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RIEMANNIAN AND FINSLER STRUCTURES OF SYMMETRIC CONES

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Abstract. The theory of domains of positivity or symmetric cones is closely tied to that of Euclidean Jordan algebras and plays an important role in various branches of analysis and geometry. The goal of this paper is to provide a self-contained description of Riemannian and Finsler geometries of symmetric cones and their applications.

1. Symmetric cones and Euclidean Jordan algebras

An open convex cone Ω in a real Euclidean space V that is self–dual with respect to the given inner product and is homogeneous in the sense that the group

$$G(\Omega) := \{g \in \mathrm{GL}(V) : g(\Omega) = \Omega\}$$

acts transitively on Ω is called a *symmetric cone*. Symmetric cones actually play a fundamental role in the fields, number theory, statistics, and convex programming, and the theory of symmetric cones is closely tied to that of Euclidean Jordan algebras in the form developed by Max Koecher [10]. In this paper, we survey some geometric structures in symmetric cones and give some of their applications in various fields needed the notion of "positivity".

We recall certain basic notions and well–known facts concerning Jordan algebras from the book [9] by J. Faraut and A. Korányi. A *Jordan algebra* V over the field $\mathbb{R}$ or $\mathbb{C}$ is a commutative algebra satisfying $x^2(xy) = x(x^2y)$ for all $x, y \in V$. We also assume the existence of a multiplicative identity e. Denote by L the left translation $L(x)y = xy$, and P by the quadratic representation $P(x) = 2L(x)^2 - L(x^2)$ for $x \in V$. An alternate statement of the Jordan algebra law is $(xy)x^2 = x(yx^2)$, a weak associativity condition that is strong enough to ensure that the subalgebra generated by $\{e, x\}$ in V is associative. An element $x \in V$ is said to be *invertible* if there exists an element y in the subalgebra generated by x and e such that $xy = e$. It is known that an element x in V is invertible if and only if $P(x)$ is invertible. In this case, $P(x)^{-1} = P(x^{-1})$. If x and y are invertible, then $P(x)y$ is invertible and $(P(x)y)^{-1} = P(x^{-1})y^{-1}$.

A *Euclidean* Jordan algebra is a finite–dimensional real Jordan algebra V equipped with an *associative* inner product $\langle\cdot|\cdot\rangle$, i.e., satisfying $\langle xy|z\rangle = \langle y|xz\rangle$ for all $x, y, z \in V$. The space $\mathrm{Sym}(n, \mathbb{R})$ of $n \times n$ real symmetric matrices is a Euclidean Jordan algebra with Jordan product $(1/2)(XY + YX)$ and inner product

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$\langle X, Y\rangle = \mathrm{tr}(XY)$. The spectral theorem (see Theorem III.1.2 of [9]) of a Euclidean Jordan algebra V states that for $x \in V$ there exist a Jordan frame, i.e., a complete system of orthogonal primitive idempotents $c_1, \dots, c_r$, where r is the rank of V, and real numbers $\lambda_1, \dots, \lambda_r$, the eigenvalues of x, such that $x = \sum_{i=1}^r \lambda_i c_i$. Due to the the power associative property $x^p \cdot x^q = x^{p+q}$, see [9], the exponential map $\exp : V \to V, \exp(x) = \sum_{n=0}^\infty x^n/n!$ is well defined. Likewise as for the special case $V = \mathrm{Sym}(n, \mathbb{R})$, the Jordan algebra exponential map is a bijection between its domain of definition V and its image $\Omega := \exp V$. In fact, Ω coincides with the interior of the set of square elements of V, and this is equal to the set of invertible squares of V.

A fundamental theorem of Euclidean Jordan algebras asserts that (i) Ω is a symmetric cone, and (ii) every symmetric cone in a real Euclidean space arises in this way. In the case of the Jordan algebra $V = \mathrm{Sym}(n, \mathbb{R})$ the Jordan algebra exponential is simply the matrix exponential map, and hence the corresponding symmetric cone is $\Omega = \mathrm{Sym}(n, \mathbb{R})^+$, the open convex cone of positive definite $n \times n$ matrices. Irreducible symmetric cones have been completely classified, and the remaining cases consist of (i) the cones of positive definite Hermitian and Hermitian quaternion $n \times n$ matrices, (ii) the Lorentzian cones, and (iii) a 27–dimensional exceptional cone. General symmetric cones are Cartesian products of these. The connected component $\mathrm{Aut}(V)_\circ$ of the identity id_V in the Jordan algebra automorphism group $\mathrm{Aut}(V)$ is a subgroup of the orthogonal group $\mathrm{O}(\Omega)$ with respect to the given inner product. Ω is irreducible if and only if V is simple, and in this case we have $\mathrm{Aut}(V)_\circ = \mathrm{O}(\Omega)$. For all of these statements, see [9] and the references therein.

In the special case $V = \mathrm{Sym}(n, \mathbb{R})$ the general formula $P(x)y = 2x(xy) - x^2 y$ reduces to $P(X)Y = XYX$, where the multiplication in the right hand side of this equation is the usual matrix multiplication, not the Jordan multiplication. A key tool for generalising $\mathrm{Sym}(n, \mathbb{R})$ to arbitrary Euclidean Jordan algebras is to consistently replace expressions of the form XYX by $P(x)y$. Throughout this paper we will assume that V is a simple Euclidean Jordan algebra with the associative inner product $\langle x|y\rangle = \mathrm{tr}(xy)$ and that Ω is the symmetric cone associated with V.

2. Riemannian structures on symmetric cones

Let $\mathrm{Aut}(V)$ be the group of Jordan automorphisms of V. Then any $g \in G := G(\Omega)_\circ$, the identity component of $G(\Omega)$, can be uniquely written as $g = P(x)k$ with $x \in \Omega$ and $k \in K := \mathrm{Aut}(V)_\circ$, and $g \in K$ if and only if $g(e) = e$. The Lie algebra $\mathfrak{g}(\Omega)$ of G is $L(V) + \mathrm{Der}(V)$, where $\mathrm{Der}(V)$ is the Lie algebra of all Jordan derivations of V. The symmetric cone Ω can be identified with the homogeneous space G/K via $gK \to g(e)$. And Ω admits a $G(\Omega)$-invariant Riemannian metric γ_x defined by:

$$\gamma_x(u, v) = \langle P(x)^{-1}u, v\rangle | x \in \Omega, u, v \in V$$

for which the Jordan inversion $j : \Omega \to \Omega, j(x) = x^{-1}$ is an involutive isometry fixing e. The pair (G, K) is a symmetric pair with respect to the involution j, and hence the geodesics passing through the point e are the curves

$$t \to (\exp tL(x))e, \ x \in V.$$

Since $P(\exp x) = \exp 2L(x)$ (see Proposition II.3.4, [9]), the geodesics passing through e are of the form $t \to \exp tx, x \in V$.

Proposition 2.1 ([19], [16]). *Let $a, b \in \Omega$. Then the unique geodesic curve passing through a and b is $t \to P(a^{1/2})(P(a^{-1/2})b)^t$ and the Riemannian metric distance $\delta(a,b)$ of a and b is given by $\delta(a,b) = \left(\sum_{i=1}^r \log^2 \lambda_i\right)^{1/2}$, where λ_i's are the eigenvalues of $P(a^{-1/2})b$. The Riemannian metric γ_x coincides with the Hessian metric of the self-scaled barrier functional $F(x) = -\log\det(x)$.*

Let us recall the Euclidean Jordan algebra $\mathrm{Sym}(n,\mathbb{R})$ of $n \times n$ real symmetric matrices with the Jordan algebra product $X \circ Y = \frac{1}{2}(XY + YX)$ and the inner product $\langle X|Y\rangle = \mathrm{tr}(XY)$. Then the geodesic curve passing through positive definite matrices A and B is $t \to A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ and he geodesic middle $A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ of A and B is known as the *geometric mean* of A and B in the matrix theory [19]. In [12] and [13], S. Lang showed that the matrix exponential function $\exp : \mathrm{Sym}(n,\mathbb{R}) \to \mathrm{Sym}(n,\mathbb{R})^+, \quad X \to \exp X = \sum_{i=0}^\infty \frac{X^n}{n!}$ is metric semi-increasing for the Riemannian metric distance on $\mathrm{Sym}^+(n,\mathbb{R})$. Hence the symmetric cone $\mathrm{Sym}(n,\mathbb{R})^+$ is a Cartan-Hadamard manifold, that is, a complete simply connected Riemannian manifold with nonpositive curvature obtained by bending the flat Euclidean space $\mathrm{Sym}(n,\mathbb{R})$ by the exponential map.

Viewing the symmetric cone Ω of a Euclidean Jordan algebra V as a symmetric Riemannian manifold of non-compact type, we have that every symmetric cone Ω is a Cartan-Hadamard manifold.

Recall that a metric space (X,δ) satisfies the *semi-parallelogram law* if for any two points $x_1, x_2 \in X$, there is a point z which satisfies for all $x \in X$:

$$\delta(x_1,x_2)^2 + 4\delta(x,z)^2 \leq 2\delta(x,x_1)^2 + 2\delta(x,x_2)^2.$$

One may see (cf. [16]) that for x_1, x_2 in a metric space (X,δ) the z (called the *midpoint* of x_1 and x_2) arising in the semi-parallelogram law is the unique point in X satisfying $\delta(x_1,z) = \delta(x_2,z) = (1/2)\delta(x_1,x_2)$.

A *Bruhat-Tits space* is a complete metric space satisfying the semi-parallelogram law. It turns out that the Riemannian metric distance δ on the symmetric cone Ω is a Bruhat-Tits metric with midpoint $a\#b$. See [5], [12], [13] and [16] for more details.

3. Finsler structures on symmetric cones

In [23], Liverani and Wojtkowski introduced a generalization of the Hilbert projective metric to the space $\mathrm{Sym}(n,\mathbb{R})^+$ by defining

$$s(A,B) := \max\{|\log\lambda| \ : \ \lambda \text{ is an eigenvalue of } B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\}.$$

They show that the metric $s(A,B)$ on $\mathrm{Sym}(n,\mathbb{R})^+$ coincides with a natural Finsler distance in the manifold $\mathrm{Sym}(n,\mathbb{R})^+$. The Finsler metric is defined by the family of norms $|X|_A := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A^{-\frac{1}{2}}XA^{-\frac{1}{2}}\}, X = X^t, A \in$

$\mathrm{Sym}(n,\mathbb{R})^+$. We observe that

$$\begin{aligned} s(A,B) &= \sup_{x\neq 0}\left|\log\frac{\langle B^{-\frac{1}{2}}AB^{-\frac{1}{2}}x|x\rangle}{||x||^2}\right| \\ &= \sup_{x\neq 0}\left|\log\frac{\langle Ax|x\rangle}{\langle Bx|x\rangle}\right| \\ &= \sup_{||x||=1}|\log\langle Ax|x\rangle - \log\langle Bx|x\rangle| \\ &= \sup_{||x||=1}|\log x^tAx - \log x^tBx| \\ &= \sup_{||x||=1}|\log \mathrm{tr}\ Axx^t - \log \mathrm{tr}\ Bxx^t| \\ &= \sup_{||x||=1}\left|\log \mathrm{tr}\frac{1}{2}(Axx^t+xx^tA) - \log \mathrm{tr}\frac{1}{2}(Bxx^t+xx^tB)\right| \\ &= \sup_{\substack{C^2=C,\\ \mathrm{tr}C=1}}|\log \mathrm{tr}\ A\circ C - \log \mathrm{tr}\ B\circ C|, \end{aligned}$$

and $|X|_A = \sup_{||x||=1}\frac{|\langle Xx|x\rangle|}{\langle Ax|x\rangle} = \sup_{\substack{C^2=C,\\ \mathrm{tr}C=1}}\frac{|\mathrm{tr}\ X\circ C|}{\mathrm{tr}\ A\circ C}$, where tr is the trace functional, and $A\circ C = \frac{1}{2}(AC+CA)$. This observation makes it possible to extend the notions $s(A,B)$ and $|\cdot|_A$ to Euclidean Jordan algebras.

Let $\mathcal{J}(V)$ be the set of all primitive idempotents of V. Then $\mathcal{J}(V)$ is a compact symmetric space of rank one ([9]). Let $a\in\Omega$. For $b\in\Omega$ and $x\in V$, we define

$$\begin{aligned} s(a,b) &:= \max\{|\log\lambda| : \lambda \text{ is an eigenvalue of } P(a^{-\frac{1}{2}})b\}, \\ |x|_a &:= \max\{|\lambda| : \lambda \text{ is an eigenvalue of } P(a^{-\frac{1}{2}})x\}. \end{aligned}$$

Note that $|x|_e = \max\{|\lambda| : \lambda$ is an eigenvalue of $x\}$, and is the spectral norm of x (see e.g. [9], Proposition X.4.1).

Since Ω is open in V, we can identity the tangent space $T_a(\Omega)$ at $a\in\Omega$ with the algebra V. The manifold Ω carries a natural Finsler metric $|\cdot|_a$ and the distance $d(a,b)$ in the Finsler metric defined by $d(a,b)=\inf\{l(\gamma):\gamma$ joins a to $b\}$, where $l(\gamma)=\int_0^1|\gamma'(t)|_{\gamma(t)}dt$ is $G(\Omega)$-invariant.

Theorem 3.1 ([17]). *The Finsler distance $d(a,b)$ coincides with $s(a,b)$. Furthermore, the Finsler metric is invariant under the group $G(\Omega)$ and the inversion.*

A common property of Riemannian and Finsler metrics on the symmetric cones is that they are invariant under the automorphism group $G(\Omega)$ and the inversion.

One computes easily that the Riemannian geodesic curve

$$\alpha(t) = P(a^{1/2})(P(a^{-1/2})b)^t$$

passing through a and b satisfies $d(\alpha(t),\alpha(t')) = |t-t'|d(a,b), \forall t,t'\in[0,1]$. This implies in particular that the Riemannian geodesic curve $\alpha(t)$ is one of shortest curves joining a and b for the Finsler metric.

We define a useful partial order on V defined by $x\leq y$ if and only if $y-x\in\overline{\Omega}$. Observe that $|x|_e=\inf\{t>0: x\leq te\}$ and each element of $G(\Omega)$ preserves the partial order. Furthermore, the inversion on Ω reverses the order: For $a,b\in\Omega$,

$a \leq b$ if and only if $b^{-1} \leq a^{-1}$. This implies that the Finsler metric distance d is exactly Thompson's metric which is defined by

$$\overline{d}(a,b) = \inf\{\log M(b/a), \log M(b/a)\},$$

where $M(a/b) = \inf\{t > 0 : a \leq tb\}$. By Lemma 1.3 of [25], there are infinitely many shortest curves passing through two points a and b, and hence the metric space (Ω, d) is not a Bruhat-Tits space (note that the midpoint for a Bruhat-Tits space is unique). This distinguishes the Finsler metric from the Riemannian metric on Ω.

In the following, we give some evidences that the Finsler metric on Ω has some sort of "nonpositive curvature". Let $a \in \Omega$. Since $\Omega = \exp V$, $a = \exp x$ for some $x \in V$. Then $\log a := x$ is well-defined since if $\exp x = \exp y$ then $\exp 2L(x) = P(\exp x) = P(\exp y) = \exp 2L(y)$. The corresponding exponential map on the tangent space $T_a(\Omega) = V$ is given by

$$\exp_a : V \to \Omega, \ \exp_a(x) = P(a^{1/2}) \exp(P(\log a^{-1/2})x)$$

with inverse $x \to P(a^{1/2}) \log(P(\log a^{-1/2})x)$. Note that

$$a\#_t b := P(a^{1/2})(P(a^{-1/2})b)^t = \exp_a(t \exp_a^{-1}(b))$$

for each $t \in [0,1]$. The following formula (a variant of Lie-Trotter product formula, see [20]): $\exp\{(1-t)x + ty\} = \lim_{n\to\infty} \left(\exp \frac{x}{n} \#_t \exp \frac{y}{n}\right)^n$, $t \in [0,1]$, particularly, when $t = 1/2$, $\exp(x+y) = \lim_{n\to\infty} \left(\exp \frac{2}{n}x \# \exp \frac{2}{n}y\right)^n$, has been used to get the following inequality (in matrix or operator cases, it is called Golden-Thompson inequality [27] or Segal's inequality [26]).

Theorem 3.2 ([20])**.** *For $x, y \in V$,*

$$|\exp(x+y)|_e \leq |P(\exp \frac{x}{2}) \exp y|_e.$$

One could show that Segal's inequality is equivalent to that the exponential mapping $\exp_a$ increases distances: $d(\exp_a x, \exp_a y) \geq |x - y|_a$. The $G(\Omega)$ and Jordan inversion-invariance of the Finsler metric and the realization of the Finsler metric distance as Thompson's metric, we have another evidence that the Finsler metric on Ω has some sort of nonpositive curvature.

Theorem 3.3 ([1])**.** *Let $a \in \Omega$. Then*

$$d(a\#_t x, a\#_t y) \leq t d(x,y)$$

for all $x, y \in \Omega$ and $t \in [0,1]$.

It turns out that every Finsler ball of $\mathrm{Sym}(n, \mathbb{R})^+$ is geodesically convex. This property holds even on the convex cone of positive definite Hermitian operators on a infinite dimensional Hilbert space [8]. Together with this fact and using the quadratic representation P on Ω, we have

Theorem 3.4 ([1])**.** *Every Finsler ball $B_a(\alpha) := \{x \in \Omega : d(a,x) \leq \alpha\}$ of the symmetric cone Ω is a convex set. That is, for all $x, y \in B_a(\alpha)$, the shortest curve $x\#_t y$ joining x and y is contained in $B_a(\alpha)$.*

Another interesting convexity property of the Finsler metric on $\mathrm{Sym}(n, \mathbb{R})^+$ is that the distance functions $d(a, x\#_t y)$ are convex [8]. We don't know yet this property holds on any symmetric cones.

Remarks (1) The symmetric cone carries a natural invariant pseudo metric defined by $p(a,b) = \log M(a/b)M(b/a), a, b \in \Omega$. This pseudo metric is called Hilbert's projective metric and is a metric on the unit sphere on Ω. One can show that the three invariant metrics, Riemannian, Finsler, and Hilbert's projective metrics on the symmetric cone Ω have a common property: the midpoint property in the sense that the geometric mean $a\#b = P(a^{1/2})(P(a^{-1/2})b)^{1/2}$ is a midpoint of a and b for these metrics.

(2) Each symmetric cone Ω corresponds to a symmetric tube domain $T_\Omega := V + i\Omega \subset V + iV$ with the symmetry $j(z) = -z^{-1}$ at ie. Let $G(T_\Omega)$ be the Lie group of all biholomorphic automorphisms of T_Ω. Then $G(T_\Omega)$ is generated by $N^+, G(\Omega)$, and j where N^+ is the vector group of all real translations $t_x(z) = x + z$, and it can viewed as the $G(\Omega)$-conformal (causal) or Kantor-Koecher-Tits group (see, [2] and [3]). The maximal parabolic Lie subgroup $\mathbf{P} := G(\Omega)(jN^+j)$ gives the conformal compactification $\mathcal{M} = G(T_\Omega)/\mathbf{P}$ of the Jordan algebra V via $V \ni x \mapsto t_x\mathbf{P} \in \mathcal{M}$. In the action of $G(T_\Omega)$ on $\mathcal{M}$, a "Lie semigroup" that is naturally related to the symmetric cone Ω occurs as the conformal compressions: $\Gamma_\Omega = \{g \in G(T_\Omega) : g \cdot \Omega \subset \Omega\}$. By the Lie theory of semigroup, it turns out ([11], [14], [15]) that Γ_Ω admits an "Ol'shanskii polar" decomposition (a semigroup variant of Cartan decomposition) and a canonical "triple" decomposition (a semigroup variant of Harish-Chandra decomposition). These factorization theories have played a key role to establish the contraction property of conformal compressions for the Riemannian and Finsler metrics ([11], [17], [18]). When $V = \mathrm{Sym}(n, \mathbb{R})$, the contraction property of conformal compressions has been used in Kalman Filtering theory ([4], [29]) and to establish Birkhoff Formula for the Riemannian and Finsler metrics [23], respectively. See ([17], [18], [22]) for their extension into any symmetric cones and see also [21] the connection between fixed points of conformal compressions and some geometric and algebraic notions of the symmetric cones: geometric mean, circumcenter, continued fraction, and quadratic equation.

(3) The study of Finsler structures on symmetric cones is motivated by the series of papers [6], [7], [8] of Corach, Porta, and Recht on the study of geometric features of domain of positive invertible elements of a C^*-algebra. The choice of a Finsler metric on this domain is necessary because of the dimension. If the C^*-algebra is finite dimensional then the corresponding cone comes from the matrix algebra. Thus our results of this note is an extension of Corach, Porta, and Recht's results in finite dimensional cases. However, there is an infinite dimensional version of symmetric cones. A complete normed real Jordan algebra V with identity e is said to be a *JB-algebra* if $||xy|| \leq ||x|||\,|y||,\ \ ||x^2|| \leq ||x^2 + y^2||$ for all $x, y \in V$. It turns out that the set Q of squares in V is a closed convex cone satisfying $V = Q - Q$ and $Q \cap -Q = \{0\}$, and the JB-algebra norm coincides with the order unit norm $||x|| = \inf\{t > 0 : te \pm x \in Q\}$. The open convex cone Ω of the interior of Q becomes a symmetric normed real Banach-manifold under $G(\Omega)$-invariant tangent norm by the JB-algebra norm $||\cdot||$, and hence Ω has a natural $G(\Omega)$-invariant Finsler metric (see, [28]): $||x||_a := ||P(a^{-1/2})x||,\ a \in \Omega, x \in V$. Every JB-algebra is a formally real, and hence the class of finite dimensional JB-algebras are equivalent to the class of Euclidean Jordan algebras.

Recently, K. H. Neeb [24] has studied Banach-Finsler manifolds endowed with a spray which have seminegative curvature. He considered infinite dimensional symmetric cones as typical examples of symmetric Finsler manifolds and showed that each symmetric cone carries a natural structure of Finsler symmetric space of seminegative curvature by obtaining Segal's inequality.

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