Trends in Mathematics
Information Center for Mathematical Sciences
Volume 4, Number 2,December 2001, Pages 61–66

A MULTIPLICITY RESULT OF POSITIVE RADIAL SOLUTIONS FOR AN ELLIPTIC SYSTEM WITH MULTIPARAMETERS

YONG-HOON LEE

1. INTRODUCTION

Dunninger and Wang ([1]) considered the Dirichlet elliptic system

$$\begin{aligned}&\triangle u+\lambda k_1(|x|)f(u,v)=0\\&\triangle v+\lambda k_2(|x|)g(u,v)=0,\ \text{in } \Omega\\&u|_{\partial\Omega}=0=v|_{\partial\Omega},\end{aligned}$$

where $\Omega=\{x\in\mathbf{R}^n : R_1<|x|<R_2\}$, $R_1,\ R_2>0$, λ is a positive real parameter. Assuming $f(0,0),\ g(0,0)>0,\ f$ and g nondecreasing with respect to componentwise-defined inequality in $\mathbf{R}^2_+$ with superlinear growth at ∞, they proved that there exists $\lambda^*>0$ such that the problem has at least two, at least one or no positive radial solutions according to $0<\lambda<\lambda^*,\ \lambda=\lambda^*$ or $\lambda>\lambda^*$.

In this note, we study the above problem when the problem is defined on an exterior domain and multiparameters $\lambda,\ \mu$ are given instead of single parameter λ. Let us consider

$$(P)\qquad \begin{aligned}&\triangle u+\lambda k_1(|x|)f(u,v)=0\\&\triangle v+\mu k_2(|x|)g(u,v)=0\ \text{ in } \Omega\\&u(x)=0=v(x)\qquad \text{on } |x|=r_0\\&\lim_{|x|\to\infty}u(x)=0=\lim_{|x|\to\infty}v(x),\end{aligned}$$

where $\Omega=\{x\in\mathbf{R}^n : |x|>r_0\},\ r_0>0$ and $\lambda,\mu>0,\ n\geq 3$. In what follows, we assume that $k_i\in C([r_0,\infty),\mathbf{R}^+)$, not vanishing identically 0 on any subinterval of $[r_0,\infty)$ and $f,g\in C(\mathbf{R}^2_+,\mathbf{R}^+_0)$, where we denote $\mathbf{R}^+=[0,\infty),\ \mathbf{R}^+_0=(0,\infty)$ and $\mathbf{R}^2_+=[0,\infty)\times[0,\infty)$.

Moreover, under assumptions;

(H_1) $\int_{r_o}^{\infty} rk_i(r)dr<\infty,\ i=1,2.$

(H_2) f and g are nondecreasing in $\mathbf{R}^2_+$.

(H_3) $f_\infty \triangleq \lim_{(u,v)\to\infty}\frac{f(u,v)}{u+v}=\infty,\ g_\infty\triangleq\lim_{(u,v)\to\infty}\frac{g(u,v)}{u+v}=\infty,$

2000 *Mathematics Subject Classification.* 34B15, 35J25.

Key words and phrases. semilinear elliptic system, radial solution, upper and lower solution method, fixed point index.

This work was supported by grant 2000-2-101-001-3 from the Basic Research Program of the Korean Science and Engineering Foundation.

Received May 8, 2001

we prove that there exists $(\lambda^*, \mu^*) > (0,0)$ such that problem (P) has at least two, at least one or no positive radial solutions according to $(0,0) < (\lambda, \mu) < (\lambda^*, \mu^*)$, $(\lambda, \mu) \in \Sigma$ or $(\lambda, \mu) > (\lambda^*, \mu^*)$, where $\Sigma = \{(\lambda, \mu) \in \mathbf{R}^2_+ : \lambda = \lambda^*, \mu \leq \mu^*\} \cup \{(\lambda, \mu) \in \mathbf{R}^2_+ : \lambda \leq \lambda^*, \mu = \mu^*\}$.

Proofs of our theorems are mainly based on o.d.e. technique, Upper and lower solutions argument and Fixed point index theory. For the radial solutions, Changing of variables; $r = |x|,\ s = r^{2-n}$ and $t = (r_o^{2-n} - s)/r_o^{2-n}$. We can transform problem (P) into a system of second order o.d.e's

$$(S) \qquad \begin{aligned} &u''(t) + \lambda q_1(t) f(u(t), v(t)) = 0 \\ &v''(t) + \mu q_2(t) g(u(t), v(t)) = 0, \quad 0 < t < 1 \\ &u(0) = u(1) = v(0) = v(1) = 0, \end{aligned}$$

where q_i can be given as

$$q_i(t) = \frac{r_o^2}{(n-2)^2}(1-t)^{\frac{-2(n-1)}{n-2}} k_i\big(r_o(1-t)^{\frac{-1}{n-2}}\big),\ i = 1, 2.$$

A detailed version for the proofs of our Lemmas and theorems will appear in [3].

2. UPPER AND LOWER SOLUTIONS METHOD

$$(S_1) \qquad \begin{aligned} &u''(t) + F(t, u(t), v(t)) = 0 \\ &v''(t) + G(t, u(t), v(t)) = 0, \quad 0 < t < 1 \\ &u(0) = u(1) = v(0) = v(1) = 0, \end{aligned}$$

where $F, G : D \to \mathbf{R}$ is a continuous function and $D \subset (0,1) \times \mathbf{R}^2$.

Lemma 1. *Assume that there exist $h_F, h_G \in C((0,1), \mathbf{R}^+)$ such that*

$$(C_1) \qquad |F(t,u,v)| \leq h_F(t),\ |G(t,u,v)| \leq h_G(t),$$

for all $(t,u,v) \in (0,1) \times \mathbf{R}^2$ and h_F and h_G satisfy

$$(C_2) \qquad C_F \triangleq \int_0^1 s(1-s) h_F(s) ds < \infty, \quad C_G \triangleq \int_0^1 s(1-s) h_G(s) ds < \infty.$$

Then (S_1) has a solution.

Definition 1. For $\alpha_u, \alpha_v \in C([0,1], \mathbf{R}) \cap C^2((0,1), \mathbf{R})$, we say (α_u, α_v) is a *lower solution* of (P_1) if $(t, \alpha_u(t), \alpha_v(t)) \in D$ for all $t \in (0,1)$ and

$$\begin{aligned} &\alpha_u''(t) + F(t, \alpha_u(t), \alpha_v(t)) \geq 0 \\ &\alpha_v''(t) + G(t, \alpha_u(t), \alpha_v(t)) \geq 0, \quad t \in (0,1) \\ &\alpha_u(0), \alpha_u(1), \alpha_v(0), \alpha_v(1) \leq 0. \end{aligned}$$

We also define an *upper solution* $(\beta_u, \beta_v) \in C[0,1] \cap C^2(0,1)$ if it satisfies the reverse of the above inequalities.

Definition 2. For a function $F : D \to \mathbf{R}$, we say $F(t,u,v)$ is quasi-monotone nondecreasing with respect to v (or u) if

$$F(t,u,v_1) \le F(t,u,v_2) \quad \textit{whenever } v_1 \le v_2$$
$$\big(\textit{or } F(t,u_1,v) \le F(t,u_2,v) \quad \textit{whenever } u_1 \le u_2\big).$$

Let $D_\alpha^\beta = \{(t,u,v) \in (0,1) \times \mathbf{R}^2 : \alpha_u(t) \le u \le \beta_u(t),\ \alpha_v(t) \le v \le \beta_v(t)\}$.

Theorem 1. (Fundamental theorem of upper and lower solutions method)
Let (α_u, α_v) and (β_u, β_v) be a lower solution and an upper solution of (S_1) respectively such that
(a_1) $(\alpha_u(t), \alpha_v(t)) \le (\beta_u(t), \beta_v(t))$ *for all* $t \in [0,1]$
(a_2) $D_\alpha^\beta \subset D$.
Assume also that there exist $h_F, h_G \in C((0,1), \mathbf{R}^+)$ such that
(a_3) $|F(t,u,v)| \le h_F(t),\ |G(t,u,v)| \le h_G(t)$, *for all* $(t,u,v) \in D_\alpha^\beta$
and h_F and h_G satisfy
(a_4) $\int_0^1 s(1-s)h_F(s)ds < \infty,\quad \int_0^1 s(1-s)h_G(s)ds < \infty.$
(a_5) *$F(t,u,v)$ and $G(t,u,v)$ are quasi-monotone nondecreasing w.r.t. v and u respectively.*
Then problem (S_1) has at least one solution (u,v) such that

$$(\alpha_u(t), \alpha_v(t)) \le (u(t), v(t)) \le (\beta_u(t), \beta_v(t)), \quad \textit{for all } t \in [0,1].$$

3. EXISTENCE

$$(S) \qquad \begin{aligned} &u''(t) + \lambda q_1(t) f(u(t), v(t)) = 0 \\ &v''(t) + \mu q_2(t) g(u(t), v(t)) = 0,\ 0 < t < 1 \\ &u(0) = u(1) = v(0) = v(1) = 0. \end{aligned}$$

In what follows, we assume that $q_i \in C((0,1), \mathbf{R}^+),\ i = 1,2$ do not vanish identically on any subinterval of $(0,1)$.

Theorem 2. *Assume*
(H) *q_i are singular at $t = 0$ and 1 and $\int_0^1 s(1-s)q_i(s)ds < \infty,\ i = 1,2.$*
(H_2') *f and g are quasi-monotone nondecreasing w.r.t. v and u respectively.*
(H_3) $f_\infty = \infty = g_\infty$
Then there exists $(\lambda^*, \mu^*) > (0,0)$ such that (S) has at least one positive solution for $(0,0) < (\lambda,\mu) \le (\lambda^*, \mu^*)$ and no solution for $(\lambda,\mu) > (\lambda^*, \mu^*)$.

OPERATOR SET-UP

$$A_\lambda(u,v)(t) \triangleq \lambda \int_0^1 K(s,t) q_1(s) f(u(s), v(s)) ds.$$
$$B_\mu(u,v)(t) \triangleq \mu \int_0^1 K(s,t) q_2(s) g(u(s), v(s)) ds.$$
$$T_{\lambda,\mu}(u,v)(t) \triangleq \big(A_\lambda(u,v)(t), B_\mu(u,v)(t)\big).$$

Then by (H), $T_{\lambda,\mu} : X \to X$ is well-defined and problem (S) is equivalent to

$$(u,v) = T_{\lambda,\mu}(u,v) \quad \text{on } X.$$

Define cones $\mathcal{C}$ and $\mathcal{K}$ as follows;

$$\mathcal{C} = \{(u,v) \in X : u, v \geq 0\},$$
$$\mathcal{K} = \{(u,v) \in \mathcal{C} : \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} (u(t)+v(t)) \geq \frac{1}{4}\|(u,v)\|\}.$$

Lemma 2. *Assume (H). Then for all $(\lambda,\mu) > (0,0)$, $T_{\lambda,\mu}$ is completely continuous on X and $T_{\lambda,\mu}(\mathcal{C}) \subset \mathcal{K}$.*

Lemma 3. *Assume (H). Then the problem, for $i = 1,2$*

$$(L_i) \qquad \begin{aligned} &u''(t) + t(1-t)q_i(t) = 0, \quad 0 < t < 1 \\ &u(0) = 0 = u(1) \end{aligned}$$

has a unique solution $\varphi_i \in C^1[0,1] \cap C^2(0,1)$ satisfying

$$c_i\varphi_i(t) \leq t(1-t) \leq d_i\varphi_i(t) \quad \text{for some } c_i, d_i > 0, \ i = 1,2.$$

Lemma 4. *Assume (H). Let (u,v) be a solution of (S) and φ_i be a solution of (L_i). Then*

$$\int_0^1 u''(s)\varphi_1(s)ds = \int_0^1 u(s)\varphi_1''(s)ds$$
$$\int_0^1 v''(s)\varphi_2(s)ds = \int_0^1 v(s)\varphi_2''(s)ds.$$

Lemma 5. *Assume (H) and also assume either $f_\infty = \infty$ or $g_\infty = \infty$. Let $\mathcal{R}$ be a compact rectangle of $(0,\infty)\times(0,\infty)$. Then for all $(\lambda,\mu) \in \mathcal{R}$, there exists a constant $b_{\mathcal{R}} > 0$ such that all possible positive solutions (u,v) of (S) at (λ,μ) satisfy*

$$\|(u,v)\| < b_{\mathcal{R}}.$$

Proof of Theorem 2. The proof consists of three claims.

Claim 1 : Problem (S) has a positive solution for certain (λ,μ).
Define $\mathcal{S} = \{(\lambda,\mu) \in \mathbf{R}^2 : (S) \text{ has a positive solution at } (\lambda,\mu)\}$, then by Claim 1, $\mathcal{S} \neq \emptyset$ and $(\mathcal{S}, \leq)$ is a partially ordered set.
Claim 2: $(\mathcal{S}, \leq)$ is bounded above.
Claim 3: $(\mathcal{S}, \leq)$ has a maximal element.
Let (λ^*,μ^*) be a maximal element of $\mathcal{S}$ and (u^*,v^*), the corresponding positive solution of (S). Then for all $(0,0) < (\lambda,\mu) \leq (\lambda^*,\mu^*)$, problem (S) has a positive solution at (λ,μ), this concludes the proof.

4. MULTIPLICITY

In this section, we show the existence of the second positive solution for $(0,0) < (\lambda,\mu) < (\lambda^*,\mu^*)$, where (λ^*,μ^*) is a maximal element of $\mathcal{S}$. Let (u^*,v^*) be a positive solution of (S) at (λ^*,μ^*).

Lemma 6. *For $(0,0) < (\lambda,\mu) < (\lambda^*,\mu^*)$, there exists $\epsilon_0 > 0$ such that $(u^*+\epsilon, v^*+\epsilon)$ is an upper solution of (S) at (λ,μ), for all $0 < \epsilon < \epsilon_0$.*

We notice that for $0 < \epsilon \le \epsilon_0$, $(u^*_\epsilon, v^*_\epsilon)$ satisfies

$$u^*_\epsilon(t) > \lambda \int_0^1 K(t,s)q_1(s)f(u^*_\epsilon(s), v^*_\epsilon(s))ds$$
$$v^*_\epsilon(t) > \mu \int_0^1 K(t,s)q_2(s)g(u^*_\epsilon(s), v^*_\epsilon(s))ds.$$

Lemma 7. ([2]) *Let X be a Banach space, K a cone in X and Ω bounded open in X. Let $0 \in \Omega$ and $T : K \cap \bar{\Omega} \to K$ be condensing. Suppose that $Tx \ne \nu x$, for all $x \in K \cap \partial\Omega$ and all $\nu \ge 1$. Then*

$$i(T, K\cap\Omega, K) = 1.$$

Lemma 8. ([2]) *Let X be a Banach space and K a cone in X. For $r > 0$, define $K_r = \{x \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is a compact map such that $Tx \ne x$ for $x \in \partial K_r$. If $\|x\| \le \|Tx\|$, for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0.$$

Theorem 3. *Assume (H), (H_2) and (H_3). Then there exists $(\lambda^*,\mu^*) > (0,0)$ such that problem (S) has at least two positive solution for $(0,0) < (\lambda,\mu) < (\lambda^*,\mu^*)$, at least one positive solution for $(\lambda,\mu) \in \Sigma$ and no positive solution for $(\lambda,\mu) > (\lambda^*,\mu^*)$, where $\Sigma = \{(\lambda,\mu) \in \mathbf{R}^2_+ : \lambda = \lambda^*, \mu \le \mu^*\} \cup \{(\lambda,\mu) \in \mathbf{R}^2_+ : \lambda \le \lambda^*, \mu = \mu^*\}$.*

Proof. By Theorem 2, it is enough to show the existence of the second positive solution of (S) for $(\lambda,\mu) < (\lambda^*,\mu^*)$. Let (λ,μ) with $(0,0) < (\lambda,\mu) < (\lambda^*,\mu^*)$ be given and $u^*_\epsilon(t) = u^*(t)+\epsilon,\ v^*_\epsilon(t) = v^*(t)+\epsilon$, where ϵ is given in Lemma 6. Let $\Omega = \{(u,v) \in X : -\epsilon < u(t) < u^*_\epsilon(t),\ -\epsilon < v(t) < v^*_\epsilon(t),\ t \in [0,1]\}$, then

$$i(T_{\lambda,\mu}, \mathcal{K}\cap\Omega, \mathcal{K}) = 1,$$

where $\mathcal{K} = \{(u,v) \in \mathcal{C} : \min_{\frac14 \le t \le \frac34}(u(t)+v(t)) \ge \frac14\|(u,v)\|\}$.
Let $R = \max\{b_{\mathcal{R}}, 4R_1, \|(u^*_\epsilon, v^*_\epsilon)\|\}$, where $b_{\mathcal{R}}$ is given in Lemma 5 with $\mathcal{R}$ a compact rectangle of $(0,\infty)\times(0,\infty)$ containing (λ,μ), and R_1 is given in (2) in the proof of Lemma 5. Let $K_R = \{(u,v) \in \mathcal{K} : \|(u,v)\| < R\}$, then

$$i(T_{\lambda,\mu}, K_R, \mathcal{K}) = 0.$$

Consequently by the additivity of the fixed point index,

$$0 = i(T_{\lambda,\mu}, K_R, \mathcal{K}) = i(T_{\lambda,\mu}, \mathcal{K}\cap\Omega, \mathcal{K}) + i(T_{\lambda,\mu}, K_R \setminus \overline{\mathcal{K}\cap\Omega}, \mathcal{K}).$$

Since $i(T_{\lambda,\mu}, \mathcal{K}\cap\Omega, \mathcal{K}) = 1$, $i(T_{\lambda,\mu}, K_R \setminus \overline{\mathcal{K}\cap\Omega}, \mathcal{K}) = -1$ and thus $T_{\lambda,\mu}$ has a fixed point on $\mathcal{K}\cap\Omega$ and another on $K_R \setminus \overline{\mathcal{K}\cap\Omega}$, and this completes the proof.

Theorem 4. *Assume* (H_2) *and* (H_3). *If* q_i *satisfy*
(H') $\quad q_i : [0,1) \to \mathbf{R}^+$ *are singular at* 1 *and* $\int_0^1 (1-s)q_i(s)ds < \infty$.
Then the result of Theorem 3 is true. Similarly, if q_i *satisfy*
(H'') $\quad q_i : (0,1] \to \mathbf{R}^+$ *is singular at* 0 *and* $\int_0^1 sq(s)ds < \infty$.
Then the result of Theorem 3 is also true.

Theorem 5. (Regular Case) *Let* $q_i \in C([0,1], \mathbf{R}^+))$. *If* f *and* g *satisfies* (H_2) *and* (H_3), *then the result of Theorem 3 is true.*

Case I. $\quad \lim_{r\to\infty} r^{2(n-1)} k_i(r) < \infty$.
In this case, $\lim_{t\to 1^-} q_i(t) < \infty$ so that q_i can be extended continuously on $[0,1]$ and the problem becomes regular. Thus applying Theorem 5, we obtain desired result described below.

Case II. $\quad \lim_{r\to\infty} r^{2(n-1)} k_i(r) = \infty$.
In this case, $\lim_{t\to 1^-} q_i(t) = \infty$ and the problem becomes singular. To apply Theorem 4, we need to add condition (H_1).

Since functions k_i in Case I also satisfies (H_1), we obtain the following corollary unifying both cases.

Corollary 1. *Assume* (H_1), (H_2) *and* (H_3). *Then there exists* $(\lambda^*, \mu^*) > (0,0)$ *such that problem* (P) *has at least two positive radial solutions for* $(0,0) < (\lambda,\mu) < (\lambda^*,\mu^*)$, *at least one positive radial solution for* $(\lambda,\mu) \in \Sigma$ *and no positive radial solution for* $(\lambda,\mu) > (\lambda^*,\mu^*)$, *where* $\Sigma = \{(\lambda,\mu) \in \mathbf{R}^2_+ : \lambda = \lambda^*, \mu \leq \mu^*\} \cup \{(\lambda,\mu) \in \mathbf{R}^2_+ : \lambda \leq \lambda^*, \mu = \mu^*\}$.

References

1. D.R. Dunninger and H. Wang, *Multiplicity of positive radial solutions for an elliptic system on an annulus domain*, Nonlin. Anal. T.M.A. 42 (5), Nov. 2000, pp 803-811..
2. D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, FL, 1988.
3. Y.H. Lee, *A multiplicity result of positive radial solutions for a multiparameter elliptic system on an elliptic system on an exterior domain*, Nonlin. Anal. T.M.A., in press.

Department of Mathematics Pusan National University Pusan 609-735, Korea
E-mail address: yhlee@hyowon.cc.pusan.ac.kr