In the paper we investigate the hyperbolicity, semi-hyperbolicity and cone field conditions for compact invariant sets of C^1 diffeomorphisms. We give an independent proof of the main theorem of [6] and propose some improvement on this result.

1. Introduction

As is well-known, computer assisted methods play an important role in proving some strong results that defy analytical investigation (see, e.g., [7, 8, 9, 12, 13, 14, 15, 16]).

One of the most recent directions in this field is the strict numerical verification of the hyperbolicity condition. The main problem that should be dealt with on this occasion is to fulfill the classical assumption on the existence of an invariant splitting, which cannot be directly examined with help of appropriate computer programs only.

So far, there were obtain a few significant results concerning hyperbolicity of the Hénon map. In [4] using the notion of a cone field Hruska proved hyperbolicity of the complex Hénon attractor with certain parameter values. Another approach was presented by Arai [1], who applied the notion of quasi-hyperbolicity and show the existence of a large set of parameter values for which the chain recurrent set of the Hénon map is hyperbolic. The most recent concept, due to Mazur and Tabor [5],

2000 *Mathematics Subject Classification.* Primary: 37D05, 37D20, Secondary: 37M99.
Key words and phrases. Hyperbolicity, semi-hyperbolicity, cone field, numerical computations.

is based on the notion of semi-hyperbolicity introduced in [2, 3] for more general case of (not necessarily invertible) Lipshitz mappings.

In this paper, which is (in some sense) continuation of [6], we investigate the relation between the hyperbolicity, semi-hyperbolicity and cone field conditions. In Section 2 we give an independent proof of the main theorem of [6] as well as in Section 3 we propose some improvement on this result involving, so called, variable constants approach that will probably be more useful for strict numerical computations.

2. Hyperbolicity, semi-hyperbolicity and cone fields

At first we recall the notion of hyperbolicity. Let $f\colon U \to f(U) \subset \mathbb{R}^n$ be a C^1 diffeomorphism. Here and later on U and $\|\cdot\|$ denote an open set and an original norm in $\mathbb{R}^n$, respectively.

Definition 2.1. An f-invariant compact subset K of U is said to be (λ_s, λ_u)-hyperbolic, where

$$0 < \lambda_s < 1 < \lambda_u,$$

if for each $x \in K$ there exist a norm $\|\cdot\|_x$ on $\mathbb{R}^n$ and a splitting $\mathbb{R}^n = E^s_x \oplus E^u_x$ with corresponding projections P^s_x and $P^u_x = I - P^s_x$ such that

[H0] the norms $\|\cdot\|_x$ are uniformly equivalent, i.e., $c^{-1}\|v\|_x \leq \|v\| \leq c\|v\|_x$ for some constant $c > 0$ and all $x \in K$, $v \in \mathbb{R}^n$;

[H1] the projections are uniformly bounded, i.e., $\max\{\|P^s_x\|, \|P^u_x\|\} \leq h$ for some constant $h > 0$ and all $x \in K$;

[H2] the splitting is invariant, i.e., $D_x f E^{s,u}_x = E^{s,u}_{f(x)}$ for $x \in K$;

[H3] $\|D_x f P^s_x v\|_{f(x)} \leq \lambda_s \|P^s_x v\|_x$ and $\|D_x f P^u_x v\|_{f(x)} \geq \lambda_u \|P^u_x v\|_x$ for $x \in K$, $v \in \mathbb{R}^n$.

Remark 2.1. The condition [H1] is equivalent (see [11]) to the following one:

[H1'] the projections are continuously dependent on the points of K, i.e., the functions $x \mapsto P^s_x$ and $x \mapsto P^u_x$ are continuous in all $x \in K$.

If we do not require the invariance of the splitting spaces and use some (a priori) weaker conditions instead, we obtain the notion of semi-hyperbolicity.

Definition 2.2. An f-invariant compact subset K of U is said to be $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semi-hyperbolic, where

$$0 < \lambda_s < 1 < \lambda_u \ \text{ and } \ (1 - \lambda_s)(\lambda_u - 1) > \mu_s \mu_u > 0,$$

if for each $x \in K$ there exist a norm $\|\cdot\|_x$ on $\mathbb{R}^n$ and a splitting $\mathbb{R}^n = E^s_x \oplus E^u_x$ with corresponding projections P^s_x and $P^u_x = I - P^s_x$ such that the conditions [H0], [H1] hold and

[SH2] the dimension of the splitting spaces is constant over individual orbits, i.e, $\dim E^{s,u}_x = \dim E^{s,u}_{f(x)}$ for $x \in K$;

[SH3] $\|P^s_{f(x)} D_x f P^s_x v\|_{f(x)} \le \lambda_s \|P^s_x v\|_x, \quad \|P^s_{f(x)} D_x f P^u_x v\|_{f(x)} \le \mu_s \|P^u_x v\|_x,$

$\|P^u_{f(x)} D_x f P^s_x v\|_{f(x)} \le \mu_u \|P^s_x v\|_x, \quad \|P^u_{f(x)} D_x f P^u_x v\|_{f(x)} \ge \lambda_u \|P^u_x v\|_x$

for $x \in K$, $v \in \mathbb{R}^n$.

Obviously, if the set K is (λ_s, λ_u)-hyperbolic, it is also $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semi-hyperbolic for sufficiently small constants $\mu_s, \mu_u > 0$ (the norms and the splitting spaces do not change).

Now we proceed to the notion of expanding and co-expanding cone field.

Definition 2.3. Let ε be a positive constant, $\mathbb{R}^n = E^s \oplus E^u$ be a splitting with corresponding projections P^s, $P^u = I - P^s$ and $\|\!|\cdot|\!\|$ be a norm on $\mathbb{R}^n$. The set

$$C_\varepsilon(E^s, E^u, \|\!|\cdot|\!\|) = \{v \in \mathbb{R}^n \mid \|\!|P^s v|\!\| \le \varepsilon \|\!|P^u v|\!\|\}$$

is called a cone in $\mathbb{R}^n$.

For a set $K \subset \mathbb{R}^n$ a collection of cones

$$\mathcal{C} = \{\mathcal{C}_x\}_{x \in K} = \{C_{\varepsilon(x)}(E^s_x, E^u_x, \|\!|\cdot|\!\|_x)\}_{x \in K},$$

determined by the splittings $\mathbb{R}^n = E^s_x \oplus E^u_x$, the norms $\|\!|\cdot|\!\|_x$ on $\mathbb{R}^n$ satisfying the condition [H0] and the positive real-valued function $\varepsilon(x)$ defined for each $x \in K$, forms a cone field over K.

Definition 2.4. Let $\mathcal{C} = \{\mathcal{C}_x\} = \{C_{\varepsilon(x)}(E^s_x, E^u_x, \|\!|\cdot|\!\|_x)\}$ be a cone field over an f-invariant compact set $K \subset U$. The diffeomorphism f is said to be strongly (λ_s, λ_u)-dominating on $\mathcal{C}$ over K, where

$$\lambda_s^{-1} \lambda_u > 1,$$

if for each $x \in K$ there exists a norm $\|\cdot\|_x$ on $\mathbb{R}^n$ such that [H0], [H1], [SH2] hold and:

[SD3] $\|D_{f(x)} f^{-1} v\|_x \ge \lambda_s^{-1} \|v\|_{f(x)}$, $\|D_x f w\|_{f(x)} \ge \lambda_u \|w\|_x$ for $x \in K$, $v \in \mathbb{R}^n \setminus \mathcal{C}_{f(x)}$, $w \in \mathcal{C}_x$.

If, additionally, $\lambda_s < 1$ and $\lambda_u > 1$ we say that f is λ_u-expanding and λ_s-co-expanding on $\mathcal{C}$ over K.

Remark 2.2. In the above definition one can replace the condition [SD3] by the following one:

[D3] $\inf_{x\in K}\left\{\inf_{v\in\mathbb{R}^n\setminus\mathcal{C}_{f(x)}}\frac{\|D_{f(x)}f^{-1}v\|_x}{\|v\|_{f(x)}}\cdot\inf_{w\in\mathcal{C}_x\setminus\{0\}}\frac{\|D_xfw\|_{f(x)}}{\|w\|_x}\right\}>1.$ x

In this case we say that f is dominating on $\mathcal{C}$ over K. Obviously, the condition [D3] is weaker than [SD3].

The following theorem corresponds to the main result of [6].

Theorem 2.5. *Let $K \subset U$ be a compact f-invariant set.*
(i) *If K is $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semi-hyperbolic according to splittings $\mathbb{R}^n = E^s_x \oplus E^u_x$ and norms $\|\!| \cdot |\!\|_x$ on $\mathbb{R}^n$, then f is $(\lambda_u - \varepsilon\mu_u)$-expanding and $(\lambda_s + \varepsilon^{-1}\mu_s)$-co-expanding on the cone field*

$$\mathcal{C} = \{\mathcal{C}_x\} = \{C_{\varepsilon(x)}(E^s_x, E^u_x, \|\!| \cdot |\!\|_x)\}$$

over K, where

$$\varepsilon(x) = \varepsilon = \frac{c\mu_s}{1-\lambda_s} \quad \text{for all } x \in K, \tag{1}$$

and c is any constant satisfying

$$1 < c < \frac{(1-\lambda_s)(\lambda_u - 1)}{\mu_s\mu_u}.$$

(ii) *If f is λ_u-expanding and λ_s-co-expanding on a cone field $\mathcal{C} = \{\mathcal{C}_x\}$ over K, then K is (λ_s, λ_u)-hyperbolic with respect to some invariant splittings $\mathbb{R}^n = E^s_x \oplus E^u_x$ satisfying $E^s_x \subset \mathbb{R}^n \setminus \mathcal{C}_x$ and $E^u_x \subset \mathcal{C}_x$ for $x \in K$.*

Proof. (i) The conditions [H1], [SH2] are obviously held and

$$\lambda_s + \tfrac{\mu_s}{\varepsilon} = \lambda_s + \tfrac{1-\lambda_s}{c} < \lambda_s + 1 - \lambda_s = 1,$$
$$\lambda_u - \varepsilon\mu_u = \lambda_u - \tfrac{c\mu_s\mu_u}{1-\lambda_s} > \lambda_u - \tfrac{(1-\lambda_s)(\lambda_u-1)}{1-\lambda_s} = 1,$$

hence it remains to show the existence of a family of norms $\|\cdot\|_x$ $(x \in K)$ on $\mathbb{R}^n$ satisfying [H0] and the following estimates for each $x \in K$, $v \in \mathbb{R}^n \setminus \mathcal{C}_{f(x)}$, $w \in \mathcal{C}_x$:

$$\|D_{f(x)}f^{-1}v\|_x \geq (\lambda_s + \frac{\mu_s}{\varepsilon})^{-1}\|v\|_{f(x)}, \ \|D_xfw\|_{f(x)} \geq (\lambda_u - \varepsilon\mu_u)\|w\|_x.$$

Take $x \in K$ and put

$$\|v\|_x = \max\{\varepsilon^{-1}\|\!|P^s_xv|\!\|_x, \|\!|P^u_xv|\!\|_x\} \ \text{ for } \ v \in \mathbb{R}^n.$$

Then the norms $\|\cdot\|_x$ satisfy the condition [H0] and for all $v, w \in \mathbb{R}^n$ such that

$$\|\!|P^s_xw|\!\|_x \leq \varepsilon\|\!|P^u_xw|\!\|_x, \ \|\!|P^s_{f(x)}v|\!\|_{f(x)} > \varepsilon\|\!|P^u_{f(x)}v|\!\|_{f(x)},$$

we have:

$$\begin{aligned}
\|D_x f w\|_{f(x)} &\geq \|P^u_{f(x)} D_x f w\|_{f(x)} \geq \|P^u_{f(x)} D_x f P^u_x w\|_{f(x)} - \|P^u_{f(x)} D_x f P^s_x w\|_{f(x)} \\
&\geq \lambda_u \|P^u_x w\|_x - \mu_u \|P^s_x w\|_x \geq \lambda_u \|P^u_x w\|_x - \mu_u \varepsilon \|P^u_x w\|_x \\
&= (\lambda_u - \varepsilon \mu_u)\|w\|_x, \\
\varepsilon \|v\|_{f(x)} &= \|P^s_{f(x)} v\|_{f(x)} = \|P^s_{f(x)} D_x f D_{f(x)} f^{-1} v\|_{f(x)} \\
&\leq \|P^s_{f(x)} D_x f P^s_x D_{f(x)} f^{-1} v\|_{f(x)} + \|P^s_{f(x)} D_x f P^u_x D_{f(x)} f^{-1} v\|_{f(x)} \\
&\leq \lambda_s \|P^s_x D_{f(x)} f^{-1} v\|_x + \mu_s \|P^u_x D_{f(x)} f^{-1} v\|_x \\
&\leq (\varepsilon \lambda_s + \mu_s)\|D_{f(x)} f^{-1} v\|_x,
\end{aligned}$$

which completes the proof of (i).

(ii) Since f is strongly (λ_s, λ_u)-dominating on $\mathcal{C}$ over K, from Theorem 1.2 of [10] for all $x \in K$ we obtain a splitting $\mathbb{R}^n = E^s_x \oplus E^u_x$ such that [H1], [H2] hold and $E^s_x \subset \mathbb{R}^n \setminus \mathcal{C}_x$, $E^u_x \subset \mathcal{C}_x$. Hence, the condition [H3] follows directly from [SD3], which finishes the proof of (ii). □

An immediate consequence of the above theorem is the following

Corollary 2.6. *Every $(\lambda_s, \lambda_u, \mu_s, \mu_u)$-semi-hyperbolic set $K \subset U$ is also $(\lambda_s + \varepsilon^{-1}\mu_s, \lambda_u - \varepsilon\mu_u)$-hyperbolic, where ε is a constant given by (1).*

3. Variable constants approach

In this section we improve the previous results by applying a simpler (from point of view of possible numerical applications) version of the semi-hyperbolicity condition.

Definition 3.1. Let $\langle\lambda_s\rangle = \{\lambda^x_s\}_{x\in K}$, $\langle\lambda_u\rangle = \{\lambda^x_u\}_{x\in K}$, $\langle\mu_s\rangle = \{\mu^x_s\}_{x\in K}$ and $\langle\mu_u\rangle = \{\mu^x_u\}_{x\in K}$ be collections of positive constants satisfying

$$\lambda^x_s < 1 < \lambda^x_u, \quad m_s m_u > 1,$$

where

$$m_s = \inf_{x\in K} \frac{1 - \lambda^x_s}{\mu^x_s}, \quad m_u = \inf_{x\in K} \frac{\lambda^x_u - 1}{\mu^x_u}.$$

An f-invariant compact subset K of U is said to be $(\langle\lambda_s\rangle, \langle\lambda_u\rangle, \langle\mu_s\rangle, \langle\mu_u\rangle)$-semi-hyperbolic if for each $x \in K$ there exist a norm $\|\cdot\|_x$ on $\mathbb{R}^n$ and a splitting $\mathbb{R}^n = E^s_x \oplus E^u_x$ with corresponding projections P^s_x and $P^u_x = I - P^s_x$ such that the conditions [H0], [H1], [SH2] hold and

[VSH3] $\|P^s_{f(x)} D_x f P^s_x v\|_{f(x)} \leq \lambda^x_s \|P^s_x v\|_x$, $\quad \|P^s_{f(x)} D_x f P^u_x v\|_{f(x)} \leq \mu^x_s \|P^u_x v\|_x$,

$\|P^u_{f(x)} D_x f P^s_x v\|_{f(x)} \leq \mu^x_u \|P^s_x v\|_x$, $\quad \|P^u_{f(x)} D_x f P^u_x v\|_{f(x)} \geq \lambda^x_u \|P^u_x v\|_x$

for $x \in K$, $v \in \mathbb{R}^n$.

Obviously, a semi-hyperbolic set in the classical sense of Definition 2.2 is semi-hyperbolic in the sense of the above definition, with respect to the same norms and splittings. The following example shows that the inverse does not hold.

Example 3.1. Consider two open sets $U_1, U_2 \subset \mathbb{R}^n$ with disjoint closures and take any points $x_1 \in U_1$, $x_2 \in U_2$. It is easy to see that we can define a diffeomorphism $f\colon U_1 \cup U_2 \to f(U_1 \cup U_2)$ such that the set $K = \{x_1, x_2\}$ forms a periodic orbit (hence K is f-invariant) and

$$D_{x_1} f = \begin{bmatrix} \frac{1}{4} & 1 \\ \frac{1}{16} & \frac{5}{4} \end{bmatrix}, \quad D_{x_2} f = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & 2 \end{bmatrix}.$$

Put

$$\|\cdot\|_{x_1} = \|\cdot\|_{x_2} = |\cdot|, \quad E^s_{x_1} = E^s_{x_2} = \mathbb{R} \times \{0\} \text{ and } E^u_{x_1} = E^u_{x_2} = \{0\} \times \mathbb{R}. \tag{2}$$

Then for each $v \in \mathbb{R}^2$ we obtain the following estimates:

$$\begin{aligned}
&|P^s_{x_2} D_{x_1} f P^s_{x_1} v| = \tfrac{1}{4}|P^s_{x_1} v| \le \tfrac{3}{4}|P^s_{x_1} v|, && |P^s_{x_2} D_{x_1} f P^u_{x_1} v| = 1|P^u_{x_1} v|,\\
&|P^u_{x_2} D_{x_1} f P^s_{x_1} v| = \tfrac{1}{16}|P^s_{x_1} v| \le \tfrac{1}{4}|P^s_{x_1} v|, && |P^u_{x_2} D_{x_1} f P^u_{x_1} v| = \tfrac{5}{4}|P^u_{x_1} v|,\\
&|P^s_{x_1} D_{x_2} f P^s_{x_2} v| = \tfrac{3}{4}|P^s_{x_2} v|, && |P^s_{x_1} D_{x_2} f P^u_{x_2} v| = \tfrac{1}{4}|P^u_{2_1} v| \le 1|P^u_{2_1} v|,\\
&|P^u_{x_1} D_{x_2} f P^s_{x_2} v| = \tfrac{1}{4}|P^s_{x_2} v|, && |P^u_{x_1} D_{x_2} f P^u_{x_2} v| = 2|P^u_{x_2} v| \ge \tfrac{5}{4}|P^u_{x_2} v|.
\end{aligned}$$

Since

$$\min\left\{\tfrac{1-\frac{1}{4}}{1}, \tfrac{1-\frac{3}{4}}{\frac{1}{4}}\right\} \cdot \min\left\{\tfrac{\frac{5}{4}-1}{\frac{1}{16}}, \tfrac{2-1}{\frac{1}{4}}\right\} = \tfrac{3}{4} \cdot 4 = 3 > 1,$$

$$(1 - \tfrac{3}{4})(\tfrac{5}{4} - 1) = \tfrac{1}{16} \not> \tfrac{1}{4} = 1 \cdot \tfrac{1}{4},$$

we conclude that K is $(\{\frac{1}{4}, \frac{3}{4}\}, \{\frac{5}{4}, 2\}, \{1, \frac{1}{4}\}, \{\frac{1}{16}, \frac{1}{4}\})$-semi-hyperbolic, but it cannot be semi-hyperbolic with respect to (2) in the sense of Definition 2.2.

Now we are ready to formulate the main result of this paper.

Theorem 3.2. *If a compact f-invariant set $K \subset U$ is $(\langle\lambda_s\rangle, \langle\lambda_u\rangle, \langle\mu_s\rangle, \langle\mu_u\rangle)$-semi-hyperbolic according to splittings $\mathbb{R}^n = E^s_x \oplus E^u_x$ and norms $\|\cdot\|_x$ on $\mathbb{R}^n$, then f is λ^*_u-expanding and λ^*_s-co-expanding on the cone field*

$$\mathcal{C} = \{\mathcal{C}_x\} = \{C_{\varepsilon(x)}(E^s_x, E^u_x, \|\cdot\|_x)\}$$

over K, where

$$\begin{gathered}
\lambda^*_s = \sup\nolimits_{x\in K}(\lambda^x_s + \varepsilon^{-1}\mu^x_s), \quad \lambda^*_u = \sup\nolimits_{x\in K}(\lambda^x_u - \varepsilon\mu^x_u),\\
\varepsilon(x) = \varepsilon = \tfrac{c}{m_s} \text{ for all } x \in K,
\end{gathered} \tag{3}$$

and c is any constant satisfying

$$1 < c < m_s m_u.$$

Proof. The proof is essentially the same as the proof of Theorem 2.5. We only need to show that

$$\lambda_s^* < 1 \ \text{ and } \ \lambda_u^* > 1.$$

Let $\delta < 1$ be a constant satisfying

$$c < \delta m_s m_u.$$

Then for each $x \in K$ we obtain the following estimates:

$$\lambda_s^x + \tfrac{\mu_s^x}{\varepsilon} = \lambda_s^x + \tfrac{\mu_s^x m_s}{c} \le \lambda_s^x + \tfrac{\mu_s^x(1-\lambda_s^x)}{\mu_s^x c} = \lambda_s^x(1 - \tfrac{1}{c}) + \tfrac{1}{c} < \tfrac{1}{c} < 1,$$

$$\lambda_u^x - \varepsilon\mu_u^x = \lambda_u^x - \tfrac{c\mu_u^x}{m_s} > \lambda_u^x - \tfrac{\delta m_s(\lambda_u^x - 1)\mu_u^x}{m_s \mu_u^x} = \lambda_u^x(1-\delta) + \delta > \delta > 1,$$

and, consequently,

$$\lambda_s^* \le \frac{1}{c} < 1, \ \ \lambda_u^* \ge \delta > 1,$$

which gives the conclusion. □

Corollary 3.3. *Every $(\langle\lambda_s\rangle, \langle\lambda_u\rangle, \langle\mu_s\rangle, \langle\mu_u\rangle)$-semi-hyperbolic set $K \subset U$ is also $(\lambda_s^*, \lambda_u^*)$-hyperbolic, where λ_s^* and λ_u^* are the constants given by (3).*

References

[1] Z. Arai, *On Hyperbolic Plateaus of the Hénon Maps*, Experiment. Math. 16 (2007), 181–188.

[2] P. Diamond, P. E. Kloeden, V. S. Kozyakin, A. V. Pokrovskii, *Semi-hyperbolic mappings*, J. Nonlinear Sci. 5 (1995), 419–431.

[3] P. Diamond, P. E. Kloeden, V. S. Kozyakin, A. V. Pokrovskii, Semi-hyperbolic mappings, Manuscript.

[4] S. L. Hruska, *A Numerical Method for Constructing the Hyperbolic Structure of Complex Hénon Mappings*, Found. Comput. Math. 6 (2006), 427-455.

[5] M. Mazur, J. Tabor, *Computational hyperbolicity*, submitted.

[6] M. Mazur, J. Tabor, P. Kościelniak, *Semi-hyperbolicity and hyperbolicity,* Discrete Contin. Dynam. Syst. 20 (2008), 1029–1038.

[7] K. Mischaikow, M. Mrozek, *Chaos in the Lorenz equations: a computer-assisted proof*, Bull. Amer. Math. Soc. 32 (1995), 66–72.

[8] K. Mischaikow, M. Mrozek, *Chaos in the Lorenz equations: a computer-assisted proof. II. Details*, Math. Comp. 67 (1998), 1023–1046.

[9] K. Mischaikow, M. Mrozek, A. Szymczak, *Chaos in the Lorenz equations: a computer-assisted proof. III. Classical parameter values*, J. Differential Equations 169 (2001), 17–56.

[10] S. Newhouse, *Cone-fields, domination, and hyperbolicity,* in: Modern dynamical systems and applications, 419–432, Cambridge Univ. Press, Cambridge, 2004.

[11] K. Palmer, Shadowing in dynamical systems. Theory and applications, Kluwer Academic Publishers, Dordrecht, 2000.

[12] P. Pilarczyk, *Topological-numerical approach to the existence of periodic trajectories in ODE's*, Discrete Contin. Dyn. Syst. 2003, suppl., 701–708.

[13] P. Pilarczyk, K. Stolot, *Excision-preserving cubical approach to the algorithmic computation of the discrete Conley index*, Topology Appl. 155 (2008), 1149-1162

[14] D. Wilczak, P. Zgliczyński, *Heteroclinic connections between periodic orbits in planar restricted circular three-body problem – a computer assisted proof*, Comm. Math. Phys. 234 (2003), 37–75.

[15] P. Zgliczyński, *Rigorous numerics for dissipative partial differential equations. II. Periodic orbit for the Kuramoto-Sivashinsky PDE – a computer-assisted proof*, Found. Comput. Math. 4 (2004), 157–185.

[16] P. Zgliczyński, K. Mischaikow, *Rigorous numerics for partial differential equations: the Kuramoto-Sivashinsky equation*, Found. Comput. Math. 1 (2001), 255–288.

Jagiellonian University, Institute of Mathematics, Reymonta 4, 30-059 Kraków, Poland

E-mail address: Marcin.Mazur@im.uj.edu.pl