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OPEN PROBLEMS IN TOEPLITZ OPERATOR THEORY

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Abstract. In this article we give open problems concerning algebraic properties, subnormality, hyponormality, and spectral properties of Toeplitz operators.

CONTENTS

Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H},\mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H},\mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_{\mathcal{H}}$, where N is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If T is subnormal then T is also hyponormal. Recall that the Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \ldots\}$. An element $f \in L^2(\mathbb{T})$ is said to be analytic if $f \in H^2(\mathbb{T})$, and co-analytic if $f \in L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$. If P denotes the projection operator $L^2(\mathbb{T}) \to H^2(\mathbb{T})$, then for every $\varphi \in L^\infty(\mathbb{T})$, the operators T_φ and H_φ on $H^2(\mathbb{T})$ defined by

$$T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi(g) := (I - P)(\varphi g) \qquad (g \in H^2(\mathbb{T}))$$

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are called the *Toeplitz operator* and the *Hankel operator*, respectively, with symbol φ. In this article we give open problems on Toeplitz operators and some related problems.

LIST OF PROBLEMS

Problem 1. Find a necessary and sufficient condition that the product $T_{\varphi_1} \cdots T_{\varphi_n}$ of Toeplitz operators be a Toeplitz operator.

Problem 2. Let T_φ be a hyponormal Toeplitz operator. Find a necessary and sufficient condition that T_φ^2 be hyponormal. More generally, if T_φ and T_ψ are hyponormal Toeplitz operators, for which symbols φ and ψ, is $T_\varphi T_\psi$ hyponormal?

Problem 3.

(1) If ψ is a Riemann map between simply connected domains, does it follow that $T_{\psi+\alpha\bar{\psi}}$ is subnormal for some α with $0 < \alpha < 1$?

(2) Conversely, if $T_{\psi+\alpha\bar{\psi}}$ is subnormal for some α with $0 < \alpha < 1$, does it follow that ψ is a Riemann map between simply connected domains?

(3) More generally, for which $f \in H^\infty$ is there λ, $0 < \lambda < 1$ with $T_{f+\lambda\bar{f}}$ subnormal?

Problem 4.

(1) Let $T \equiv W_\alpha$ be the weighted shift with weight sequence $\alpha = \{\alpha_k\}_{k=0}^\infty$ with

$$\alpha_k = \left(\sum_{j=0}^{k} \alpha^{2j}\right)^{\frac{1}{2}}.$$

and let $S := T + \lambda T^*$ $(\lambda \in \mathbb{C})$. Find a necessary and sufficient condition in terms of λ for S to be (weakly) k-hyponormal.

(2) Make an analogue theory with the Bergman shift T or a recursively generated weighted shift T and an operator S_λ in place of T and $T + \lambda T^*$ in the above setting.

Problem 5. Let $0 < \alpha < 1$ be given and let ψ be a Riemann map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. Let $\varphi = \psi + \alpha\bar{\psi}$. and let T_φ be the corresponding Toeplitz operator on H^2. Find a necessary and sufficient condition in terms of λ for T_φ to be (weakly) k-hyponormal.

Problem 6. If T_φ is a 2-hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that T_φ is analytic? If the answer is affirmative, is φ a linear function of a finite Blaschke product?

Problem 7. Let $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$. Find the classes of φ satisfying

(i) T_φ is a hyponormal operator.

(ii) For every zero ζ of $z^m\varphi$ such that $|\zeta| > 1$, the number $1/\bar{\zeta}$ is a zero of $z^m\varphi$ in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ.

Problem 8. Is every p-hyponormal Toeplitz operator hyponormal?

Problem 9. Determine the hyponormality of block Toeplitz operators T_Φ $(\Phi \in L^\infty \otimes M_n)$.

Problem 10.

(1) Does there exist a Toeplitz operator that is polynomially hyponormal but not subnormal?

(2) Is every polynomially hyponormal Toeplitz operator rationally hyponormal?

(3) Is every Toeplitz operator a von-Neumann operator?

Problem 11. Identify subsets $\mathfrak{S}$ of $L^\infty(\mathbb{T})$ for which the spectrum σ is continuous when restricted to the set of Toeplitz operators with symbols in $\mathfrak{S}$.

§1. Algebraic Properties of Toeplitz Operators

In [**BH**, Theorem 8] it was shown that a necessary and sufficient condition that the product $T_\varphi T_\psi$ of two Toeplitz operators be a Toeplitz operator is that either φ be co-analytic or ψ be analytic; if the condition is satisfied, then $T_\varphi T_\psi = T_{\varphi\psi}$. What can we say about the product $T_{\varphi_1} \cdots T_{\varphi_n}$?

Problem 1. *Find a necessary and sufficient condition that the product $T_{\varphi_1} \cdots T_{\varphi_n}$ of Toeplitz operators be a Toeplitz operator.*

Here is a partial-strategy to Problem 1.

Theorem 1.1. *If T_φ is a Toeplitz oprator such that $T_{\overline{z}\varphi}$ is one-one then a necessary condition for $T_\varphi S$ to be a Toeplitz operator for an operator $S \in \mathcal{L}(H^2)$ is that S is an analytic Toeplitz operator.*

Proof. Let (α_{ij}) be the matrix of S. If the Fourier expansion of φ is $\varphi = \sum a_i z^i$, so that the matrix of T_φ is (a_{i-j}), then a straightforward calculation shows that if (β_{i-j}) is the matrix of $T_\varphi S$ then

$$\beta_{i,j} = \sum_{k=1}^{\infty} a_{i+2-k}\alpha_{k,j},$$

whenever $i, j \geq 1$. Since $\beta_{i,j} = \beta_{i+1,j+1}$ for each $i, j \geq 1$, we have the following equation:

$$\begin{pmatrix} a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \ddots \\ a_3 & a_2 & a_1 & a_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \alpha_{1,j} \\ \alpha_{2,j} - \alpha_{1,j-1} \\ \alpha_{3,j} - \alpha_{2,j-1} \\ \alpha_{4,j} - \alpha_{3,j-1} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{pmatrix} \quad \text{for each } j \geq 2.$$

Since the Toeplitz matrix in the left-hand side is the matrix of $T_{\overline{z}\varphi}$, it follows from the injectivity of $T_{\overline{z}\varphi}$ that

$$\begin{pmatrix} \alpha_{1,j} \\ \alpha_{2,j} - \alpha_{1,j-1} \\ \alpha_{3,j} - \alpha_{2,j-1} \\ \alpha_{4,j} - \alpha_{3,j-1} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{pmatrix} \quad \text{for each } j \geq 2,$$

which implies that for $j \geq 2$,

$$\alpha_{1+k,j+k} = 0 \quad \text{for each } k \geq 2$$

and for $i \geq j$,

$$\alpha_{i,j} = \alpha_{i+1,j+1}.$$

This shows that S must be an anlytic Toeplitz operator. $\square$

Remark. Theorem 1.1 may be false if the condition "$T_{\overline{z}\varphi}$ is one-one" is dropped. For example, consider

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & 1 & 0 & 0 & \ddots \\ 0 & \frac{1}{2} & 1 & 0 & \ddots \\ 0 & 0 & \frac{1}{2} & 1 & \ddots \\ 0 & 0 & 0 & \frac{1}{2} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & \dots \\ -\frac{1}{2} & \frac{5}{4} & -\frac{1}{2} & 0 & \ddots \\ 0 & -\frac{5}{8} & \frac{5}{4} & -\frac{1}{2} & \ddots \\ 0 & \frac{1}{16} & -\frac{5}{8} & \frac{5}{4} & \ddots \\ 0 & \frac{1}{32} & \frac{1}{16} & -\frac{5}{8} & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & \dots \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & \ddots \\ -\frac{1}{4} & 0 & 1 & -\frac{1}{2} & 0 & \ddots \\ 0 & -\frac{1}{4} & 0 & 1 & -\frac{1}{2} & \ddots \\ 0 & 0 & -\frac{1}{4} & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

It is well known that there is a hyponormal operator whose square is not hyponormal (e.g., $U^* + 2U$; see [**Ha3**, Problem 209]). Since $U^* + 2U$ is a Toeplitz operator, the square of a hyponormal Toeplitz operator need not be hyponormal. Hence the following seems to be interesting:

Problem 2. *Let T_φ be a hyponormal Toeplitz operator. Find a necessary and sufficient condition that T_φ^2 be hyponormal. More generally, if T_φ and T_ψ are hyponormal Toeplitz operators, for which symbols φ and ψ, is $T_\varphi T_\psi$ hyponormal?*

§2. Subnormality of Toeplitz Operators

We make a brief survey on answers to Problem 5 of Halmos's 1970 lectures "Ten problems in Hilbert space" (cf. [**Ha1**],[**Ha2**]):

Is every subnormal Toeplitz operator either normal or analytic?

Even though the above problem was already answered negatively by Cowen and Long [**CoL**], it seems to be interesting to consider the following problem:

Which Toeplitz operators are subnormal ?

The Halmos's problem was answered affirmatively for trigonometric Toeplitz operators [**ItW**] and for quasinormal Toeplitz operators [**AIW**]. In 1976, Abrahamse [**Ab**] gave a general sufficient condition for the answer to the Halmos's problem to be yes.

Theorem 2.1 ([Ab] Theorem). *If*
(i) T_φ *is hyponormal;*
(ii) φ *or* $\bar\varphi$ *is of bounded type (i.e.,* φ *or* $\bar\varphi$ *is a quotient of two analytic functions);*
(iii) *ker* $[T_\varphi^*, T_\varphi]$ *is invariant for* T_φ*,*
then T_φ *is normal or analytic.*

Since $\ker[T^*, T]$ is invariant for every subnormal operator T, Theorem 2.1 answers Halmos's problem affirmatively when φ or $\bar\varphi$ is of bounded type. Also, in [**Ab**], Abrhamse proposed a question for a strategy to answer Halmos's problem:

Is the Bergman shift unitarily equivalent to a Toeplitz operator ?

To review an answer to this question, recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called *weights*), the *(unilateral) weighted shift* W_α associated with α is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is straightforward to check that W_α can never be *normal*, and that W_α is *hyponormal* if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The Bergman shift is a weighted shift W_α with weights

$$\alpha := \left\{\sqrt{\frac{n}{n+1}}\right\}_{n=1}^\infty .$$

It is well-known that the Bergman shift is subnormal. In 1983, Sun Shunhua [**Shu**] showed that if a Toeplitz operator T_φ is unitarily equivalent to a hyponormal weighted shift W_α with weight sequence α, then α must be of the form

$$\alpha = \left\{(1-\beta^{2n+2})^{\frac{1}{2}}||T_\varphi||\right\}_{n=0}^\infty \quad \text{for some } \beta \ (0 < \beta < 1), \tag{2.1.1}$$

thus answering Abrahamse's question in the negative. Cowen and Long [**CoL**] showed that a unilateral weighted shift with weight sequence of the form (2.1.1) must be subnormal (also see [**Fa2**]). Consequently, we have:

Theorem 2.2 ([Sun], [Cow2]). *Every hyponormal Toeplitz operator which is unitarily equivalent to a weighted shift must be subnormal.*

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At last, in 1984, Cowen and Long [**CoL**] constructed the symbol φ for which T_φ is unitarily equivalent to the weighted shift with weight sequence (2.1.1). This answered the Halmos's problem negatively.

Theorem 2.3 ([CoL],[Cow2]). *Let $0<\alpha<1$ be given and let ψ be a Riemann map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. Let $\varphi=\psi+\alpha\bar\psi$. and let T_φ be the corresponding Toeplitz operator on H^2. Then T_φ is a weighted shift with weight sequence*

$$\alpha_n=(1-\alpha^2)^{\frac{3}{2}}\left(\sum_{j=0}^{n}\alpha^{2j}\right)^{\frac{1}{2}}$$

and is subnormal but neither normal nor analytic.

Problem 3.

(1) *If ψ is a Riemann map between simply connected domains, does it follow that $T_{\psi+\alpha\,\bar\psi}$ is subnormal for some α with $0<\alpha<1$?*

(2) *Conversely, if $T_{\psi+\alpha\,\bar\psi}$ is subnormal for some α with $0<\alpha<1$, does it follow that ψ is a Riemann map between simply connected domains?*

(3) *([**Cow2**, Question 3]) More generally, for which $f\in H^\infty$ is there λ, $0<\lambda<1$ with $T_{f+\lambda\,\bar f}$ subnormal?*

After Theorem 2.3, one turned their attentions to hyponormality of Toeplitz operators. The hyponormality of Toeplitz operators has been studied by M. Abrahamse [**Ab**], C. Cowen [**Cow1**],[**Cow2**], P. Fan [**Fa1**], C. Gu [**Gu**], T. Ito and T. Wong [**ItW**], T. Nakazi and K. Takahashi [**NaT**], D. Yu [**Yu**], K. Zhu [**Zhu**], D. Farenick, the authors and others (cf. [**FL1**],[**FL2**],[**CuL1**],[**HKL1**],[**HKL2**],[**KL**]). An elegant theorem of C. Cowen [**Cow3**] characterizes the hyponormality of a Toeplitz operator T_φ on $H^2(\mathbb{T})$ by properties of the symbol $\varphi\in L^\infty(\mathbb{T})$. K. Zhu [**Zhu**] reformulated Cowen's criterion and then showed that the hyponormality of T_φ with polynomial symbols φ can be decided by a method based on the classical interpolation theorem of I. Schur [**Sch**]. We shall use a variant of Cowen's theorem [**Cow3**] that was first proposed by Nakazi and Takahashi [**NaT**].

Cowen's Theorem. *Suppose $\varphi\in L^\infty(\mathbb{T})$ is arbitrary and write*

$$\mathcal{E}(\varphi)=\{k\in H^\infty(\mathbb{T}):||k||_\infty\le 1 \text{ and } \varphi-k\overline{\varphi}\in H^\infty(\mathbb{T})\}.$$

Then T_φ is hyponormal if and only if the set $\mathcal{E}(\varphi)$ is nonempty.

On the other hand, the Bram-Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j}(T^ix_j,T^jx_i)\ge 0$$

for all finite collections $x_0,x_1,\cdots,x_k\in\mathcal{H}$ ([**Br**],[**Con**, II.1.9]). Using the Choleski algorithm for operator matrices, it is easy to see that this is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \dots & T^{*k}\\ T & T^*T & \dots & T^{*k}T\\ \vdots & \vdots & \ddots & \vdots\\ T^k & T^*T^k & \dots & T^{*k}T^k\end{pmatrix}\ge 0 \qquad (\text{all } k\ge 1). \tag{2.3.1}$$

Condition (2.3.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (2.3.1) for $k = 1$ is equivalent to the hyponormality of T, while subnormality requires the validity of (2.3.1) for all k. If we denote by $[A, B] := AB - BA$ the commutator of two operators A and B, and if we define T to be *k-hyponormal* whenever the $k \times k$ operator matrix

$$M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k \tag{2.3.2}$$

is positive, or equivalently, the $(k+1)\times(k+1)$ operator matrix in (2.3.1) is positive, then the Bram-Halmos criterion can be rephrased as saying that T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([**CMX**]). Now it seems to be interesting to understand the gap between k-hyponormality and subnormality for Toeplitz operators.

In [**CuL1**], the following was shown:

Theorem 2.4 ([CuL1]). *Every trigonometric Toeplitz operator whose square is hyponormal must be normal or analytic. Hence, in particular, every 2-hyponormal trigonometric Toeplitz operator is subnormal.*

It is well known ([**Cu**]) that there is a gap between hyponormality and 2–hyponormality for weighted shifts. Theorem 2.4 also shows that there is a big gap between hyponormality and 2-hyponormality for Toeplitz operators. For example, if

$$\varphi(z) = \sum_{n=-m}^{N} a_n z^n \quad (m < N)$$

is such that T_φ is hyponormal then by Theorem 2.4, T_φ is never 2-hyponormal because T_φ is neither analytic nor normal (recall that if $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ is such that T_φ is normal then $m = N$ (cf. [**FL1**])).

We can extend Theorem 2.4. First of all we observe:

Proposition 2.5 ([CuL2]). *If $T \in \mathcal{L}(\mathcal{H})$ is 2-hyponormal then*

$$T\big(ker[T^*, T]\big) \subseteq ker[T^*, T]. \tag{2.5.1}$$

Proof. Suppose that $[T^*, T]f = 0$. Since T is 2-hyponormal, it follows from (2.3.2) that (cf. [**CMX**, Lemma 1.4])

$$|([T^{*2}, T]g, f)|^2 \leq ([T^*, T]f, f)([T^{*2}, T^2]g, g) \quad \text{for all } g \in \mathcal{H}.$$

By assumption, we have that for all $g \in \mathcal{H}$, $0 = ([T^{*2}, T]g, f) = (g, [T^{*2}, T]^* f)$, so that $[T^{*2}, T]^* f = 0$, i.e., $T^*T^2 f = T^2 T^* f$. Therefore,

$$[T^*, T]Tf = (T^*T^2 - TT^*T)f = (T^2T^* - TT^*T)f = T[T^*, T]f = 0,$$

which proves (2.5.1). □

Corollary 2.6. *If T_φ is 2-hyponormal and if φ or $\bar{\varphi}$ is of bounded type then T_φ is normal or analytic, so that T_φ is subnormal.*

Proof. This follows at once from Theorem 2.1 and Proposition 2.5. □

Corollary 2.7. *If T_φ is a 2-hyponormal operator such that $\mathcal{E}(\varphi)$ contains at least two elements then T_φ is normal or analytic, so that T_φ is subnormal.*

Proof. This follows from Corollary 2.6 and the fact ([**NaT**, Proposition 8]) that if $\mathcal{E}(\varphi)$ contains at least two elements then φ is of bounded type. □

From Corollaries 2.6 and 2.7, we can see that if T_φ is 2-hyponormal but not subnormal then φ is not of bounded type and $\mathcal{E}(\varphi)$ consists of exactly one element.

In [**Cow1**] the following was established.

Theorem 2.8. *Let $T \equiv W_\alpha$ be the weighted shift with weight sequence $\alpha = \{\alpha_k\}_{k=0}^\infty$ with*

$$\alpha_k = \left(\sum_{j=0}^{k} \alpha^{2j}\right)^{\frac{1}{2}}.$$

and let $S := T + \lambda T^*$ ($\lambda \in \mathbb{C}$). Then we have:
(1) *S is hyponormal if and only if $|\lambda| \le 1$;*
(2) *S is subnormal if and only if $\lambda = 0$ or $|\lambda| = \alpha^k$ for some $k = 0, 1, 2, \cdots$.*

Proof. (1) From a straightforward calculation.
(2) See [**Cow1**, Theorem 2.3]. □

Recall ([**At**],[**CMX**],[**CoS**]) that $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if

$$LS((T, T^2, \cdots, T^k)) := \left\{\sum_{j=1}^{k} \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k\right\}$$

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is *weakly positive*, i.e., ([**CMX**])

$$(2.8.1) \qquad (M_k(T)\begin{pmatrix}\lambda_0 x\\ \vdots\\ \lambda_k x\end{pmatrix}, \begin{pmatrix}\lambda_0 x\\ \vdots\\ \lambda_k x\end{pmatrix}) \ge 0 \qquad \text{for } x \in \mathcal{H} \text{ and } \lambda_0, \cdots, \lambda_k \in \mathbb{C}.$$

If $k = 2$ then T is said to be *quadratically hyponormal*, and if $k = 3$ then T is said to be *cubically hyponormal*. Similarly, $T \in \mathcal{L}(\mathcal{H})$ is said to be *polynomially hyponormal* if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that k-hyponormal $\Rightarrow$ weakly k-hyponormal, but the converse is not true in general. The classes of (weakly) k-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([**Cu1**],[**Cu2**],[**CF1**], [**CF2**],[**CF3**],[**CuL1**],[**CMX**], [**DPY**],[**McCP**]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle: in fact, even subnormality for Toeplitz operators has not yet been characterized (cf. [**Ha1**], [**Cow2**]). For weighted shifts, positive results appear in [**Cu1**] and [**CF3**], although no concrete example of a weighted shift which is polynomially hyponormal but not subnormal has yet been found (the existence of such weighted shifts was established in [**CP1**] and [**CP2**]).

Thus the following problem seems to be interesting:

Problem 4.
(1) *Let S be defined as in Theorem 2.8. Find a necessary and sufficient condition in terms of λ for S to be (weakly) k-hyponormal.*
(2) *Make an analogue theory with the Bergman shift T or a recursively generated weighted shift T and an operator S_λ in place of T and $T+\lambda T^*$ in Theorem 2.8.*

Also in [**Cow1**] the following was established:

Theorem 2.9 ([Cow1]). *Let $0<\alpha<1$ be given and let ψ be a Riemann map of the unit disk onto the interior of the ellipse with vertices $\pm(1+\alpha)i$ and passing through $\pm(1-\alpha)$. Let $\varphi=\psi+\alpha\bar{\psi}$. and let T_φ be the corresponding Toeplitz operator on H^2. Then*
(i) *T_φ is hyponormal if and only if $|\lambda|\le 1$;*
(2) *T_φ is subnormal if and only if*

$$\lambda=\alpha \quad or \quad \lambda=\frac{\alpha^k e^{i\theta}+\alpha}{1+\alpha^{k+1}e^{i\theta}} \quad (k=0,1,2,\cdots;\ 0\le\theta<\pi).$$

Problem 5. *Let T_φ be defined as in Theorem 2.9. Find a necessary and sufficient condition in terms of λ for T_φ to be (weakly) k-hyponormal.*

§3. Self-commutators of Toeplitz Opeators

In [**AIW**] it was shown that every subnormal Toeplitz operator with rank-one self-commutator is a linear function of some inner function χ, where $\chi(z)=\frac{z-\alpha}{1-\bar{\alpha}z}$ for some $|\alpha|<1$. Also K. Clancy has shown that every pure subnormal operator with rank-one self-commutator is a linear function of the unilateral shift (cf. Indiana Univ. Math. J. 23 (1973)). Also, in [**CuL2**] it was shown that every pure 2-hyponormal operator with rank-one self-commutator is a linear function of the unilateral shift. McCarthy and Yang [**McCYa**] classified all rationally cyclic subnormal operators with finite rank self-commutators. However it remains still open what are the pure subnormal operators with finite rank self-commutators.

Now the following question comes up at once:

Problem 6. *If T_φ is a 2-hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that T_φ is analytic?*

For affirmativity to Problem 6 we shall give a partial answer. To do this we recall Theorem 15 in [**NaT**] which states that if T_φ is subnormal and $\varphi=q\bar{\varphi}$, where q is a finite Blaschke product then T_φ is normal or analytic. But from a careful examination of the proof of the theorem we can see that its proof uses subnormality assumption only for the fact that $\ker[T_\varphi^*,T_\varphi]$ is invariant under T_φ. Thus in view of Proposition 2.5, the theorem is still valid for "2–hyponormal" in place of "subnormal". We thus have:

Theorem 3.1 ([CuL4]). *If T_φ is 2–hyponormal and $\varphi=q\bar{\varphi}$, where q is a finite Blaschke product then T_φ is normal or analytic.*

We now give a partial answer to Problem 6.

Theorem 3.2 ([CuL4]). *Suppose* $\log|\varphi|$ *is not integrable. If* T_φ *is a 2–hyponormal operator with nonzero finite rank self-commutator then* T_φ *is analytic.*

Proof. If T_φ is hyponormal such that $\log|\varphi|$ is not integrable then by an argument of [**NaT**, Theorem 4], $\varphi = q\bar{\varphi}$ for some inner function q. Also if T_φ has a finite rank self-commutator then by [**NaT**, Theorem 10], there exists a finite Blaschke product $b \in \mathcal{E}(\varphi)$. If $q \neq b$, so that $\mathcal{E}(\varphi)$ contains at least two elements, then by Corollary 2.7, T_φ is normal or analytic. If instead $q = b$ then by Theorem 3.1, T_φ is also normal or analytic. □

Theorem 3.2 reduces Problem 6 to the class of Toeplitz operators such that $\log|\varphi|$ is integrable. If $\log|\varphi|$ is integrable then there exists an outer function e such that $|\varphi| = |e|$. Thus we may write $\varphi = ue$, where u is a unimodular function. Since by the Douglas-Rudin theorem (cf. [**Ga**, p.192]), every unimodular function can be approximated by quotients of inner functions, it follows that if $\log|\varphi|$ is integrable then φ can be approximated by functions of bounded type. Therefore if we could obtain such a sequence ψ_n converging to φ such that T_{ψ_n} is 2–hyponormal with finite rank self-commutator for each n, then we would answer Problem 6 affirmatively. On the other hand, if T_φ attains its norm then by a result of Brown and Douglas [**BD**], φ is of the form $\varphi = \lambda\frac{\psi}{\theta}$ with $\lambda > 0$, ψ and θ inner. Thus φ is of bounded type. Therefore by Corollary 2.7, if T_φ is 2–hyponormal and attains its norm then T_φ is normal or analytic. However we were not able to decide that if T_φ is a 2–hyponormal operator with finite rank self-commutator then T_φ attains its norm.

§4. Hyponormality of Toeplitz Operators

Nakazi and Takahashi [**NaT**, Corollary 5] showed that if $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ is a trigonometric polynomial with $m \le N$ and if for every zero ζ of $z^m\varphi$ such that $|\zeta| > 1$, the number $1/\bar{\zeta}$ is a zero of $z^m\varphi$ in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ, then T_φ is hyponormal. But the converse is not true in general. To see this consider the following trigonometric polynomial: $\varphi(z) = z^{-2}(z-2)(z-1)(z-\frac{1}{5})(z-\frac{1}{3})$. Then $\varphi(z) = \frac{2}{15}z^{-2} - \frac{19}{15}z^{-1} + \frac{55}{15} - \frac{53}{15}z + z^2$. Using an argument of P. Fan [**Fa1**, Theorem 1] – for every trigonometric polynomial φ of the form $\varphi(z) = \sum_{n=-2}^{2} a_n z^n$,

$$T_\varphi \text{ is hyponormal } \iff \left|\det \begin{pmatrix} \frac{a_{-1}}{a_1} & \frac{a_{-2}}{a_2} \end{pmatrix}\right| \le |a_2|^2 - |a_{-2}|^2,$$

a straightforward calculation shows that T_φ is hyponormal. In [**HKL1**] it was considered how the converse of the above result due to Nakazi and Takahashi survives for arbitrary trigonometric polynomials. The main result of [**HKL1**] is as follows. Suppose $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ with $m \le N$ and write

$$\mathfrak{F} := \{\zeta, 1/\bar{\zeta} : \text{ the complex numbers } \zeta \text{ and } 1/\bar{\zeta} \text{ are zeros of } z^m\varphi\}.$$

If $\mathfrak{F}$ contains at least $(N+1)$ elements then the following statements are equivalent.

(i) T_φ is a hyponormal operator.

(ii) For every zero ζ of $z^m\varphi$ such that $|\zeta| > 1$, the number $1/\bar{\zeta}$ is a zero of $z^m\varphi$ in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ.

Moreover, in the cases where T_φ is a hyponormal operator, the rank of the selfcommutator of T_φ is computed from the formula $\operatorname{rank}[T_\varphi^*, T_\varphi] = N - m + Z_{\mathbb{D}} - Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$, where $Z_{\mathbb{D}}$ and $Z_{\mathbb{C}\setminus\overline{\mathbb{D}}}$ are the number of zeros of $z^m\varphi$ in $\mathbb{D}$ and in $\mathbb{C}\setminus\overline{\mathbb{D}}$ counting multiplicity.

The above result can be easily applied for Toeplitz operators with polynomial and circulat-type symbols (cf. [**FL2**]). The crucial point for the converse of Nakazi-Takahashi Theorem is to find the classes φ such that T_φ is hyponormal if and only if $\varphi = k\,\bar\varphi$ for some $k \in \mathcal{E}(\varphi)$. For example, if $\log|\varphi|$ is not integrable, then φ belongs to that class (cf. [**NaT**, Theorem 4]). However, if φ is a trigonometric polynomial then $\log|\varphi|$ is integrable. Thus the above result can not be applied for trigonometric Toeplitz operators. We now have:

Problem 7. *Let $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$. Find the classes of φ satisfying*
(i) *T_φ is a hyponormal operator.*
(ii) *For every zero ζ of $z^m\varphi$ such that $|\zeta| > 1$, the number $1/\bar\zeta$ is a zero of $z^m\varphi$ in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of ζ.*

§5. p-hyponormality of Toeplitz Operators

An operator T is called p-hyponormal (cf. [**Xi**]) if $(T^*T)^p - (TT^*)^p \geq 0$. If $p = 1$, T is hyponormal and if $p = \frac{1}{2}$, T is semi-hyponormal. There are many examples which provide a gap between p-hyponormality and hyponormality. But p-hyponormality and hyponormality coincide for weighted shifts. One might also guess that every p-hyponormal Toeplitz operator is hyponormal. We have not yet found a counter example.

Problem 8. *Is every p-hyponormal Toeplitz operator hyponormal?*

As a strategy try with $U^* + \alpha U$.

As a related problem, we have: *Find an example of an operator T such that T is p-hyponormal but $T - \lambda$ is not p-hyponormal for some λ.* Here is a strategy.

Try with an example of Cho and Jin [**ChJ**]: on $M_2(\mathbb{C}) \otimes \ell_2(\mathbb{Z})$,

$$T := \begin{pmatrix} \ddots & & & \vdots & & & & \\ \ddots & 0 & & \vdots & & & & \\ & C & 0 & \vdots & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & C & \vdots & 0 & & & \\ & & & \vdots & D & 0 & & \\ & & & \vdots & & D & 0 & \\ & & & \vdots & & & \ddots & \ddots \end{pmatrix} \quad \text{with some } C, D \in M_2(\mathbb{C}).$$

Cho and Jin gave an example of a semi-hyponormal non-quasihyponormal operator with $C = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix} 3 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Also M.Y. Lee and S.H. Lee gave an example of a semi-hyponormal non-$*$-paranormal operator with $C = \left(\begin{smallmatrix} 4 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix} 5 & 4 \\ 4 & 5 \end{smallmatrix}\right)$.

§6. Hyponormality of Block Toeplitz Operators

It is very complicated to Determine the hyponormality of the block Toeplitz operators T_Φ ($\Phi \in L^\infty \otimes M_n$), i.e.,

$$T_\Phi = \begin{pmatrix} A_0 & A_{-1} & A_{-2} & \dots & \dots \\ A_1 & A_0 & A_{-1} & A_{-2} & \\ A_2 & A_1 & A_0 & A_{-1} & \ddots \\ \vdots & A_2 & A_1 & A_0 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} T_{\varphi_{11}} & T_{\varphi_{12}} & \dots & T_{\varphi_{1n}} \\ T_{\varphi_{21}} & T_{\varphi_{22}} & \dots & T_{\varphi_{2n}} \\ \vdots & \vdots & & \vdots \\ T_{\varphi_{n1}} & T_{\varphi_{n2}} & \dots & T_{\varphi_{nn}} \end{pmatrix}.$$

Any criterion for hyponormality of T_Φ have not been forund yet. We first consider a problem to determine the hyponormality of the block Toeplitz operator of the following form

$$\begin{pmatrix} a_0A & a_{-1}A & a_{-2}A & \dots & \dots \\ a_1A & a_0A & a_{-1}A & a_{-2}A & \\ a_2A & a_1A & a_0A & a_{-1}A & \ddots \\ \vdots & a_2A & a_1A & a_0A & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{6.0.1}$$

where A is a $n \times n$ matrix. Thus (6.0.1) is the operator $T_\varphi \otimes A$, where $\varphi \in L^\infty(\mathbb{T})$ has the Foureier series expansion $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.

Our criterion for hyponormality of $T_\varphi \otimes A$ is as follows.

Theorem 6.1. *If $\varphi \in L^\infty(\mathbb{T})$ and $A \in M_n$, then the following are equivalent:*
(i) *$T_\varphi \otimes A$ is hyponormal;*
(ii) *T_φ is hyponormal and A is normal.*

Proof. Observe

$$[(T_\varphi \otimes A)^*, (T_\varphi \otimes A)] = T_\varphi^* T_\varphi \otimes [A^*, A] + [T_\varphi^*, T_\varphi] \otimes AA^*. \tag{6.1.1}$$

Thus if A is normal and T_φ is hyponormal then

$$[(T_\varphi \otimes A)^*, (T_\varphi \otimes A)] = [T_\varphi^*, T_\varphi] \otimes AA^* \geq 0,$$

which gives (ii) $\Rightarrow$ (i). For the implication (i) $\Rightarrow$ (ii), let P_m be the orthogonal projection of $H^2(\mathbb{T})$ onto $\bigvee\{e_0, \cdots, e_{m-1}\}$. If $S \in \mathcal{L}(H^2)$ then $P_m U^{*m} S U^m P_m$ represents a $m \times m$ principal submatrix consisting $\{m, \cdots, 2m-1\}$ columns of the matrix of S. Thus if $S \geq 0$ then $P_m U^{*m} S U^m P_m \geq 0$ for every $m = 0, 1, \cdots$. On the other hand, a straightforward calculation shows that

$$\alpha_m := ([T_\varphi^*, T_\varphi] e_m,\ e_m) = \sum_{k=m+1}^{\infty} \left(|a_k|^2 - |a_{-k}|^2\right)$$

and

$$\beta_m := (T_\varphi^* T_\varphi e_m,\ e_m) = \sum_{k=-m}^{\infty} |a_k|^2.$$

Thus $\{\beta_m\}$ is a monotonically increasing sequence of positive numbers. Since $||\varphi||_\infty \geq ||\varphi||_2$, it follows that $\alpha_m \to 0$ as $m \to \infty$. Observe

$$\begin{aligned} &P_{mn}U^{*mn}[(T_\varphi \otimes A)^*,\ (T_\varphi \otimes A)]U^{*mn}P_{mn} \\ &= \beta_m\,[A^*,\ A] + \alpha_m(AA^*) \geq 0 \quad \text{for every } m = 0, 1, 2 \cdots . \end{aligned}$$

Assume to the contrary that there exists $x \in \mathbb{C}^n$ with $||x|| = 1$ such that $([A^*, A]x,\ x) < 0$. But then

$$\beta_m\,([A^*, A]x,\ x)\ \geq\ -\alpha_m\,||A^*x||^2. \tag{6.1.2}$$

Letting $m \to \infty$, we have a contradiction because the right-hand side of (6.1.2) converges to 0, while the left-hand side is negative and monotonically decresing. Therefore we should have that $[A^*, A] \geq 0$, which implies that A is normal. Therefore from (6.1.1) we have

$$[(T_\varphi \otimes A)^*, (T_\varphi \otimes A)] = [T_\varphi^*, T_\varphi] \otimes AA^* \geq 0,$$

which implies that $[T_\varphi^*, T_\varphi] \geq 0$, i.e., T_φ is hyponormal. □

Example 6.2. Consider the following operator on ℓ_2:

$$T = \begin{pmatrix} 0 &&&&&&&&& \\ 0 & 0 &&&&&&&& \\ 1 & i & 0 &&&&&&& \\ i & 1 & 0 & 0 &&&&&& \\ 0 & 0 & 1 & i & 0 &&&&& \\ & 0 & i & 1 & 0 & 0 &&&& \\ && 0 & 0 & 1 & i & 0 &&& \\ &&& 0 & i & 1 & 0 & 0 && \\ &&&& \ddots && \ddots && \ddots & \\ &&&&& \ddots && \ddots && \ddots \end{pmatrix}$$

Then $T = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \otimes U$, so that by Theorem 7.1, T is hyponormal.

To determine the hyponormality of T_Φ $(\Phi \in L^\infty \otimes M_n)$, one might mimic the Cowen's theorem; i.e., T_Φ is hyponormal if and only if

$$\mathcal{E}(\Phi) := \{K \in H^\infty \otimes M_n : \Phi - K\,\Phi^* \in H^\infty \otimes M_n \ \text{ and } \ ||K|| = 1\}.$$

is nonempty. But this fails. For example, if $\varphi \in H^\infty(\mathbb{T})$, put

$$\Phi = \begin{pmatrix} \varphi & \bar\varphi \\ 0 & \varphi \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\Phi - K\Phi^* = \begin{pmatrix} \varphi & \bar\varphi \\ 0 & \varphi \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar\varphi & 0 \\ \varphi & \bar\varphi \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \varphi \end{pmatrix} \in H^\infty \otimes M_2.$$

But since

$$[T_\Phi^*, T_\Phi] = \begin{pmatrix} T_\varphi^* & 0 \\ T_\varphi & T_\varphi^* \end{pmatrix} \begin{pmatrix} T_\varphi & T_\varphi^* \\ 0 & T_\varphi \end{pmatrix} - \begin{pmatrix} T_\varphi & T_\varphi^* \\ 0 & T_\varphi \end{pmatrix} \begin{pmatrix} T_\varphi^* & 0 \\ T_\varphi & T_\varphi^* \end{pmatrix} = \begin{pmatrix} -T_\varphi T_\varphi^* & 0 \\ 0 & T_\varphi^* T_\varphi \end{pmatrix},$$

it follows that T_Φ is not hyponormal if $\varphi \neq 0$.

For another example, consider

$$\Phi = \begin{pmatrix} i & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$\Phi - K\Phi^* = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix} \in H^\infty \otimes M_2.$$

But since

$$[T_\Phi^*, T_\Phi] = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix},$$

and hence $\sigma([T_\Phi^*, T_\Phi]) = \{-2, 2\}$ it follows that T_Φ is not hyponormal.

We now have:

Problem 9. *Determine the hyponormality of block Toeplitz operators T_Φ ($\Phi \in L^\infty \otimes M_n$).*

§7. Spectral Properties of Toeplitz Operators

Recall that $T \in \mathcal{L}(\mathcal{H})$ is called a *von-Neumann operator* if $\sigma(T)$ is a spectral set for T. It is well-known that T is a von-Neumann operator if and only if $q(T)$ is normaloid (i.e., norm equals spectral radius) for every rational function q with poles outside $\sigma(T)$. Thus if T is *rationally* hyponormal, i.e., $q(T)$ is hyponormal for every rational function q with poles outside $\sigma(T)$, then T is a von-Neumann operator.

On the other hand, although the existence of a non-subnormal polynomially hyponormal weighted shift was established in [**CP1**] and [**CP2**], it is still open whether the implication "polynomially hyponormal ⇒ subnormal" can be disproved with a *Toeplitz operator*.

Problem 10.

(1) *Does there exist a Toeplitz operator that is polynomially hyponormal but not subnormal?*
(2) *Is every polynomially hyponormal Toeplitz operator rationally hyponormal?*
(3) *Is every Toeplitz operator a von-Neumann operator?*

Remark. As we mentioned above, Curto and Putinar [**CP1**], [**CP2**] shows that there exists an operator that is polynomially hyponormal but not 2-hyponormal (and hence not subnormal). McCarthy and Yang [**McCYa**] also showed that there exists an operator that is polynomially hyponormal but not subnormal if and only if there exists a weighted shift that is polynomially hyponormal but not subnormal. Consequently, combining two results shows that there exists a weighted shift that is polynomially hyponormal but not subnormal. However such *weighted shifts* have not been yet found even though they exist.

We thus have:

Problem 10 – Re1. *Find a weighted shift that is polynomially hyponormal but not subnormal.*

Also it is still open whether the implication "polynomially hyponormal ⇒ 2-hyponormal" can be disproved with a weighted shift. We thua have:

Problem 10 – Re2. *Is there a weighted shift that is polynomially hyponormal but not 2-hyponormal ?*

As related problems, Curto and Putinar [**CP2**] raised the following problems:

Problem 10 – Re3.

(1) *Are the classes of polynomially hyponormal, rationally (with n distinct poles) hyponormal, and analytically hyponormal operators all differenr ?*
(2) *Classify the polynomially hyponormal operators with finite rank self-commutator.*
(3) *What is the dilation and extension theory for polynomially hyponormal operators ?*
(4) *Is there an analogue of Berger's theorem for polynomially hyponormal weighted shifts ? Alternatively, is there a matricial characterization of polynomial hyponormality for weighted shifts which parallels one for subnormal shifts ?*

Let $\mathbf{K}$ denote the set, equipped with the Hausdorff metric, of all compact subsets of $\mathbb{C}$. If $\mathfrak{A}$ is a unital Banach algebra then the function $\sigma : \mathfrak{A} \to \mathbf{K}$ that maps each $T \in \mathfrak{A}$ to its spectrum $\sigma(T)$ is upper semicontinuous. In noncommutative algebras we generally have points at which the spectrum is not continuous. The work of Newburgh [**New**] contains the fundamental results on spectral continuity in general Banach algebras. J. Conway and B. Morrel [**CoM**] have undertaken a detailed study of spectral continuity in the case where the Banach algebra is the C^*-algebra of all operators acting on a complex separable Hilbert space. It is known that when restricting the spectrum to certain subsets, the spectrum becomes a continuous function on the set. The set of normal operators is perhaps the most immediate of such results ([**New**], [**Ha3**, Solution 105]). Recently, this result was extended for the set of p-hyponormal operators ([**HL2**]). Also in [**FL1**] and [**HL1**], the continuity of the spectrum was considered when the function is restricted to certain subsets of Toeplitz operators. Very recently, in [**BGS**], it was shown that the spectrum is discontinuous on the entire manifold of Toeplitz operators. In spite of this result, the following is still a challenging and interesting problem.

Problem 11. *Identify subsets $\mathfrak{S}$ of $L^\infty(\mathbb{T})$ for which the spectrum σ is continuous when restricted to the set of Toeplitz operators with symbols in $\mathfrak{S}$.*

If $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ then the *reduced minimum modulus* of T is defined by (cf. [**Ap**])

$$\gamma(T) = \begin{cases} \inf\big\{||Tx|| : \ \mathrm{dist}\,(x,\, N(T)) = 1\big\} & \text{if } T \neq 0 \\ 0 & \text{if } T = 0. \end{cases}$$

Thus $\gamma(T) > 0$ if and only if T has closed nonzero range (cf. [**Ap**],[**Go**]). If $T \in \mathcal{L}(\mathcal{H})$ is a non-zero operator then we can see ([**Ap**]) that $\gamma(T) = \inf\big(\sigma(|T|) \setminus \{0\}\big)$, where $|T|$ denotes $(T^*T)^{\frac{1}{2}}$. Thus we have that $\gamma(T) = \gamma(T^*)$. The reduced minimum modulus of an invertible operator is often the distance from 0 to its spectrum. For example this is the case for hyponormal operators.

We would like to pose:

Sub-Problem. *Find the subset $\mathfrak{S}$ of $L^\infty(\mathbb{T})$ such that if $\varphi \in \mathfrak{S}$ then*

$$\text{(7.0.1)} \qquad \mathit{dist}\Big(\lambda,\ \sigma(T_\varphi)\Big) = \gamma(T_\varphi - \lambda) \quad \textit{for every } \lambda \notin \sigma(T_\varphi).$$

For example, (7.0.1) holds for every hyponormal operator $T \in \mathcal{L}(\mathcal{H})$ in place of T_φ: for if T is an invertible hyponormal operator then since T^{-1} is also hyponormal it follows

$$\gamma(T) = \frac{1}{||T^{-1}||} = \frac{1}{\max_{\lambda\in\sigma(T)}\left|\frac{1}{\lambda}\right|} = \min_{\lambda\in\sigma(T)} |\lambda| = \text{dist}\Big(0,\ \sigma(T)\Big).$$

However (7.0.1) is not true for $T \in \mathcal{L}(\mathcal{H})$ in general; in fact (7.0.1) fails for even finite dimensional operators. For example if $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $\gamma(T) = \frac{\sqrt{5}-1}{2}$, while $\text{dist}\big(0,\ \sigma(T)\big) = 1$.

Proposition 7.1. *If $\mathfrak{S}$ is a subset of L^∞ satisfying (7.0.1) then the restriction of the spectrum σ to the set of Toeplitz operators with symbols in $\mathfrak{S}$ is continuous.*

Proof. If $\{T_n\}$ is a sequence of elements in a unital Banach algebra $\mathfrak{A}$, then $\liminf_n \sigma(T_n)$ is the set of all limit points of convergent sequences of the form $\{\lambda_n\}$, where $\lambda_n \in \sigma(T_n)$ for each n. Because the set of invertible elements in $\mathfrak{A}$ is open, we conclude that $\liminf_n \sigma(T_n) \subseteq \sigma(T)$ whenever the sequence of elements T_n converges to T in $\mathfrak{A}$. Therefore proving the spectral continuity is to show equality in this relation.

Suppose that $\varphi, \varphi_n \in \mathfrak{S}$, for $n \in \mathbb{Z}^+$, are such that T_{φ_n} converges to T_φ in norm. It suffices to show that $\sigma(T_\varphi) \subseteq \liminf\, \sigma(T_{\varphi_n})$. Assume $\lambda \notin \liminf\, \sigma(T_{\varphi_n})$. Then there exists a neighborhood $\mathcal{N}(\lambda)$ of λ such that does not intersect infinitely many $\sigma(T_{\varphi_n})$. Thus we can choose a subsequence $\{\varphi_{n_k}\}_k$ of $\{\varphi_n\}$ such that for some $\epsilon > 0$, $\text{dist}\,(\lambda,\ \sigma(T_{\varphi_{n_k}})) > \epsilon$ for all $k \in \mathbb{Z}^+$. Then by (7.0.1), $\gamma(T_{\varphi_{n_k}-\lambda}) > \epsilon$ for all $k \in \mathbb{Z}^+$. Since γ is continuous at every Toeplitz operator we must have that $\gamma(T_{\varphi-\lambda}) \geq \epsilon$, which implies that $T_\varphi - \lambda$ has closed range. Since by Coburn's theorem, either $T_\varphi - \lambda$ or $(T_\varphi - \lambda)^*$ is one-one we have that $T_\varphi - \lambda$ is semi-Fredholm. Therefore by the continuity of the (semi-Fredholm) index, $\text{ind}\,(T_\varphi - \lambda) = \lim_{k\to\infty} \text{ind}\,(T_{\varphi_{n_k}} - \lambda) = 0$, which implies that $T_\varphi - \lambda$ is Fredholm of index zero. Therefore $\lambda \notin \sigma(T_\varphi)$. This completes the proof. □

References

[Ab] M.B. Abrahamse, *Subnormal Toeplitz operators and functions of bounded type*, Duke Math. J. **43** (1976), 597–604.

[AIW] I. Amemiya, T. Ito, and T.K. Wong, *On quasinormal Toeplitz operators*, Proc. Amer. Math. Soc. **50** (1975), 254–258.

[Ap] C. Apostol, *The reduced minimum modulus*, Michigan Math. J. **32** (1985), 279–294.

[At] A. Athavale, *On joint hyponormality of operators*, Proc. Amer. Math. Soc. **103** (1988), 417–423.

[BGS] A. Böttcher, S. Grudsky and I. Spitkovsky, *The spectrum is discontinuous on the manifold of Toeplitz operators*, Arch. Math. (Basel) **75** (2000), 46–52.

[Br] J. Bram, *Subnormal operators*, Duke Math. J. **22** (1955), 75–94.

[BD] A. Brown and R.G. Douglas, *Partially isometric Toeplitz operators*, Proc. Amer. Math. Soc. **16** (1965), 681–682.

[BH] A. Brown and P.R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963-1964), 89–102.

[ChJ] M. Cho and H. Jin, *On p-hyponormal operators*, Nihonkai Math.J. **6** (1995), 201–206.

[Con] J.B. Conway, *The Theory of Subnormal Operators*, Math. Surveys and Monographs, vol. 36, Amer. Math. Soc., Providence, 1991.

[CoM] J.B. Conway and B.B. Morrel, *Operators that are points of spectral continuity*, Integral Equations Operator Theory **2** (1979), 174–198.

[CoS] J.B. Conway and W. Szymanski, *Linear combination of hyponormal operators*, Rocky Mountain J. Math. **18** (1988), 695–705.

[Cow1] C. Cowen, *More subnormal Toeplitz operators*, J. Reine Angew. Math. **367** (1986), 215–219.

[Cow2] ———, *Hyponormal and subnormal Toeplitz operators*, Surveys of Some Recent Results in Operator Theory, I (J.B. Conway and B.B. Morrel, eds.), Pitman Research Notes in Mathematics, Vol **171**, Longman, 1988, pp.(155–167).

[Cow3] ———, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103** (1988), 809–812.

[CoL] C.C. Cowen and J.J. Long, *Some subnormal Toeplitz operators*, J. Reine Angew. Math. **351** (1984), 216–220..

[Cu1] R.E. Curto, *Quadratically hyponormal weighted shifts*, Integral Equations Operator Theory **13** (1990), 49–66.

[Cu2] ———, *Joint hyponormality:A bridge between hyponormality and subnormality*, Operator Theory: Operator Algebras and Applications (Durham, NH, 1988) (W.B. Arveson and R.G. Douglas, eds.), Proc. Sympos. Pure Math., Vol. **51**, part II, American Mathematical Society, Providence, (1990), Part 11, 69–91.

[CF1] R.E. Curto and L.A. Fialkow, *Recursiveness, positivity, and truncated moment problems*, Houston J. Math. **17** (1991), 603–635.

[CF2] ———, *Recursively generated weighted shifts and the subnormal completion problem*, Integral Equations Operator Theory **17** (1993), 202–246.

[CF3] ———, *Recursively generated weighted shifts and the subnormal completion problem, II*, Integral Equations Operator Theory **18** (1994), 369–426.

[CuL1] R.E. Curto and W.Y. Lee, *Joint hyponormality of Toeplitz pairs*, Memoirs Amer. Math. Soc. no. 712, Amer. Math. Soc., Providence, 2001.

[CuL2] ———, *Towards a model theory for* 2*–hyponormal operators* (Integral Equations Operator Theory (to appear).).

[CuL3] ———, *Flatness, perturbations and completions of weighted shifts* (preprint 1999).

[CuL4] ———, *Subnormality and k-hyponormality of Toeplitz operators: A brief survey and open questions* (Banach Center Publication (to appear)).

[CMX] R.E. Curto, P.S. Muhly and J. Xia, *Hyponormal pairs of commuting operators*, Contributions to Operator Theory and Its Applications (Mesa,AZ, 1987) (I. Gohberg, J.W. Helton and L. Rodman, eds.), Operator Theory: Advances and Applications, vol. 35, Birkhäuser, Basel–Boston, (1988), 1–22.

[CP1] R.E. Curto and M. Putinar, *Existence of non-subnormal polynomially hyponormal operators*, Bull. Amer. Math. Soc. (N.S.) **25** (1991), 373–378.

[CP2] ———, *Nearly subnormal operators and moment problems*, J. Funct. Anal. **115** (1993), 480–497.

[DPY] R.G. Douglas, V.I. Paulsen, and K. Yan, *Operator theory and algebraic geometry*, Bull. Amer. Math. Soc. (N.S.) **20** (1989), 67–71.

[Fa] P. Fan, *Remarks on hyponormal trigonometric Toeplitz operators*, Rocky Mountain J. Math. **13** (1983), 489–493.

[FL1] D.R. Farenick and W.Y. Lee, *Hyponormality and spectra of Toeplitz operators*, Trans. Amer. Math. Soc. **348** (1996), 4153–4174.

[FL2] ———, *On hyponormal Toeplitz operators with polynomial and circulant-type symbols*, Int. Eq. Op. Th. **29** (1997), 202–210.

[Ga] J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[Go] S. Goldberg, *Unbounded Linear Operators*, McGraw-Hill, New York, 1966.

[Gu] C. Gu, *A generalization of Cowen's characterization of hyponormal Toeplitz operators*, J. Funct. Anal. **124** (1994), 135–148.

[Ha1] P.R. Halmos, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970), 887–933.

[Ha2] ______, *Ten years in Hilbert space*, Int. Eq. Op. Th. **2** (1979), 529–564.

[Ha3] ______, *A Hilbert Space Problem Book*, Springer, New York, 1982.

[HKL1] I.S. Hwang, I.H. Kim and W.Y. Lee, *Hyponormality of Toeplitz operators with polynomial symbols*, Math. Ann. **313** (1999), 247–261.

[HKL2] ______, *Hyponormality of Toeplitz operators with polynomial symbols (II)* (preprint 1999).

[HL1] I. S. Hwang and W. Y. Lee, *On the continuity of spectra of Toeplitz operators*, Arch. Math. (Basel) **70** (1998), 66-73.

[HL2] ______, *The spectrum is continuous on the set of p-hyponormal operators*, Math. Z. **235** (2000), 151–157.

[KL] I.H. Kim and W.Y. Lee, *On hyponormal Toeplitz operators with polynomial and symmetric-type symbols*, Int. Eq. Op. Th. **32** (1998), 216–233.

[ItW] T. Ito and T.K. Wong, *Subnormality and quasinormality of Toeplitz operators*, Proc. Amer. Math. Soc. **34** (1972), 157–164.

[McCYa] J.E. McCarthy and L. Yang, *Subnormal operators and quadrature domains*, Adv. Math. **127** (1997), 52–72.

[McCP] S. McCullough and V. Paulsen, *A note on joint hyponormality*, Proc. Amer. Math. Soc. **107** (1989), 187–195.

[NaT] T. Nakazi and K. Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc. **338** (1993), 753–767.

[New] J.D. Newburgh, *The variation of spectra*, Duke Math. J. **18** (1951), 165–176.

[Sch] I. Schur, *Über Potenzreihen die im Innern des Einheitskreises beschränkt*, J. Reine Angew. Math. **147** (1917), 205–232.

[Sun] S. Sun, *Bergman shift is not unitarily equivalent to a Toeplitz operator*, Kexue Tongbao (English Ed.) **28** (1983), 1027–1030.

[Xi] D. Xia, *Spectral Theory of Hyponormal Operators*, Birkhäuser Verlag, Boston, 1983.

[Yu] D. Yu, *Hyponormal Toeplitz operators on $H^2(\mathbb{T})$ with polynomial symbols*, Nagoya Math. J. **144** (1996), 179–182.

[Zhu] K. Zhu, *Hyponormal Toeplitz operators with polynomial symbols*, Int. Eq. Op. Th. **21** (1995), 376–381.

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