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AMBROSETTI-PRODI TYPE MULTIPLICITY RESULTS FOR SEMILINEAR PARABOLIC EQUATIONS*

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ABSTRACT. Ambrosetti-Prodi type multiplicity for semilinear parabolic equations are treated

0. GENERAL INTRODUCTION

Let X be a Banach space. Consider differential equations of the form

$$Lu + N(u,s) = 0 \tag{1.1}$$

where $L : X \to X$ is a linear operator and $N : X \times R \to X$ is a nonlinear operator depending on paramator s.

An Ambrosetti-Prodi(briefly AP) type multiplicity for an equation of the form (1.1) is to find constant s_0 such that (1.1) has no solution, at least one solution, or at least two solutions according to $s < s_0, s = s_0$, or $s > s_0$.

This kinds of problem has been initiated by Ambrosetti-Prodi [1] in 1972 in the study of a Dirichlet problem to elliptic equations and developed in various directions by several authors to ordinary and partial differential equations. They considered the problem

$$\begin{aligned} \triangle u + g(u) = h(x) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.2}$$

where g is of class C^2, strictly convex and such that

$$0 < lim_{u\to-\infty} g'(u) < \lambda_1 < lim_{u\to\infty} g'(u) < \lambda_2, \tag{1.3}$$

and proved that the Holder space $C^{0,\alpha}(\bar{\Omega})$ was split into two open sets O_0 and O_2 by C^1-manifold M such that the problem (1.2) has no solution, exactly one solution, or exactly two solutions according to $h \in O_0, h \in M, or h \in O_2$.

Beger and Podolak[3] obtained a certain representation for the structure of manifold M by expressing h in the form $h = \varphi + h_1$ where h_1 is orthogonal to φ in usual $L^2(\Omega)$ sense. They proved the existence of a real number s_0 such that the conclusion of the AP type theorem respectively hold according to $s < s_0, s = s_0, or s > s_0$

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Kazdan and Warner studied in [11] the problem of the form

$$\begin{aligned} \triangle u + g(x,u) &= sf(x) \text{ on } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1.4}$$

with $h \geq 0, g$ sufficiently smooth, and if (1.3) is generalized to

$$-\infty \leq lim_{u \to -\infty} sup \frac{g(x,u)}{u} < \lambda_1 < lim_{u \to \infty} inf \frac{g(x,u)}{u} \leq \infty, \tag{1.5}$$

uniformy in $x \in \bar{\Omega}$, then there exist s_0 such that the problem (1.4) has no solution if $s < s_0$, and at least one solution if $s \geq s_0$. They relaxed convexity on g but obtained the existence of at lest one solution. Once convexicity of g was not assumed, statement about pericise number of solutions are lost.

we may rewite (1.4) as follows:

$$\begin{aligned} \triangle u + \lambda_1 u + f(x,u) &= s\varphi \text{ on } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1.6}$$

then the AP type condition (1.5) becomes

$$lim_{u \to -\infty} sup \frac{f(x,u)}{u} < 0 < lim_{u \to \infty} inf \frac{f(x,u)}{u}. \tag{1.7}$$

Kannan and Ortega replace in [10] (1.7) for f locally Lipschitzian by

$$\begin{aligned} &lim_{u \to \infty} f(x,u) = \infty \text{ and} \\ &lim_{u \to -\infty} (\lambda_1 + f(x,u)) = \infty \text{ uniformly in } x \in \Omega. \end{aligned} \tag{1.8}$$

In [6], Fabry, Mawhin and Nakashama gave a notable discussion for second order ordinary differental equations with one-sided coercive nonlinearity and they particularized thir results to Lienard equations having a coercive nonlinearity.

More precisely, they considered the periodic boundary value problem

$$\begin{aligned} x''(t) + g(t, x(t)) &= s \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) &= 0, \end{aligned} \tag{1.9}$$

where $g : [0, 2\phi] \times R \to R$ is continuous, and

$$lim_{|x| \to} g(t,x) = \infty \text{ uniformly in } t \tag{1.10}$$

and they proved the existence of a real number s_0 such that (1.9) has no solution, at least one solution, at least two solutioons according to $s < s_0, s = s_0,$ or $s > s_0$.

We also note a result of Chappinelli, Mawhin and Nugari[2] when $\lambda_1 = 1$ in (1.6) for second order ordinary differential equations. They studied Dirichlet boundary value problem

$$\begin{aligned} x''(t) + x(t) + g(t, x(t)) &= s(2/\pi)^{1/2} sinx \\ x(0) = x(\pi) &= 0, \end{aligned} \tag{1.11}$$

where g is continuous and satisfying 1.10. They showed there exists $s_0 \le s_0^+$ such that (1.11) has no, at least one at least two solutions according to $s < s_0, s \in [s_0, s_0^+]$, or $s > s_0^+$.

For AP type result for periodic solutions of higher order ordinary differential equations, we refer the results of Ding and Mawhin in [3]. AP type results for Lienard systems have been done by Hirano and Kim[9], and Kim[13]. For AP type results to second order ordinary system, we refer Lee[17]. We can refer the paper of de Figueiredo[4] for the survey of AP type type problem up to 1980

Lazer and Mckenna treated AP type multiplicity result for elliptic and parabolic equations in [10]. In [9], AP type results for dissipative hyperbolic equations has been treated by Kim in [9,15]. AP type results with stability for semilinear parabolic equations are discussed in [7,8]. In our result, we discuss our problem in arbitrary dimension with the coercive growth condition on the nonlinear term g.

Now we are going to introduce an AP type multiplicity in semilinear parabolic equations[12].

1. Introduction

Let Z^+, Z, R^+ and R be the set of all positive integers, integers, nonnegative reals and reals, respectively, and let $\Omega \subseteq R^n$, $n \ge 1$, be a bounded domain with smooth boundary $\partial\Omega$ which is assumed to be of class C^2.

Let $Q = (0, 2\pi) \times \Omega$ and $L^2(Q)$ be the space of measurable Lebesgue square integrable real-valued functions on Q with usual inner product $< \cdot, \cdot >$ and corresponding norm $\| \cdot \|_2$.

By $H_0^1(\Omega)$ we mean the completion of $C_0^1(\Omega)$ with respect to the norm $\| \cdot \|_1$ defined by

$$\|\phi\|_1^2 = \int_\Omega \sum_{|\alpha| \le 1} |D^\alpha \phi(x)|^2 dx$$

$H^2(\Omega)$ stands for the usual Sobolev space ; i.e., the completion of $C^2(\bar{\Omega})$ with respect to the norm $\| \cdot \|_2$ defined by

$$\|\phi\|_2^2 = \int_\Omega \sum_{|\alpha| \le 2} |D^\alpha \phi(x)|^2 dx.$$

Let $g : R \to R$ be a continuous function. Moreover, we assume that there exist constants a_0 and b_0 such that

$$(H_1) \qquad |g(u)| \le a_0|u| + b_0 \text{ for all } u \in R.$$

The purpose of this work is to introduce AP type multiplicity result for weak solution of the nonlinear parabolic equations

$$\text{(E)} \qquad \frac{\partial u}{\partial t} - \triangle_x u - \lambda_1 u + g(u) = \frac{s\phi_1}{\sqrt{2\pi}} + h(t,x) \quad \text{in} \quad Q$$

$$(B_1) \qquad u(t,x) = 0 \quad \text{on} \quad (0, 2\pi) \times \partial\Omega$$

$$(B_2) \qquad u(0,x) = u(2\pi, x) \quad \text{on} \quad \Omega$$

where λ_1 denotes the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition and ϕ_1 is the correspanding positive normalized eigenfunction; i.e., $\phi_1(x) > 0$ on Ω and $\int_\Omega \phi_1^2(x)dx = 1$, and $h \in L^2(Q)$ with

$$\iint_Q h(t,x)\phi_1(x)dtdx = 0.$$

More precisely, the purpose is to find constants $s_0 < s_1$ such that the problem $(E)(B_1)(B_2)$ has no solution, at least one solution, or at least two solutions according to $s < s_0$, $s = s_1$ or $s > s_1$.

Here we assume the following

$$(H_2) \qquad \lim_{|u|\to\infty} \inf g(u) = +\infty,$$

$$(H_3) \qquad \lim_{u\to-\infty} \sup |\frac{g(u)}{u}| < \lambda_2 - \lambda_1.$$

2. Preliminary Results

Let's define the linear operator

$$L : DomL \subseteq L^2(Q) \to L^2(Q)$$

by

$$DomL = \{u \in L^2((0,2\pi), H^2(\Omega) \cap H_0^1(\Omega)) | \frac{\partial u}{\partial t} \in L^2(Q), \\ u(0,x) = u(2\pi,x), x \in \Omega\}$$

and

$$Lu = \frac{\partial u}{\partial t} - \Delta u - \lambda_1 u$$

Using Fourier series and Parseval inequality, we get easily

$$< Lu, \frac{\partial u}{\partial t} >= \|\frac{\partial u}{\partial t}\|_{L^2}^2 \quad \text{for all} \quad u \in DomL.$$

Hence $kerL = ker(\Delta+\lambda_1 I) = kerL^*$ since $\Delta+\lambda_1 I$ is self-adjoint and $ker(\Delta+\lambda_1 I)$ is one space dimension generated by the eigenfunction ϕ_1 . Therefore L is a closed, densely defined linear operator and $Im(L) = [kerL]^\perp$; i.e., $L^2(Q) = kerL \bigoplus ImL$. Let's consider a continous projection $P_1 : L^2(Q) \to L^2(Q)$ such that $kerP_1 = ImL$. Then $L^2(Q) = kerL \bigoplus kerP_1$. We consider another continuous projection $P_2 : L^2(Q) \to L^2(Q)$ defined by

$$(P_2 h)(t,x) = \frac{1}{2\pi} \iint_Q h(t,x)\phi_1(x)dtdx\phi_1(x).$$

Then we have $L^2(Q) = ImP_1 \bigoplus ImL$, $kerP_2 = ImL$, and $L^2(Q)/ImL$ is isomorphism to ImP_2.

Since $dim[L^2(Q)/ImL] = dim[ImP_2] = dim[kerL] = 1$, we have an isomorphism $J : ImP_2 \to kerL$.

By the closed graph theorm, the generalized right inverse of L defined by

$$K = [L|_{DomL\cap ImL}]^{-1} : ImL \to ImL$$

is continuous. If we equip the space $DomL$ with the norm

$$\|u\|_{DomL} = \iint_Q [u^2 + (\frac{\partial u}{\partial t})^2 + \sum_{|\beta|\leq 2}(D_x^\beta u)^2]dtdx,$$

then there exist a constant $c > 0$ independently of $h \in ImL,\ u = Kh$ such that

$$\|Kh\|_{DomL} \leq c\|h\|_{L^2}.$$

Therefore $K : ImL \to ImL$ is continuous and by the compact imbedding of $DomL$ in $L^2(Q)$, we have that $K : ImL \to ImL$ is compact

Lemma 2.1. *L is closed, densely defined linear operator such that $kerL = [ImL]^\perp$ and such that the right inverse $K : ImL \to ImL$ is completely continuous.*

Proof. See [18]

3. Multiplicity Result

Let us consider the following

$$(E_s^\mu) \qquad \frac{\partial u}{\partial t} - \triangle_x u - \lambda_1 u + \mu g(u) = \mu s\phi + \mu h(t,x) \quad \text{in} \quad Q$$

$$(B_1) \qquad u(t,x) = 0 \quad \text{on} \quad (0, 2\pi) \times \partial\Omega$$

$$(B_2) \qquad u(0,x) = u(2\pi, x) \quad \text{on} \quad \Omega$$

where $\mu \in [0,1]$ and $\phi(x) = \frac{\phi_1(x)}{\sqrt{2\pi}}$.

Let $L : DomL \subseteq L^2(Q) \to L^2(Q)$ be defined as before. If we define a substitution operator $N_s^\mu : L^2(Q) \to L^2(Q)$ by

$$(N_s^\mu)(t,x) = \mu g(u) - \mu s\phi - \mu h(t,x).$$

for $u \in L^2(Q)$ and $(t,x) \in Q$, then N_s^μ maps continuously into itself and take bounded sets into bounded set. Let G be any open bounded subset of $L^2(Q)$. Then $P_2 N_s^\mu : \bar{G} \to L^2(Q)$ is bounded and $K(I - P_2) : \bar{G} \to L^2(Q)$ is compact and continuous. Thus N_s^μ is L-compact on $\bar{G}$.

The coincidence degree $D_L(L + N_s^\mu, G)$ is well defined and constant in μ if $Lu + N_s^\mu \neq 0$ for $\mu \in [0,1],\ s \in R$ and $u \in DomL \cap \partial G$. It is easy to check that (u, μ) is a weak solution of (E_s^μ) if and only if $u \in DomL$ and

$$(3.1_s^\mu) \qquad Lu + N_s^\mu u = 0$$

¿From (H_2) and (H_3), we may assume that

$$m = \inf_{u\in R} g(u) > -\infty$$

and there exist $a \in (0, \lambda_2 - \lambda_1)$ and $b \geq 0$ such that

$$|g(u)| \leq a|u| + b \text{ for all } \ u \leq 0.$$

Here we have the following

Lemma 3.1. *If (H_1) (H_2) and (H_3) are satisfied, then, for each $s^* \in R^+$, there exists $M(s^*) > 0$ independently of μ such that*

$$\|\tilde{u}\|_{L^2} \le M(s^*)$$

holds for each possible weak solution $u = \alpha\phi + \tilde{u}$, with $\alpha \in R$ and $\tilde{u} \in ImL$, of (3.1_s^μ) where $s \le s^*$, $\mu \in]0.1]$.

Lemma 3.2. *If (H_1), (H_2) and (H_3) are satisfied, then, for each $s^* \in R^+$, there exists $\gamma(s^*) > 0$ independently of μ such that*

$$|\bar{u}| \le \gamma(s^*)$$

holds for each possible weak solution $u = \bar{u} + \tilde{u}$, with $\bar{u} = \alpha\phi(x)$, $\alpha \in R$ and $\tilde{u} \in ImL$, of (3.1_s^μ) where $s \le s^*$, $\mu \in]0,1]$.

Lemma 3.3. *If (H_1), (H_2) and (H_3) are satisfied, then, for each $s^* \in R^+$, we can find an open bounded set $G(s^*)$ in $L^2(Q)$ such that, for each open bounded set G in $L^2(Q)$ such that $G \supseteq G(s^*)$, we have*

$$D_L(L + N_s^1, G) = 0 \quad \textit{for all} \quad |s| \le s^*.$$

Proof. By Fatous lemme and (H_2), it is easy to see there exists $\bar{r}(s^*) > 0$ such that, for $|\alpha| > \bar{r}(s^*)$, we have

$$\iint_Q g(\alpha\phi(x))\phi(x)dtdx > s^*.$$

Let

$$G(s^*) = \{u \in L^2(Q) | -\tilde{r}(s^*)\phi(x) < \alpha\phi(x) < \tilde{r}(s^*)\phi(x) \text{ for } x \in \Omega, \|\tilde{u}\|_{L^2} < M\}$$

where $u = \alpha\phi(x) + \tilde{u}$ with $\tilde{r}(s^*) > \max\{r(s^*), r_0(s^*), \bar{r}(s^*)\}$ and $\tilde{M} > M$ which are given in Lemma 3.1 and Lemma 3.3. Let

$$s_0 = d \min_{u \in R} g(u)$$

where $d = 2\pi \int_\Omega \phi(x)dx$
If $(3.1_{\bar{s}}^\mu)$ has a solution u for some $\bar{s} \in R$ and $\mu \in]0,1]$, then by taking the inner product with ϕ on the both sides of the equation $(3.1_{\bar{s}}^\mu)$, we have

$$s_0 \le \iint_Q g(u(t,x))\phi(x)dtdx = \bar{s}.$$

Thus $(3.1_{\bar{s}}^\mu)$ has no solution for $\bar{s} < s_0$.

Hence for each open bounded set $G \supseteq G(s^*)$, we have

$$D_L(L + N_{\bar{s}}^1, G) = 0 \quad \text{for} \quad \bar{s} < s_0.$$

Choose $\bar{s} < s_0$ and define

$$F : (D(L) \cap G) \times [0,1] \to L^2(\Omega) \quad \text{by}$$

$$F(u,\mu) = Lu + N_{(1-\mu)\bar{s}+\mu s}(u) \quad \text{for} \quad |s| \le s^*.$$

They by Lemma 3.1 and Lemma 3.2, we have

$$0 \notin F(D(L) \cap \partial G) \times [0,1] \quad \text{for} \quad |s| \le s^*.$$

By the homotopy invariance of degree, we have, for all $|s| \le s^*$,

$$\begin{aligned} D_L(L + N_s^1, G) &= D_L(F(\cdot,1), G) \\ &= D_L(F(\cdot,0), G) \\ &= D_L(L + N_{\bar{s}}^1, G) \\ &= 0 \end{aligned}$$

and the proof is completed.

Lemma 3.4. *If (H_1), (H_2) and (H_3) are satisfied, then there exists $s_1 > s_0$ such that, for each $s^* > s_1$, we can find an open bounded set $\Delta(G(s^*))$ in $L^2(Q)$ on which*

$$|D_L(L + N_s^1, \Delta(G(s^*))| = 1$$

for all $s_1 < s \le s^*$.

Proof. Let

$$g(\alpha_0\phi(x_0) + \tilde{u}_0) = \min_{\substack{x\in\bar{\Omega} \\ |\alpha|\le\tilde{\gamma}(s^*) \\ |\tilde{u}|\le\tilde{M}}} g(\alpha\phi(x) + \tilde{u})$$

and $s_1 = \max_{\|\tilde{u}\|_{L^2}\le\tilde{M}} |\iint_Q g(\alpha_0\phi(x) + \tilde{u}(t,x)\phi(x)dtdx|$.

Define

$$\Delta(G(s^*)) = \{u \in L^2(Q) | \alpha_0\phi(x) < \alpha\phi(x) < \tilde{\gamma}(s^*)\phi(x) \text{ for } x \in \Omega, \|\tilde{u}\|_{L^2} < \tilde{M}\}$$

where $\tilde{\gamma}(s^*)$ and $\tilde{M}$ are given in Lemma 3.3.

If $s > s_1$, $\mu \in]0,1]$ and (u,μ) is a possible solution of (3.1_s^μ) such that $u \in \partial\Delta(G(s^*))$, then by (B_1), Lemma 3.1 and Lemma 3.2, we have necessary $u = \alpha_0\phi(x) + \tilde{u}$ or $u = \tilde{\gamma}(s^*)\phi(x)$. If $u = \alpha_0\phi(x) + \tilde{u}$ with $\|\tilde{u}\|_{L^2} < \tilde{M}$, then, by taking the inner product with ϕ on the both sides of (3.1_s^μ), we have

$$\iint_Q g(\alpha_0\phi(x) + \tilde{u}(t,x))\phi(x)dtdx = s.$$

But

$$s_1 \ge \iint_Q g(\alpha_0\phi(x) + \tilde{u}(t,x))\phi(x)dtdx = s$$

which is impossible. If $u = \tilde{\gamma}(s^*)\phi(x) + \tilde{u}$ with $\|\tilde{u}\|_{L^2} < \tilde{M}$, then, by the result in the proof of Lemma 3.3, we have

$$s = \iint_Q g(\tilde{\gamma}(s^*)\phi(x) + \tilde{u})\phi(x)dtdx > s^*$$

which is also impossible

Thus for $s \geq s_1$, and $\mu \in]0,1]1$, $D_L(L + N_s^\mu, \Delta(G(s^*)))$ is well defined and

$$D_L(L + N_s^\mu, \Delta(G(s^*))) = D_B(JP_2 N_s^\mu, \Delta(G(s^*)) \cap kerL, 0),$$

where $P_2 N_s^\mu : L^2(\Omega) \to kerL$ is an operator defined by

$$(P_2 N_s^\mu u)(t,x) = \frac{1}{2\pi}[\mu \iint_Q g(u(t,x))\phi(x)dtdx - s]\phi(x).$$

Now let $T : kerL \to R$ be defined by

$$T(\alpha\phi(x)) = \alpha.$$

Then, for $\mu = 1$,

$$\begin{aligned} D_L(L + N_s^1, \Delta(G(s^*))) &= D_B(JP_2 N_s^1, \Delta(G(s^*)) \cap kerL, 0) \\ &= D_B(T(JP_2 N_s^1)T^{-1}, T(\Delta(G(s^*)) \cap kerL), 0). \end{aligned}$$

If we let $J : ImP_2 \to kerL$ be the identity map, then the operator $\Phi = T(JP_2 N_s^1)T^{-1}$ will be defined by

$$\Phi(\alpha) = \iint_Q g(\alpha\phi(x))\phi(x)dtdx - s.$$

Thus, for $s_1 < s \leq s^*$,we have

$$\Phi(\alpha_0) = \iint_Q g(\alpha_0\phi(x))\phi(x)dtdx - s < s_1 - s < 0$$

and by the choice of $\tilde{\gamma}(s^*)$, we have

$$\begin{aligned} \Phi(\tilde{\gamma}(s^*) &= \iint_Q [g(\tilde{\gamma}(s^*)\phi(x))\phi(x)]dtdx - s \\ &> s^* - s \\ &\geq 0. \end{aligned}$$

Therefore $|D_L(L + N_s^1, \Delta(G(s^*))| = 1$ and the proof is completed.

Theorem.. *Assume* (H_1), (H_2) *and* (H_3). *Then there exist real numbers* $s_0 \leq s_1$ *such that*

(i) $(E)(B_1)(B_2)$ *has no solution for* $s < s_0$.
(ii) $(E)(B_1)(B_2)$ *has at least one solution for* $s = s_1$.
(iii) $(E)(B_1)(B_2)$ *has at least two solution for* $s > s_1$.

Proof.
Let s_0 and s_1 be constants defined in Lemma 3.3 and Lemma 3.4. Part (i) has been proved in Lemma 3.3. For part (iii), if $s > s_1$ then we can choose $G \supsetneq \Delta(G(s))$, where G and $\Delta(G(s))$ are defined in Lemma 3.3 and Lemma 3.4, respectively.

By the additivits of degree, we have

$$0 = (D_L(L + N_s^1, G) = D_L(L + N_s^1, \Delta(G(s))) + D_L(L + N_s^1, G - \overline{\Delta(G(s))})$$

and hence, by Lemma 3.3,

$$|D_L(L + N_s^1, G - \overline{\Delta(G(s))})| = 1.$$

Therefore (3.1_s^1) has one solution in $\Delta(G(s))$ and one in $G - \overline{\Delta(G(s))}$. For part (ii), let $\{s_{(n)}\}$ be a sequence in R with $s_{(1)} > s_{(2)} > \cdots > s_1$ such that $s_{(n)} \to s_1$ and let $\{u_n\}$ be the corresponding sequence of solutions of (3.1_s^1).

Then $u_n = \alpha_n \phi(x) + \tilde{u}_n$ with $\alpha_n \in R$ and $\tilde{u}_n \in ImL$.

By Lemma 3.2, we have a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ which converges to some α in R.

On the other hand, by (H_1), Lemma 3.1 and Lemma3.2, we can see that $\{Lu_{n_k}\}$ is a bounded sequence in $ImL \subseteq L^2(Q)$. Since $K : ImL \to ImL$ is a compact operator, and $\tilde{u}_{n_k} = K(Lu_{n_k})$, we have a subsequence say again $\{\tilde{u}_{n_k}\}$ converging to $\tilde{u}$ in $DomL \cap ImL$.

Therefore, we have a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which converges to $u = \alpha\phi + \tilde{u}$ with $\alpha \in R$ and $\tilde{u} \in ImL$.

Since L is closed operator, $u \in DomL$ and u is a solution of (3.1_s^1) for $s = s_1$. This complete our proof.

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