OPEN PROBLEMS IN TOEPLITZ OPERATOR THEORY

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ABSTRACT. In this article we give open problems concerning algebraic properties, subnormality, hyponormality, and spectral properties of Toeplitz operators.

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INTRODUCTION

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \succeq TT^*$, and subnormal if $T = N|_\Omega$, where $N$ is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If $T$ is subnormal then $T$ is also hyponormal. Recall that the Hilbert space $L^2(\mathbb{T})$ has a canonical orthonormal basis given by the trigonometric functions $e_n(z) = z^n$, for all $n \in \mathbb{Z}$, and that the Hardy space $H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \ldots\}$. An element $f \in L^2(\mathbb{T})$ is said to be analytic if $f \in H^2(\mathbb{T})$, and co-analytic if $f \in L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$. If $P$ denotes the projection operator $L^2(\mathbb{T}) \to H^2(\mathbb{T})$, then for every $\varphi \in L^\infty(\mathbb{T})$, the operators $T_\varphi$ and $H_\varphi$ on $H^2(\mathbb{T})$ defined by

$$T_\varphi g := P(\varphi g) \quad \text{and} \quad H_\varphi(g) := (I - P)(\varphi g) \quad (g \in H^2(\mathbb{T}))$$

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are called the *Toeplitz operator* and the *Hankel operator*, respectively, with symbol \( \varphi \). In this article we give open problems on Toeplitz operators and some related problems.

**LIST OF PROBLEMS**

**Problem 1.** Find a necessary and sufficient condition that the product \( T_{\varphi_1} \cdots T_{\varphi_n} \) of Toeplitz operators be a Toeplitz operator.

**Problem 2.** Let \( T_\varphi \) be a hyponormal Toeplitz operator. Find a necessary and sufficient condition that \( T_\varphi^2 \) be hyponormal. More generally, if \( T_\varphi \) and \( T_\psi \) are hyponormal Toeplitz operators, for which symbols \( \varphi \) and \( \psi \), is \( T_\varphi T_\psi \) hyponormal?

**Problem 3.**
1. If \( \psi \) is a Riemann map between simply connected domains, does it follow that \( T_{\psi+\alpha \overline{\psi}} \) is subnormal for some \( \alpha \) with \( 0 < \alpha < 1 \)?
2. Conversely, if \( T_{\psi+\alpha \overline{\psi}} \) is subnormal for some \( \alpha \) with \( 0 < \alpha < 1 \), does it follow that \( \psi \) is a Riemann map between simply connected domains?
3. More generally, for which \( f \in H^\infty \) is there \( \lambda \), \( 0 < \lambda < 1 \), with \( T_f + \lambda T_f^* \) subnormal?

**Problem 4.**
1. Let \( T \equiv W_\alpha \) be the weighted shift with weight sequence \( \alpha = \{\alpha_k\}_{k=0}^\infty \) with
   \[
   \alpha_k = \left( \sum_{j=0}^{k} \alpha^{2j} \right)^{\frac{1}{2}}.
   \]
   and let \( S := T + \lambda T^* \) (\( \lambda \in \mathbb{C} \)). Find a necessary and sufficient condition in terms of \( \lambda \) for \( S \) to be (weakly) \( k \)-hyponormal.
2. Make an analogue theory with the Bergman shift \( T \) or a recursively generated weighted shift \( T \) and an operator \( S_\lambda \) in place of \( T \) and \( T + \lambda T^* \) in the above setting.

**Problem 5.** Let \( 0 < \alpha < 1 \) be given and let \( \psi \) be a Riemann map of the unit disk onto the interior of the ellipse with vertices \( \pm (1 + \alpha)i \) and passing through \( \pm (1 - \alpha) \). Let \( \varphi = \psi + \alpha \overline{\psi} \) and let \( T_\varphi \) be the corresponding Toeplitz operator on \( H^2 \). Find a necessary and sufficient condition in terms of \( \lambda \) for \( T_\varphi \) to be (weakly) \( k \)-hyponormal.

**Problem 6.** If \( T_\varphi \) is a 2-hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that \( T_\varphi \) is analytic? If the answer is affirmative, is \( \varphi \) a linear function of a finite Blaschke product?

**Problem 7.** Let \( \varphi(z) = \sum_{n=-m}^N a_n z^n \). Find the classes of \( \varphi \) satisfying
   (i) \( T_\varphi \) is a hyponormal operator.
   (ii) For every zero \( \zeta \) of \( z^m \varphi \) such that \( |\zeta| > 1 \), the number \( 1/\overline{\zeta} \) is a zero of \( z^m \varphi \) in the open unit disk \( \mathbb{D} \) of multiplicity greater than or equal to the multiplicity of \( \zeta \).

**Problem 8.** Is every \( p \)-hyponormal Toeplitz operator hyponormal?

**Problem 9.** Determine the hyponormality of block Toeplitz operators \( T_\Phi \) (\( \Phi \in L^\infty \otimes M_n \)).
Problem 10.
(1) Does there exist a Toeplitz operator that is polynomially hyponormal but not subnormal?
(2) Is every polynomially hyponormal Toeplitz operator rationally hyponormal?
(3) Is every Toeplitz operator a von-Neumann operator?

Problem 11. Identify subsets $S$ of $L^\infty(T)$ for which the spectrum $\sigma$ is continuous when restricted to the set of Toeplitz operators with symbols in $S$.

§1. Algebraic Properties of Toeplitz Operators

In [BH, Theorem 8] it was shown that a necessary and sufficient condition that the product $T_\varphi T_\psi$ of two Toeplitz operators be a Toeplitz operator is that either $\varphi$ be co-analytic or $\psi$ be analytic; if the condition is satisfied, then $T_\varphi T_\psi = T_\varphi \psi$. What can we say about the product $T_\varphi_1 \cdots T_\varphi_n$?

Problem 1. Find a necessary and sufficient condition that the product $T_\varphi_1 \cdots T_\varphi_n$ of Toeplitz operators be a Toeplitz operator.

Here is a partial-strategy to Problem 1.

Theorem 1.1. If $T_\varphi$ is a Toeplitz operator such that $T_\varphi$ is one-one then a necessary condition for $T_\varphi S$ to be a Toeplitz operator for an operator $S \in L(H^2)$ is that $S$ is an analytic Toeplitz operator.

Proof. Let $(\alpha_{ij})$ be the matrix of $S$. If the Fourier expansion of $\varphi$ is $\varphi = \sum a_i z^i$, so that the matrix of $T_\varphi$ is $(\alpha_{i-j})$, then a straightforward calculation shows that if $(\beta_{i-j})$ is the matrix of $T_\varphi S$ then

$$\beta_{i,j} = \sum_{k=1}^{\infty} a_{i+2-k} \alpha_{k,j},$$

whenever $i, j \geq 1$. Since $\beta_{i,j} = \beta_{i+1,j+1}$ for each $i, j \geq 1$, we have the following equation:

$$
\begin{pmatrix}
  a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\
  a_2 & a_1 & a_0 & a_{-1} & \cdots \\
  a_3 & a_2 & a_1 & a_0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots \\
  \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\begin{pmatrix}
  \alpha_{1,j} \\
  \alpha_{2,j} - \alpha_{1,j-1} \\
  \alpha_{3,j} - \alpha_{2,j-1} \\
  \alpha_{4,j} - \alpha_{3,j-1} \\
  \vdots \\
\end{pmatrix} =
\begin{pmatrix}
  0 \\
  0 \\
  0 \\
  \vdots \\
\end{pmatrix}
$$

for each $j \geq 2$.

Since the Toeplitz matrix in the left-hand side is the matrix of $T_\varphi$, it follows from the injectivity of $T_\varphi$ that

$$
\begin{pmatrix}
  \alpha_{1,j} \\
  \alpha_{2,j} - \alpha_{1,j-1} \\
  \alpha_{3,j} - \alpha_{2,j-1} \\
  \alpha_{4,j} - \alpha_{3,j-1} \\
  \vdots \\
\end{pmatrix} =
\begin{pmatrix}
  0 \\
  0 \\
  0 \\
  \vdots \\
\end{pmatrix}
$$

for each $j \geq 2$. 

which implies that for \( j \geq 2 \),

\[
\alpha_{1+k,j+k} = 0 \quad \text{for each } k \geq 2
\]

and for \( i \geq j \),

\[
\alpha_{i,j} = \alpha_{i+1,j+1}.
\]

This shows that \( S \) must be an analytic Toeplitz operator. \( \square \)

**Remark.** Theorem 1.1 may be false if the condition “\( T_z \varphi \) is one-one” is dropped. For example, consider

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & \ldots \\
\frac{1}{2} & 1 & 0 & 0 & \ldots \\
0 & \frac{1}{2} & 1 & 0 & \ldots \\
0 & 0 & \frac{1}{2} & 1 & \ldots \\
0 & 0 & 0 & \frac{1}{2} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
1 & -\frac{1}{2} & 0 & 0 & \ldots \\
\frac{1}{2} & \frac{5}{4} & -\frac{1}{2} & 0 & \ldots \\
0 & -\frac{5}{8} & \frac{5}{4} & -\frac{1}{2} & \ldots \\
0 & \frac{1}{16} & -\frac{5}{8} & \frac{5}{4} & -\frac{1}{2} & \ldots \\
0 & \frac{1}{512} & \frac{1}{16} & -\frac{5}{8} & \frac{5}{4} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & -\frac{1}{2} & 0 & 0 & \ldots \\
0 & 1 & -\frac{1}{2} & 0 & \ldots \\
-\frac{1}{4} & 0 & 1 & -\frac{1}{2} & 0 & \ldots \\
0 & -\frac{1}{4} & 0 & 1 & -\frac{1}{2} & \ldots \\
0 & 0 & -\frac{1}{4} & 0 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

It is well known that there is a hyponormal operator whose square is not hyponormal (e.g., \( U^* + 2U \); see [Ha3, Problem 209]). Since \( U^* + 2U \) is a Toeplitz operator, the square of a hyponormal Toeplitz operator need not be hyponormal. Hence the following seems to be interesting:

**Problem 2.** Let \( T_\varphi \) be a hyponormal Toeplitz operator. Find a necessary and sufficient condition that \( T_\varphi^2 \) be hyponormal. More generally, if \( T_\varphi \) and \( T_\psi \) are hyponormal Toeplitz operators, for which symbols \( \varphi \) and \( \psi \), is \( T_\varphi T_\psi \) hyponormal?

### §2. Subnormality of Toeplitz Operators

We make a brief survey on answers to Problem 5 of Halmos’s 1970 lectures “Ten problems in Hilbert space” (cf. [Ha1],[Ha2]):

Is every subnormal Toeplitz operator either normal or analytic?
Even though the above problem was already answered negatively by Cowen and Long [CoL], it seems to be interesting to consider the following problem:

Which Toeplitz operators are subnormal?

The Halmos’s problem was answered affirmatively for trigonometric Toeplitz operators [ItW] and for quasinormal Toeplitz operators [AIW]. In 1976, Abrahamse [Ab] gave a general sufficient condition for the answer to the Halmos’s problem to be yes.

**Theorem 2.1 ([Ab] Theorem).** If

(i) $T_\varphi$ is hyponormal;
(ii) $\varphi$ or $\overline{\varphi}$ is of bounded type (i.e., $\varphi$ or $\overline{\varphi}$ is a quotient of two analytic functions);
(iii) $\ker[T_\varphi^*, T_\varphi]$ is invariant for $T_\varphi$,

then $T_\varphi$ is normal or analytic.

Since $\ker[T^*, T]$ is invariant for every subnormal operator $T$, Theorem 2.1 answers Halmos’s problem affirmatively when $\varphi$ or $\overline{\varphi}$ is of bounded type. Also, in [Ab], Abrahamse proposed a question for a strategy to answer Halmos’s problem:

Is the Bergman shift unitarily equivalent to a Toeplitz operator?

To review an answer to this question, recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called weights), the (unilateral) weighted shift $W_\alpha$ associated with $\alpha$ is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $\ell^2$. It is straightforward to check that $W_\alpha$ can never be normal, and that $W_\alpha$ is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The Bergman shift is a weighted shift $W_\alpha$ with weights

$$\alpha := \left\{ \frac{n}{n+1} \right\}_{n=1}^\infty.$$

It is well-known that the Bergman shift is subnormal. In 1983, Sun Shunhua [Shu] showed that if a Toeplitz operator $T_\varphi$ is unitarily equivalent to a hyponormal weighted shift $W_\alpha$ with weight sequence $\alpha$, then $\alpha$ must be of the form

$$\alpha = \left\{ (1 - \beta^{2n+2})^{\frac{1}{2}} ||T_\varphi|| \right\}_{n=0}^\infty$$

for some $\beta$ ($0 < \beta < 1$),

thus answering Abrahamse’s question in the negative. Cowen and Long [CoL] showed that a unilateral weighted shift with weight sequence of the form (2.1.1) must be subnormal (also see [Fa2]). Consequently, we have:

**Theorem 2.2 ([Sun], [Cow2]).** Every hyponormal Toeplitz operator which is unitarily equivalent to a weighted shift must be subnormal.

At last, in 1984, Cowen and Long [CoL] constructed the symbol $\varphi$ for which $T_\varphi$ is unitarily equivalent to the weighted shift with weight sequence (2.1.1). This answered the Halmos’s problem negatively.
Theorem 2.3 ([CoL],[Cow2]). Let \(0 < \alpha < 1\) be given and let \(\psi\) be a Riemann map of the unit disk onto the interior of the ellipse with vertices \(\pm(1+\alpha)i\) and passing through \(\pm(1-\alpha)i\). Let \(\varphi = \psi + \alpha\bar{\psi}\), and let \(T_\varphi\) be the corresponding Toeplitz operator on \(H^2\). Then \(T_\varphi\) is a weighted shift with weight sequence

\[
\alpha_n = (1 - \alpha^2)^{\frac{j}{2}} \left( \sum_{j=0}^{n} \alpha^{2j} \right)^{\frac{1}{2}}
\]

and is subnormal but neither normal nor analytic.

Problem 3.

(1) If \(\psi\) is a Riemann map between simply connected domains, does it follow that \(T_\psi + \alpha\bar{\psi}\) is subnormal for some \(\alpha\) with \(0 < \alpha < 1\)?

(2) Conversely, if \(T_\psi + \alpha\bar{\psi}\) is subnormal for some \(\alpha\) with \(0 < \alpha < 1\), does it follow that \(\psi\) is a Riemann map between simply connected domains?

(3) ([Cow2, Question 3]) More generally, for which \(f \in H^\infty\) is there \(\lambda\), \(0 < \lambda < 1\) with \(T_f + \lambda\bar{f}\) subnormal?

After Theorem 2.3, one turned their attentions to hyponormality of Toeplitz operators. The hyponormality of Toeplitz operators has been studied by M. Abramamse [Ab], C. Cowen [Cow1],[Cow2], P. Fan [Fa1], C. Gu [Gu], T. Ito and T. Wong [ItW], T. Nakazi and K. Takahashi [NaT], D. Yu [Yu], K. Zhu [Zhu], D. Farenick, the authors and others (cf. [FL1],[FL2],[CuL1],[HKL1],[HKL2],[KL]).

An elegant theorem of C. Cowen [Cow3] characterizes the hyponormality of a Toeplitz operator \(T_\varphi\) on \(H^2(T)\) by properties of the symbol \(\varphi \in L^\infty(T)\). K. Zhu [Zhu] reformulated Cowen’s criterion and then showed that the hyponormality of \(T_\varphi\) with polynomial symbols \(\varphi\) can be decided by a method based on the classical interpolation theorem of I. Schur [Sch]. We shall use a variant of Cowen’s theorem [Cow3] that was first proposed by Nakazi and Takahashi [NaT].

Cowen’s Theorem. Suppose \(\varphi \in L^\infty(T)\) is arbitrary and write

\[
\mathcal{E}(\varphi) = \{ k \in H^\infty(T) : ||k||_\infty \leq 1 \ \text{and} \ \varphi - k\varphi \in H^\infty(T) \}.
\]

Then \(T_\varphi\) is hyponormal if and only if the set \(\mathcal{E}(\varphi)\) is nonempty.

On the other hand, the Bram-Halmos criterion for subnormality states that an operator \(T\) is subnormal if and only if

\[
\sum_{i,j}(T^i x_j, T^j x_i) \geq 0
\]

for all finite collections \(x_0, x_1, \cdots, x_k \in \mathcal{H}\) ([Br],[Con, II.1.9]). Using the Choleski algorithm for operator matrices, it is easy to see that this is equivalent to the following positivity test:

\[
\begin{pmatrix}
I & T & T^2 & \cdots & T^k \\
T & T^2 & T^3 & \cdots & T^k T \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
T^k & T^k T & \cdots & T^k T^k
\end{pmatrix} \geq 0 \quad (\text{all } k \geq 1).
\]
Condition (2.3.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (2.3.1) for \( k = 1 \) is equivalent to the hyponormality of \( T \), while subnormality requires the validity of (2.3.1) for all \( k \). If we denote by \([A, B] := AB - BA\) the commutator of two operators \( A \) and \( B \), and if we define \( T \) to be \( k \)-hyponormal whenever the \( k \times k \) operator matrix

\[
M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k
\]

is positive, or equivalently, the \((k+1) \times (k+1)\) operator matrix in (2.3.1) is positive, then the Bram-Halmos criterion can be rephrased as saying that \( T \) is subnormal if and only if \( T \) is \( k \)-hyponormal for every \( k \geq 1 \) ([CMX]). Now it seems to be interesting to understand the gap between \( k \)-hyponormality and subnormality for Toeplitz operators.

In [CuL1], the following was shown:

**Theorem 2.4 ([CuL1]).** Every trigonometric Toeplitz operator whose square is hyponormal must be normal or analytic. Hence, in particular, every 2-hyponormal trigonometric Toeplitz operator is subnormal.

It is well known ([Cu]) that there is a gap between hyponormality and 2-hyponormality for weighted shifts. Theorem 2.4 also shows that there is a big gap between hyponormality and 2-hyponormality for Toeplitz operators. For example, if

\[
\varphi(z) = \sum_{n=-m}^{N} a_n z^n \quad (m < N)
\]

is such that \( T_\varphi \) is hyponormal then by Theorem 2.4, \( T_\varphi \) is never 2-hyponormal because \( T_\varphi \) is neither analytic nor normal (recall that if \( \varphi(z) = \sum_{n=-m}^{N} a_n z^n \) is such that \( T_\varphi \) is normal then \( m = N \) (cf. [FL1]).

We can extend Theorem 2.4. First of all we observe:

**Proposition 2.5 ([CuL2]).** If \( T \in \mathcal{L}(\mathcal{H}) \) is 2-hyponormal then

\[
(T(\ker[T^*,T]) \subseteq \ker[T^*,T].
\]

**Proof.** Suppose that \([T^*,T]f = 0\). Since \( T \) is 2-hyponormal, it follows from (2.3.2) that (cf. [CMX, Lemma 1.4])

\[
|([T^*f, T^*g])|^2 \leq ([T^*, T]f, f)([T^*g, T]g, g) \quad \text{for all } g \in \mathcal{H}.
\]

By assumption, we have that for all \( g \in \mathcal{H}, 0 = ([T^*f, T]g, f) = (g, [T^*f, T]^*f) = (g, T^*f, f) = T^*f, f \). Therefore,

\[
[T^*, T]f = (T^*T^2 - TT^*T)f = (T^2T^* - TT^*T)f = T[T^*, T]f = 0,
\]

which proves (2.5.1). \(\square\)

**Corollary 2.6.** If \( T_\varphi \) is 2-hyponormal and if \( \varphi \) or \( \overline{\varphi} \) is of bounded type then \( T_\varphi \) is normal or analytic, so that \( T_\varphi \) is subnormal.

**Proof.** This follows at once from Theorem 2.1 and Proposition 2.5. \(\square\)
Corollary 2.7. If \( T_\varphi \) is a 2-hyponormal operator such that \( \mathcal{E}(\varphi) \) contains at least two elements then \( T_\varphi \) is normal or analytic, so that \( T_\varphi \) is subnormal.

Proof. This follows from Corollary 2.6 and the fact ([NaT, Proposition 8]) that if \( \mathcal{E}(\varphi) \) contains at least two elements then \( \varphi \) is of bounded type. \( \square \)

From Corollaries 2.6 and 2.7, we can see that if \( T_\varphi \) is 2-hyponormal but not subnormal then \( \varphi \) is not of bounded type and \( \mathcal{E}(\varphi) \) consists of exactly one element.

Theorem 2.8. Let \( T \equiv W_\alpha \) be the weighted shift with weight sequence \( \alpha = \{\alpha_k\}_{k=0}^\infty \)

\[
\alpha_k = \left( \sum_{j=0}^k \alpha_j \right)^{\frac{1}{2}}.
\]

and let \( S := T + \lambda T^\ast \) (\( \lambda \in \mathbb{C} \)). Then we have:

1. \( S \) is hyponormal if and only if \( |\lambda| \leq 1 \);
2. \( S \) is subnormal if and only if \( \lambda = 0 \) or \( |\lambda| = \alpha_k \) for some \( k = 0, 1, 2, \ldots \).

Proof. (1) From a straightforward calculation.

(2) See [Cow1, Theorem 2.3]. \( \square \)

Recall ([At],[CMX],[CoS]) that \( T \in \mathcal{L}(\mathcal{H}) \) is said to be weakly \( k \)-hyponormal if

\[
LS((T, T^2, \ldots, T^k)) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{C}^k \right\}
\]

consists entirely of hyponormal operators, or equivalently, \( M_k(T) \) is weakly positive, i.e., ([CMX])

\[
(2.8.1) \quad (M_k(T) \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_0 x \\ \vdots \\ \lambda_k x \end{pmatrix}) \geq 0 \quad \text{for } x \in \mathcal{H} \text{ and } \lambda_0, \ldots, \lambda_k \in \mathbb{C}.
\]

If \( k = 2 \) then \( T \) is said to be quadratically hyponormal, and if \( k = 3 \) then \( T \) is said to be cubically hyponormal. Similarly, \( T \in \mathcal{L}(\mathcal{H}) \) is said to be polynomially hyponormal if \( p(T) \) is hyponormal for every polynomial \( p \in \mathbb{C}[z] \). It is known that \( k \)-hyponormal \( \Rightarrow \) weakly \( k \)-hyponormal, but the converse is not true in general. The classes of (weakly) \( k \)-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([Cu1],[Cu2],[CF1],[CF2],[CF3],[CuL1],[CMX],[DPY],[McCP]). The study of this gap has been only partially successful. For example, such a gap is not yet well described for Toeplitz operators on the Hardy space of the unit circle: in fact, even subnormality for Toeplitz operators has not yet been characterized (cf. [Ha1],[Cow2]). For weighted shifts, positive results appear in [Cu1] and [CF3], although no concrete example of a weighted shift which is polynomially hyponormal but not subnormal has yet been found (the existence of such weighted shifts was established in [CP1] and [CP2]).

Thus the following problem seems to be interesting:
Problem 4.  
(1) Let \( S \) be defined as in Theorem 2.8. Find a necessary and sufficient condition in terms of \( \lambda \) for \( S \) to be (weakly) \( k \)-hyponormal. 
(2) Make an analogue theory with the Bergman shift \( T \) or a recursively generated weighted shift \( T \) and an operator \( S_\lambda \) in place of \( T \) and \( T + \lambda T^* \) in Theorem 2.8.

Also in \([\text{Cow1}]\) the following was established:

Theorem 2.9 (\([\text{Cow1}]\)). Let \( 0 < \alpha < 1 \) be given and let \( \psi \) be a Riemann map of the unit disk onto the interior of the ellipse with vertices \( \pm (1 + \alpha)i \) and passing through \( \pm (1 - \alpha) \). Let \( \varphi = \psi + \alpha \bar{\psi} \) and let \( T_\varphi \) be the corresponding Toeplitz operator on \( H^2 \). Then

(i) \( T_\varphi \) is hyponormal if and only if \( |\lambda| \leq 1 \);
(ii) \( T_\varphi \) is subnormal if and only if

\[
\lambda = \alpha \quad \text{or} \quad \lambda = \frac{\alpha^k e^{i\theta} + \alpha}{1 + \alpha^{k+1} e^{i\theta}} \quad (k = 0, 1, 2, \cdots; 0 \leq \theta < \pi).
\]

Problem 5. Let \( T_\varphi \) be defined as in Theorem 2.9. Find a necessary and sufficient condition in terms of \( \lambda \) for \( T_\varphi \) to be (weakly) \( k \)-hyponormal.

§3. Self-commutators of Toeplitz Operators

In \([\text{AIW}]\) it was shown that every subnormal Toeplitz operator with rank-one self-commutator is a linear function of some inner function \( \chi \), where \( \chi(z) = \frac{z - \alpha}{1 - \bar{\alpha}z} \) for some \( |\alpha| < 1 \). Also K. Clancy has shown that every pure subnormal operator with rank-one self-commutator is a linear function of the unilateral shift (cf. Indiana Univ. Math. J. 23 (1973)). Also, in \([\text{CuL2}]\) it was shown that every pure 2-hyponormal operator with rank-one self-commutator is a linear function of the unilateral shift. McCarthy and Yang \([\text{McCYa}]\) classified all rationally cyclic subnormal operators with finite rank self-commutators. However it remains still open what are the pure subnormal operators with finite rank self-commutators.

Now the following question comes up at once:

Problem 6. If \( T_\varphi \) is a 2-hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that \( T_\varphi \) is analytic?

For affirmativity to Problem 6 we shall give a partial answer. To do this we recall Theorem 15 in \([\text{NaT}]\) which states that if \( T_\varphi \) is subnormal and \( \varphi = q\bar{\varphi} \), where \( q \) is a finite Blaschke product then \( T_\varphi \) is normal or analytic. But from a careful examination of the proof of the theorem we can see that its proof uses subnormality assumption only for the fact that \( \ker [T_\varphi^*, T_\varphi] \) is invariant under \( T_\varphi \). Thus in view of Proposition 2.5, the theorem is still valid for “2–hyponormal” in place of “subnormal”. We thus have:

Theorem 3.1 (\([\text{CuL4}]\)). If \( T_\varphi \) is 2-hyponormal and \( \varphi = q\bar{\varphi} \), where \( q \) is a finite Blaschke product then \( T_\varphi \) is normal or analytic.

We now give a partial answer to Problem 6.
Theorem 3.2 ([CuL4]). Suppose $\log |\varphi|$ is not integrable. If $T_\varphi$ is a 2–hyponormal operator with nonzero finite rank self-commutator then $T_\varphi$ is analytic.

Proof. If $T_\varphi$ is hyponormal such that $\log |\varphi|$ is not integrable then by an argument of [NaT, Theorem 4], $\varphi = q\hat{\varphi}$ for some inner function $q$. Also if $T_\varphi$ has a finite rank self-commutator then by [NaT, Theorem 10], there exists a finite Blaschke product $b \in \mathcal{E}(\varphi)$. If $q \neq b$, so that $\mathcal{E}(\varphi)$ contains at least two elements, then by Corollary 2.7, $T_\varphi$ is normal or analytic. If instead $q = b$ then by Theorem 3.1, $T_\varphi$ is also normal or analytic. 

Theorem 3.2 reduces Problem 6 to the class of Toeplitz operators such that $\log |\varphi|$ is integrable. If $\log |\varphi|$ is integrable then there exists an outer function $e$ such that $|\varphi| = |e|$. Thus we may write $\varphi = ue$, where $u$ is a unimodular function. Since by the Douglas-Rudin theorem (cf. [Ga, p.192]), every unimodular function can be approximated by quotients of inner functions, it follows that if $\log |\varphi|$ is integrable then $\varphi$ can be approximated by functions of bounded type. Therefore if we could obtain such a sequence $\psi_n$ converging to $\varphi$ such that $T_{\psi_n}$ is 2–hyponormal with finite rank self-commutator for each $n$, then we would answer Problem 6 affirmatively. On the other hand, if $T_\varphi$ attains its norm then by a result of Brown and Douglas [BD], $\varphi$ is of the form $\varphi = \lambda \frac{e}{\theta}$ with $\lambda > 0$, $\psi$ and $\theta$ inner. Thus $\varphi$ is of bounded type. Therefore by Corollary 2.7, if $T_\varphi$ is 2–hyponormal and attains its norm then $T_\varphi$ is normal or analytic. However we were not able to decide that if $T_\varphi$ is a 2–hyponormal operator with finite rank self-commutator then $T_\varphi$ attains its norm.

§4. Hyponormality of Toeplitz Operators

Nakazi and Takahashi [NaT, Corollary 5] showed that if $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ is a trigonometric polynomial with $m \leq N$ and if for every zero $\zeta$ of $z^m \varphi$ such that $|\zeta| > 1$, the number $1/\zeta$ is a zero of $z^m \varphi$ in the open unit disk $D$ of multiplicity greater than or equal to the multiplicity of $\zeta$, then $T_\varphi$ is hyponormal. But the converse is not true in general. To see this consider the following trigonometric polynomial: $\varphi(z) = z^{-2}(z - 2)(z - 1)(z - \frac{1}{2})(z - \frac{1}{4})$. Then $\varphi(z) = \frac{25}{12}z^{-2} - \frac{13}{12}z^{-1} + \frac{5}{12}z - \frac{5}{12}z + z^2$. Using an argument of P. Fan [Fa1, Theorem 1] – for every trigonometric polynomial $\varphi$ of the form $\varphi(z) = \sum_{n=-2}^{2} a_n z^n$, 

$$T_\varphi \text{ is hyponormal } \iff \det \left( \begin{array}{cc} a_{-1} & a_{-2} \\ a_1 & a_2 \end{array} \right) \leq |a_2|^2 - |a_{-2}|^2,$$

a straightforward calculation shows that $T_\varphi$ is hyponormal. In [HKL1] it was considered how the converse of the above result due to Nakazi and Takahashi survives for arbitrary trigonometric polynomials. The main result of [HKL1] is as follows. Suppose $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$ with $m \leq N$ and write 

$$\mathcal{F} := \{ \zeta, 1/\zeta : \text{ the complex numbers } \zeta \text{ and } 1/\zeta \text{ are zeros of } z^m \varphi \}.$$

If $\mathcal{F}$ contains at least $(N + 1)$ elements then the following statements are equivalent.

(i) $T_\varphi$ is a hyponormal operator.

(ii) For every zero $\zeta$ of $z^m \varphi$ such that $|\zeta| > 1$, the number $1/\zeta$ is a zero of $z^m \varphi$ in the open unit disk $D$ of multiplicity greater than or equal to the multiplicity of $\zeta$. 

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Moreover, in the cases where $T_{\varphi}$ is a hyponormal operator, the rank of the selfcommutator of $T_{\varphi}$ is computed from the formula $\text{rank } [T_{\varphi}^*, T_{\varphi}] = N - m + Z_D - Z_{C \setminus D}$, where $Z_D$ and $Z_{C \setminus D}$ are the number of zeros of $z^m \varphi$ in $\mathbb{D}$ and in $\mathbb{C} \setminus \overline{\mathbb{D}}$ counting multiplicity.

The above result can be easily applied for Toeplitz operators with polynomial and circulant-type symbols (cf. [FL2]). The crucial point for the converse of Nakazi-Takahashi Theorem is to find the classes $\varphi$ such that $T_{\varphi}$ is hyponormal if and only if $\varphi = k\overline{\varphi}$ for some $k \in \mathcal{E}(\varphi)$. For example, if $\log |\varphi|$ is not integrable, then $\varphi$ belongs to that class (cf. [NaT, Theorem 4]). However, if $\varphi$ is a trigonometric polynomial then $\log |\varphi|$ is integrable. Thus the above result can not be applied for trigonometric Toeplitz operators. We now have:

**Problem 7.** Let $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$. Find the classes of $\varphi$ satisfying

(i) $T_{\varphi}$ is a hyponormal operator.
(ii) For every zero $\zeta$ of $z^m \varphi$ such that $|\zeta| > 1$, the number $1/\zeta$ is a zero of $z^m \varphi$ in the open unit disk $\mathbb{D}$ of multiplicity greater than or equal to the multiplicity of $\zeta$.

§5. $p$-Hyponormality of Toeplitz Operators

An operator $T$ is called $p$-hyponormal (cf. [Xi]) if $(T^*T)^p - (TT^*)^p \geq 0$. If $p = 1$, $T$ is hyponormal and if $p = \frac{1}{2}$, $T$ is semi-hyponormal. There are many examples which provide a gap between $p$-hyponormality and hyponormality. But $p$-hyponormality and hyponormality coincide for weighted shifts. One might also guess that every $p$-hyponormal Toeplitz operator is hyponormal. We have not yet found a counter example.

**Problem 8.** Is every $p$-hyponormal Toeplitz operator hyponormal?

As a strategy try with $U^* + \alpha U$.
As a related problem, we have: Find an example of an operator $T$ such that $T$ is $p$-hyponormal but $T - \lambda$ is not $p$-hyponormal for some $\lambda$. Here is a strategy.

Try with an example of Cho and Jin [ChJ]: on $M_2(\mathbb{C}) \otimes \ell_2(\mathbb{Z})$,

$$
T := \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
\ddots & 0 & C & 0 & \ddots & \ddots & \ddots & \\
\ddots & C & 0 & \ddots & \ddots & \ddots & \ddots & \\
\ddots & \ddots & C & 0 & \ddots & \ddots & \ddots & \\
\ddots & \ddots & \ddots & D & 0 & \ddots & \ddots & \\
\ddots & \ddots & \ddots & \ddots & D & 0 & \ddots & \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
$$

with some $C, D \in M_2(\mathbb{C})$.

Cho and Jin gave an example of a semi-hyponormal non-quasihyponormal operator with $C = \begin{pmatrix} 2 & 0 \\
0 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 1 \\
1 & 2 \end{pmatrix}$. Also M.Y. Lee and S.H. Lee gave an example of a semi-hyponormal non-$s$-paranormal operator with $C = \begin{pmatrix} 4 & 2 \\
2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 4 \\
4 & 5 \end{pmatrix}$.
§6. Hyponormality of Block Toeplitz Operators

It is very complicated to determine the hyponormality of the block Toeplitz operators $T_{\Phi} (\Phi \in L^\infty \otimes M_n)$, i.e.,

$$T_{\Phi} = \begin{pmatrix} A_0 & A_{-1} & A_{-2} & \cdots & \cdots \\ A_1 & A_0 & A_{-1} & A_{-2} & \cdots \\ \vdots & A_2 & A_1 & A_0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} T_{\varphi_{11}} & T_{\varphi_{12}} & \cdots & T_{\varphi_{1n}} \\ T_{\varphi_{21}} & T_{\varphi_{22}} & \cdots & T_{\varphi_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ T_{\varphi_{n1}} & T_{\varphi_{n2}} & \cdots & T_{\varphi_{nn}} \end{pmatrix}.$$ 

Any criterion for hyponormality of $T_{\Phi}$ have not been found yet. We first consider a problem to determine the hyponormality of the block Toeplitz operator of the following form

$$(6.0.1) \begin{pmatrix} a_0A & a_{-1}A & a_{-2}A & \cdots & \cdots \\ a_1A & a_0A & a_{-1}A & a_{-2}A & \cdots \\ a_2A & a_1A & a_0A & a_{-1}A & \ddots \\ \vdots & a_2A & a_1A & a_0A & \ddots \end{pmatrix},$$

where $A$ is a $n \times n$ matrix. Thus (6.0.1) is the operator $T_{\varphi} \otimes A$, where $\varphi \in L^\infty(\mathbb{T})$ has the Fourier series expansion $\varphi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.

Our criterion for hyponormality of $T_{\varphi} \otimes A$ is as follows.

**Theorem 6.1.** If $\varphi \in L^\infty(\mathbb{T})$ and $A \in M_n$, then the following are equivalent:

(i) $T_{\varphi} \otimes A$ is hyponormal;

(ii) $T_{\varphi}$ is hyponormal and $A$ is normal.

**Proof.** Observe

$$(6.1.1) \quad [(T_{\varphi} \otimes A)^*, (T_{\varphi} \otimes A)] = T_{\varphi}^* T_{\varphi} \otimes [A^*, A] + [T_{\varphi}^*, T_{\varphi}] \otimes AA^*.$$ 

Thus if $A$ is normal and $T_{\varphi}$ is hyponormal then

$$[(T_{\varphi} \otimes A)^*, (T_{\varphi} \otimes A)] = [T_{\varphi}^*, T_{\varphi}] \otimes AA^* \geq 0,$$

which gives (ii) $\Rightarrow$ (i). For the implication (i) $\Rightarrow$ (ii), let $P_m$ be the orthogonal projection of $H^2(\mathbb{T})$ onto $\bigvee \{e_0, \cdots, e_{m-1}\}$. If $S \in \mathcal{L}(H^2)$ then $P_m U^* S U^m P_m$ represents a $m \times m$ principal submatrix consisting $\{m, \cdots, 2m-1\}$ columns of the matrix of $S$. Thus if $S \geq 0$ then $P_m U^* S U^m P_m \geq 0$ for every $m = 0, 1, \cdots$. On the other hand, a straightforward calculation shows that

$$\alpha_m := (T_{\varphi}^* T_{\varphi}^* e_m, e_m) = \sum_{k=m+1}^{\infty} (|a_k|^2 - |a_{-k}|^2)$$

and

$$\beta_m := (T_{\varphi}^* T_{\varphi} e_m, e_m) = \sum_{k=-m}^{\infty} |a_k|^2.$$
Thus \( \{ \beta_m \} \) is a monotonically increasing sequence of positive numbers. Since \( \| \varphi \|_\infty \geq \| \varphi \|_2 \), it follows that \( \alpha_m \to 0 \) as \( m \to \infty \). Observe

\[
P_{mn} U^{mn} [(T_\varphi \otimes A)^*, (T_\varphi \otimes A)] U^{mn} P_{mn} = \beta_m [A^*, A] + \alpha_m (AA^*) \geq 0 \quad \text{for every } m = 0, 1, 2 \cdots .
\]

Assume to the contrary that there exists \( x \in \mathbb{C}^n \) with \( \| x \| = 1 \) such that \( ([A^*, A]x, x) < 0 \). But then

\[
(6.1.2) \quad \beta_m ([A^*, A]x, x) \geq -\alpha_m \| A^* x \|^2.
\]

Letting \( m \to \infty \), we have a contradiction because the right-hand side of (6.1.2) converges to 0, while the left-hand side is negative and monotonically decreasing. Therefore we should have that \( [A^*, A] \geq 0 \), which implies that \( A \) is normal. Therefore from (6.1.1) we have

\[
[(T_\varphi \otimes A)^*, (T_\varphi \otimes A)] = [T_\varphi^*, T_\varphi] \otimes AA^* \geq 0,
\]

which implies that \( [T_\varphi^*, T_\varphi] \geq 0 \), i.e., \( T_\varphi \) is hyponormal. \qed

**Example 6.2.** Consider the following operator on \( \ell_2 \):

\[
T = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
0 & 1 & i & 0 & \cdots \\
i & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & i & 0 & \cdots \\
0 & i & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & i & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots 
\end{pmatrix}
\]

Then \( T = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \otimes U \), so that by Theorem 7.1, \( T \) is hyponormal.

To determine the hyponormality of \( T_\Phi \) (\( \Phi \in L^\infty \otimes M_n \)), one might mimic the Cowen’s theorem; i.e., \( T_\Phi \) is hyponormal if and only if

\[
\mathcal{E}(\Phi) := \{ K \in H^\infty \otimes M_n : \Phi - K \Phi^* \in H^\infty \otimes M_n \text{ and } \| K \| = 1 \}
\]

is nonempty. But this fails. For example, if \( \varphi \in H^\infty(\mathbb{T}) \), put

\[
\Phi = \begin{pmatrix} \varphi & \bar{\varphi} \\ 0 & \varphi \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Then

\[
\Phi - K\Phi^* = \begin{pmatrix} \varphi & \bar{\varphi} \\ 0 & \varphi \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\varphi} & 0 \\ \varphi & \varphi \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in H^\infty \otimes M_2.
\]
But since 
\[ [T_\Phi^*, T_\Phi] = \begin{pmatrix} T_\Phi & 0 \\ 0 & T_\Phi^* \end{pmatrix} - \begin{pmatrix} T_\Phi^* & 0 \\ 0 & T_\Phi \end{pmatrix} = \begin{pmatrix} -T_\Phi^* T_\Phi & 0 \\ 0 & T_\Phi^* T_\Phi \end{pmatrix}, \]
it follows that \( T_\Phi \) is not hyponormal if \( \varphi \neq 0 \).

For another example, consider
\[ \Phi = \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \]
Then
\[ \Phi - K \Phi^* = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix} \in H^\infty \otimes M_2. \]

But since
\[ [T_\Phi^*, T_\Phi] = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}, \]
and hence \( \sigma([T_\Phi^*, T_\Phi]) = \{-2, 2\} \) it follows that \( T_\Phi \) is not hyponormal.

We now have:

**Problem 9.** Determine the hyponormality of block Toeplitz operators \( T_\Phi \) \((\Phi \in L^\infty \otimes M_n)\).

§7. Spectral Properties of Toeplitz Operators

Recall that \( T \in \mathcal{L}(\mathcal{H}) \) is called a von-Neumann operator if \( \sigma(T) \) is a spectral set for \( T \). It is well-known that \( T \) is a von-Neumann operator if and only if \( q(T) \) is normaloid (i.e., norm equals spectral radius) for every rational function \( q \) with poles outside \( \sigma(T) \). Thus if \( T \) is rationally hyponormal, i.e., \( q(T) \) is hyponormal for every rational function \( q \) with poles outside \( \sigma(T) \), then \( T \) is a von-Neumann operator.

On the other hand, although the existence of a non-subnormal polynomially hyponormal weighted shift was established in [CP1] and [CP2], it is still open whether the implication “polynomially hyponormal \( \Rightarrow \) subnormal” can be disproved with a Toeplitz operator.

**Problem 10.**
1. Does there exist a Toeplitz operator that is polynomially hyponormal but not subnormal?
2. Is every polynomially hyponormal Toeplitz operator rationally hyponormal?
3. Is every Toeplitz operator a von-Neumann operator?

**Remark.** As we mentioned above, Curto and Putinar [CP1], [CP2] shows that there exists an operator that is polynomially hyponormal but not 2-hyponormal (and hence not subnormal). McCarthy and Yang [McCYa] also showed that there exists an operator that is polynomially hyponormal but not subnormal if and only if there exists a weighted shift that is polynomially hyponormal but not subnormal. Consequently, combining two results shows that there exists a weighted shift that is polynomially hyponormal but not subnormal. However such weighted shifts have not been yet found even though they exist.

We thus have:
Problem 10 – Re1. Find a weighted shift that is polynomially hyponormal but not subnormal.

Also it is still open whether the implication “polynomially hyponormal \(\Rightarrow\) 2-hyponormal” can be disproved with a weighted shift. We thus have:

Problem 10 – Re2. Is there a weighted shift that is polynomially hyponormal but not 2-hyponormal?

As related problems, Curto and Putinar [CP2] raised the following problems:

Problem 10 – Re3.
1. Are the classes of polynomially hyponormal, rationally (with \(n\) distinct poles) hyponormal, and analytically hyponormal operators all different?
2. Classify the polynomially hyponormal operators with finite rank self-commutator.
3. What is the dilation and extension theory for polynomially hyponormal operators?
4. Is there an analogue of Berger’s theorem for polynomially hyponormal weighted shifts? Alternatively, is there a matricial characterization of polynomial hyponormality for weighted shifts which parallels one for subnormal shifts?

Let \(K\) denote the set, equipped with the Hausdorff metric, of all compact subsets of \(\mathbb{C}\). If \(\mathfrak{A}\) is a unital Banach algebra then the function \(\sigma : \mathfrak{A} \to K\) that maps each \(T \in \mathfrak{A}\) to its spectrum \(\sigma(T)\) is upper semicontinuous. In noncommutative algebras we generally have points at which the spectrum is not continuous. The work of Newburgh [New] contains the fundamental results on spectral continuity in general Banach algebras. J. Conway and B. Morrel [CoM] have undertaken a detailed study of spectral continuity in the case where the Banach algebra is the \(C^*\)-algebra of all operators acting on a complex separable Hilbert space. It is known that when restricting the spectrum to certain subsets, the spectrum becomes a continuous function on the set. The set of normal operators is perhaps the most immediate of such results ([New], [Ha3, Solution 105]). Recently, this result was extended for the set of \(p\)-hyponormal operators ([HL2]). Also in [FL1] and [HL1], the continuity of the spectrum was considered when the function is restricted to certain subsets of Toeplitz operators. Very recently, in [BGS], it was shown that the spectrum is discontinuous on the entire manifold of Toeplitz operators. In spite of this result, the following is still a challenging and interesting problem.

Problem 11. Identify subsets \(\mathfrak{S}\) of \(L^\infty(\mathbb{T})\) for which the spectrum of Toeplitz operators with symbols in \(\mathfrak{S}\) is continuous.

If \(T \in \mathcal{L}(\mathcal{H}, \mathcal{K})\) then the reduced minimum modulus of \(T\) is defined by (cf. [Ap])

\[
\gamma(T) = \begin{cases} 
\inf \{\|Tx\| : \dist(x, N(T)) = 1\} & \text{if } T \neq 0 \\
0 & \text{if } T = 0.
\end{cases}
\]

Thus \(\gamma(T) > 0\) if and only if \(T\) has closed nonzero range (cf. [Ap], [Go]). If \(T \in \mathcal{L}(\mathcal{H})\) is a non-zero operator then we can see (cf. [Ap]) that \(\gamma(T) = \inf(\sigma(|T|)) \setminus \{0\}\), where \(|T|\) denotes \((T^*T)^{\frac{1}{2}}\). Thus we have that \(\gamma(T) = \gamma(T^*)\). The reduced minimum modulus of an invertible operator is often the distance from 0 to its spectrum. For example this is the case for hyponormal operators.

We would like to pose:
Sub-Problem. Find the subset $\mathcal{S}$ of $L^\infty(T)$ such that if $\varphi \in \mathcal{S}$ then

(7.0.1) \[ \text{dist} \left( \lambda, \sigma(T_\varphi) \right) = \gamma(T_\varphi - \lambda) \quad \text{for every } \lambda \notin \sigma(T_\varphi). \]

For example, (7.0.1) holds for every hyponormal operator $T \in \mathcal{L}(\mathcal{H})$ in place of $T_\varphi$; for if $T$ is an invertible hyponormal operator then since $T^{-1}$ is also hyponormal it follows

\[ \gamma(T) = \frac{1}{||T^{-1}||} = \frac{1}{\max_{\lambda \in \sigma(T)} |\frac{1}{\lambda}|} = \min_{\lambda \in \sigma(T)} |\lambda| = \text{dist} \left( 0, \sigma(T) \right). \]

However (7.0.1) is not true for $T \in \mathcal{L}(\mathcal{H})$ in general; in fact (7.0.1) fails for even finite dimensional operators. For example if $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $\gamma(T) = \frac{\sqrt{2} - 1}{2}$, while $\text{dist} \left( 0, \sigma(T) \right) = 1$.

**Proposition 7.1.** If $\mathcal{S}$ is a subset of $L^\infty$ satisfying (7.0.1) then the restriction of the spectrum $\sigma$ to the set of Toeplitz operators with symbols in $\mathcal{S}$ is continuous.

**Proof.** If $\{T_n\}$ is a sequence of elements in a unital Banach algebra $\mathfrak{A}$, then $\liminf_n \sigma \left( T_n \right)$ is the set of all limit points of convergent sequences of the form $\{\lambda_n\}$, where $\lambda_n \in \sigma(T_n)$ for each $n$. Because the set of invertible elements in $\mathfrak{A}$ is open, we conclude that $\liminf_n \sigma(T_n) \subseteq \sigma(T)$ whenever the sequence of elements $T_n$ converges to $T$ in $\mathfrak{A}$. Therefore proving the spectral continuity is to show equality in this relation.

Suppose that $\varphi, \varphi_n \in \mathcal{S}$, for $n \in \mathbb{Z}^+$, are such that $T_{\varphi_n}$ converges to $T_\varphi$ in norm. It suffices to show that $\sigma(T_{\varphi_n}) \subseteq \liminf \sigma(T_{\varphi_n})$. Assume $\lambda \notin \liminf \sigma(T_{\varphi_n})$. Then there exists a neighborhood $N(\lambda)$ of $\lambda$ such that does not intersect infinitely many $\sigma(T_{\varphi_n})$. Thus we can choose a subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$ such that for some $\epsilon > 0$, $\text{dist} \left( \lambda, \sigma(T_{\varphi_{n_k}}) \right) > \epsilon$ for all $k \in \mathbb{Z}^+$. Then by (7.0.1), $\gamma(T_{\varphi_{n_k}} - \lambda) > \epsilon$ for all $k \in \mathbb{Z}^+$. Since $\gamma$ is continuous at every Toeplitz operator we must have that $\gamma(T_{\varphi_n} - \lambda) \geq \epsilon$, which implies that $T_{\varphi_n} - \lambda$ has closed range. Since by Coburn’s theorem, either $T_{\varphi_n} - \lambda$ or $(T_{\varphi_n} - \lambda)^*$ is one-one we have that $T_{\varphi_n} - \lambda$ is semi-Fredholm. Therefore by the continuity of the (semi-Fredholm) index, $\text{ind} \left( T_{\varphi_n} - \lambda \right) = \lim_{k \to \infty} \text{ind} \left( T_{\varphi_{n_k}} - \lambda \right) = 0$, which implies that $T_{\varphi_n} - \lambda$ is Fredholm of index zero. Therefore $\lambda \notin \sigma(T_{\varphi_n})$. This completes the proof. \(\square\)

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