LIE GROUPS, BURAU REPRESENTATION, AND NON-CONJUGATE BRAIDS WITH THE SAME CLOSURE LINK

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ABSTRACT. We use the unitarization of the Burau representation, found by Squier, and some Lie group arguments, to extend the previous construction of infinite sequences of pairwise non-conjugate braids with the same closure link of a non-minimal number of (and at least 4) strands. We extend some results to the Lawrence-Krammer representation.

1. INTRODUCTION

The theory of braid groups took its origins in the 1930s from the work of Artin [Ar] and Alexander [Al]. By a classical theorem of Alexander, knots and links embedded in real 3-dimensional space are all realized as closures of braids. Contrarily, Markov's theorem relates braids realizing the same link by two moves. These moves are conjugacy in the braid group, and (de)stabilization.

Birman-Menasco proved that up to 3 strands at most 3 conjugacy classes of braids with the same strand number have the same closure link [BM2]. For minimal braid representations (i.e. representations realizing the braid index) in general braid groups sometimes only finitely many conjugacy classes seem to occur. There are certain links also for $n \geq 4$ with a single conjugacy class, e.g. unlinks [BM], and a further example due to Ko and Lee [KL]. Contrarily, Shinjo [Sh2] constructed many

knots with infinitely many minimal braid conjugacy classes. When we abandon minimality, Morton [Mo] discovered an infinite sequence of conjugacy classes of 4-braids with unknotted closure, and also constructed an irreducible one [Mo]. (An irreducible conjugacy class is one containing no braid which admits a destabilization as in Markov's theorem.) Then Fiedler [Fi] combined both properties and showed the existence of an infinite sequence of irreducible conjugacy classes. Recently, Shinjo [Sh] obtained the following even more general theorem for *knots*.

Theorem 1.1. *(Shinjo) If there is an $n-1$-strand braid having a knot K as closure $(n \geq 4)$, then there exists an infinite sequence of pairwise non-conjugate braids of n strands realizing K (as closure).*

Our work was motivated by attempts to extend this result to links. In this context we came to consider the Burau representation ψ_n. This representation associates to a braid β in the n-strand braid group B_n an $(n-1) \times (n-1)$ matrix with entries in $\mathbb{Z}[t^{\pm 1}]$. It remains of fundamental importance to braid and link theory (see for example [Bi, J]).

In the talk I reported about a study in [St] of the Burau representation, and some more sophisticated machinery of Lie group theory, in the realm to obtain the following result.

Theorem 1.2. *Assume $L = \hat{\beta}'$, and $\beta' \in B_{n-1}$ for $n \geq 4$, such that β' has a non-scalar Burau matrix. (That is, the Burau matrix is not a multiple, that may depend on t, of the identity.) Then there exists an infinite sequence of pairwise non-conjugate braids of n strands realizing L.*

Our approach was to study the Burau trace of the stabilization of conjugates by braids α of a given braid representation β' of L. First we show a result of independent meaning that concerns the image of the Burau representation of B_n in $GL(n-1, \mathbb{C})$ for some particular values of t.

Theorem 1.3. *When $t \in \mathbb{C}$ with $|t| = 1$ and t is close to 1, but not a root of unity, and $n \geq 3$, then $\overline{\psi_n(B_n)} \simeq U(n-1)$ as a subgroup of $GL(n-1, \mathbb{C})$.*

Here bar means closure in the usual topology of $M(n-1, \mathbb{C})$ and isomorphy is meant up to conjugation with a matrix depending on t. The proof rests fundamentally on Dynkin's seminal work [Dy]. The inclusion $\psi_n(B_n) \subset U(n-1)$ is due to Squier [Sq].

Theorem 1.2 left over a (small) number of braids in the kernel (at least for $n \geq 5$ [Bi, LP]). One could hope to deal with these by the Lawrence-Krammer

representation [Kr, Bi2]. The Lawrence-Krammer representation ρ_n of B_n into $SL(p, \mathbb{Z}[t^{\pm 1}q^{\pm 1}])$, with $p = n(n-1)/2$, has become recently of interest as the first faithful representation of braid groups. Unitarizability is found by Budney [Bu]. We obtained:

Theorem 1.4. *Assume t, q with $|t| = |q| = 1$ are chosen so that $t^a q^b = 1$ for $a, b \in \mathbb{Z}$ implies that $a = b = 0$, and the Budney form is definite at q, t. Moreover, assume that ρ_n is irreducible at q, t. Then $\overline{\rho_n(B_n)} \simeq U(p)$ (for $p = n(n-1)/2$).*

We have an extra proof of irreducibility of ρ_n. It should be pointed out that it has been proved at separate places. There is written account by M. Zinno [Zi], though it was observed also by others, incl. V. Jones, R. Budney and W. T. Song. But it appears all this material is (yet) unpublished. There is closely related work of I. Marin [Ma].

Our main motivation was again the study of braid representations of links. We apply theorem 1.4 to prove

Theorem 1.5. *Assume L is a link and $n > b(L)$. Then there exist infinitely many conjugacy classes of n-braid representations of L, except if*
(a) $n \le 3$ or (possibly)
(b) L is a $(n-1, k(n-1))$-torus link ($k \in \mathbb{Z}$). (This includes the case $k = 0$ of the $n-1$-component trivial link.)

The number $b(L)$ is the minimal number of strands of a braid representation of L, and is called *braid index* (see e.g. [Mu]). As explained, the case 1.5 is very well-known from [BM2] to need to be excluded, but we do not know anything about whether any link of case 1.5 is indeed exceptional. Still the theorem almost completely settles the (in)finiteness for $n > b(L)$.

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