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VARIOUS SHADOWINGS IN MULTIDIMENSIONAL DYNAMICAL SYSTEMS

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Abstract. We study the concepts of continuous shadowing and inverse shadowing in multidimensional dynamical systems (in particular, $\mathbb{Z}^2$-actions), and the problems related to stability are considered.

1. Introduction

Let (M, d) be a compact metric space and let $H(M)$ be the set of all homeomorphisms on M with the metric d_0 such that $d_0(f, g) = \sup\{d(f(x), g(x)) : x \in M\}$ for all $f, g \in H(M)$. Let G be an abelian group with operation $+$. A continuous map

$$\Phi : G \times M \to M$$

is said to be a *G-action* if the following are satisfied : for all $m, n \in G$

(i) $\Phi(n, \cdot) \in H(M)$;
(ii) $\Phi(\mathbf{0}, x) = x$ for $x \in M$ where $\mathbf{0}$ is the identity element of G;
(iii) $\Phi(n + m, \cdot) = \Phi(n, \Phi(m, \cdot))$.

Put $G = \mathbb{Z}^2$. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Let $I = \{1, 2\}$. Fix $n = (n_1, n_2) \in \mathbb{Z}^2, i \in I$, and $k \in \mathbb{Z}$. We denote $n+ke_i$ by the element in $\mathbb{Z}^2$ with the usual addition (for instance, $n + ke_1 = (n_1 + k, n_2)$). Each mapping $\Phi(m, \cdot)$ can be expressed as a composition of finitely many maps (with iterations) of $\{\Phi(e_i, \cdot) : i \in I\}$.

A set $\xi = \{x_n \in M : n \in \mathbb{Z}^2\}$ is said to be a *δ-pseudo orbit* of Φ if

$$d(\Phi(e_i, x_n), x_{n+e_i}) < \delta \quad \text{for all } n \in \mathbb{Z}^2, i \in I. \tag{1}$$

Definition 1.1 ([10])**.** We say that a $\mathbb{Z}^2$-action Φ on M has the *shadowing property* provided that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every δ-pseudo orbit $\{x_n : n \in \mathbb{Z}^2\}$ of Φ there is a point $x \in M$ such that

$$d(\Phi(n,x), x_n) < \varepsilon \quad \text{for all } n \in \mathbb{Z}^2.$$

Definition 1.2 ([7])**.** A $\mathbb{Z}^2$-action Φ on a compact metric space M is said to be *expansive* provided that there exists $e > 0$ such that if $d(\Phi(n,x), \Phi(n,y)) < e$ for all $n \in \mathbb{Z}^2$, then $x = y$.

Let $Z^2(M)$ be the set of all continuous $\mathbb{Z}^2$-actions on M and let $M^{\mathbb{Z}^2}$ be a compact metric space endowed with the product topology. For a constant $\delta > 0$ and $\Phi \in Z^2(M)$, let $\mathcal{P}_\Phi(\delta)$ denote the set of all δ-pseudo orbits of Φ.

A mapping $\varphi : M \to \mathcal{P}_\Phi(\delta) \subset M^{\mathbb{Z}^2}$ satisfying $\varphi(x)_{\mathbf{0}} = x, x \in M$ is called a *δ-method* for Φ where $\mathbf{0} = (0,0) \in \mathbb{Z}^2$. For convenience, we write $\varphi(x)$ for $\{\varphi(x)_n : n \in \mathbb{Z}^2\}$. If φ is continuous, then we say that φ is a *continuous δ-method* for Φ. The set of all (resp. continuous) δ-methods for Φ will be denoted by $\mathcal{T}_0(\Phi,\delta)$ (resp. $\mathcal{T}_c(\Phi,\delta)$).

Every $\Psi \in Z^2(M)$ with

$$D_0(\Psi,\Phi) = \sup\{d(\Psi(e_i,x), \Phi(e_i,x)) : x \in M, i \in I\} < \delta \tag{2}$$

induces a continuous δ-method $\varphi_\Psi : M \to M^{\mathbb{Z}^2}$ for Φ which is defined by $\varphi_\Psi(x) = \{\Psi(n,x) : n \in \mathbb{Z}^2\}$. Let $\mathcal{T}_h(\Phi,\delta)$ be the set of all continuous δ-methods φ_Ψ for Φ induced by $\Psi \in Z^2(M)$ satisfying (2).

We define a class $\mathcal{P}_\alpha(\Phi,\delta)$ by

$$\mathcal{P}_\alpha(\Phi,\delta) = \bigcup_{\varphi \in \mathcal{T}_\alpha(\Phi,\delta)} \varphi(M)$$

for $\alpha = 0, c, h$. Then $\mathcal{P}_h(\Phi,\delta) \subset \mathcal{P}_c(\Phi,\delta) \subset \mathcal{P}_0(\Phi,\delta) = \mathcal{P}_\Phi(\delta)$.

Definition 1.3. A $\mathbb{Z}^2$-action Φ on M is said to be *$\mathcal{T}_\alpha$-shadowing* ($\mathcal{T}_\alpha$-S) for each $\alpha = 0, c, h$ provided that for any $\varepsilon > 0$ there are $\delta > 0$ and a map $r : \mathcal{P}_\alpha(\Phi,\delta) \to M$ such that

$$d(\Phi(n, r(\mathbf{x})), x_n) < \varepsilon \tag{3}$$

for all $\mathbf{x} = \{x_n : n \in \mathbb{Z}^2\} \in \mathcal{P}_\alpha(\Phi,\delta)$ and $n \in \mathbb{Z}^2$.

We say that Φ is *$\mathcal{T}_\alpha$-continuous shadowing* if the map r is continuous.

Clearly we have the following relationships among the above notions.

$$\mathcal{T}_0\text{-S} \Rightarrow \mathcal{T}_c\text{-S} \Rightarrow \mathcal{T}_h\text{-S}.$$

Remark 1.1. In the above definition, $\mathcal{T}_0$-shadowing is equivalent to the usual shadowing which was introduced in [10].

Definition 1.4. A $\mathbb{Z}^2$-action Φ on M is said to be $\mathcal{T}_\alpha$*-inverse shadowing* ($\mathcal{T}_\alpha$-IS) for each $\alpha = 0, c, h$ provided that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-method $\varphi \in \mathcal{T}_\alpha(\Phi, \delta)$ there is a map $s : M \to M$ satisfying

$$d(\Phi(n, x), \varphi(s(x))_n) < \varepsilon \tag{4}$$

for all $x \in M$ and $n \in \mathbb{Z}^2$.

We say that Φ is $\mathcal{T}_\alpha$*-continuous inverse shadowing* if the map s is continuous.

Then we have the following relationships among these kinds of inverse shadowing.

$$\mathcal{T}_0\text{-IS} \Rightarrow \mathcal{T}_c\text{-IS} \Rightarrow \mathcal{T}_h\text{-IS}.$$

2. Topological Stability

Throughout this section, M is assumed as a topological manifold. (we do not assume the smoothness of M.)

Definition 2.1. A $\mathbb{Z}^2$-action Φ on a compact metric space M is said to be *topologically stable* provided that for any $\varepsilon > 0$, there is $\delta > 0$ such that if for any $\Psi \in Z^2(M)$ with $D_0(\Phi, \Psi) < \delta$, there exists a continuous map $h : M \to M$ such that $d(x, h(x)) < \varepsilon$ and $\Phi(n, h(x)) = h(\Psi(n, x))$ for all $x \in M$ and $n \in \mathbb{Z}^2$.

Theorem 2.1. *If $\Phi \in Z^2(M)$ is $\mathcal{T}_\alpha$-continuous inverse shadowing, then it is $\mathcal{T}_\alpha$-shadowing for each $\alpha = 0, c, h$.*

Theorem 2.2. *If $\Phi \in Z^2(M)$ is $\mathcal{T}_\alpha$-continuous shadowing, then it is topologically stable for each $\alpha = 0, c, h$.*

Theorem 2.3. *If $\Phi \in Z^2(M)$ is topologically stable, then it is $\mathcal{T}_h$-inverse shadowing.*

Remark 2.1. At this moment, we do not know that if Φ is topologically stable, then it is $\mathcal{T}_c$-inverse shadowing.

Theorem 2.4. *If $\Phi \in Z^2(M)$ is $\mathcal{T}_c$-continuous shadowing, then Φ is $\mathcal{T}_c$-inverse shadowing.*

Theorem 2.5. *If $\Phi \in \mathbb{Z}^2(M)$ is expansive and $\mathcal{T}_\alpha$-shadowing, then it is $\mathcal{T}_\alpha$-continuous shadowing for each $\alpha = 0, c, h$.*

We have the following corollary by Theorem 2.2, 2.4, and 2.5.

Corollary 2.1. *If $\Phi \in \mathbb{Z}^2(M)$ is expansive and $\mathcal{T}_c$-shadowing, then it is $\mathcal{T}_c$-inverse shadowing and topologically stable.*

Remark 2.2. In Theorem 2.3 and 2.4, the assumption that M is a topological manifold can not be ignored as we see in Example 2.1, that is, there exists an expansive $\mathbb{Z}^2$-action with $\mathcal{T}_\alpha$-shadowing which is not $\mathcal{T}_\alpha$-inverse shadowing for each $\alpha = 0, c, h$.

Remark 2.3. In Theorem 2.2 and 2.5, the assumption of M can be weakened by a compact metric space. Thus, Example 2.1 is topologically stable and $\mathcal{T}_\alpha$-continuous shadowing for each $\alpha = 0, c, h$.

Let $||n|| = \max\{|n_1|, |n_2|\}$ for any $n = (n_1, n_2) \in \mathbb{Z}^2$. We can generalize the usual metric on $\{0,1\}^{\mathbb{Z}}$ to a metric d on $\{0,1\}^{\mathbb{Z}^2}$ such that $d(x,y) = 1$ if $x_{\mathbf{0}} \neq y_{\mathbf{0}}$, and $d(x,y) = \frac{1}{2^k}$ if

$$k = \max\{j > 0 : x_n = y_n,\ n \in \mathbb{Z}^2, ||n|| < j\}.$$

It means that if $x_n = y_n$ for all $||n|| \leq k$, then $d(x,y) < \frac{1}{2^k}$.

Example 2.1. Let $M = \{0,1\}^{\mathbb{Z}^2}$ and define a map $\Phi : \mathbb{Z}^2 \times M \to M$ by

$$\Phi(m,x)_n = x_{n+m}$$

for each $x = \{x_n : n \in \mathbb{Z}^2\} \in M$ and $m \in \mathbb{Z}^2$. Then the $\mathbb{Z}^2$-action Φ is shadowing ([7]), and it is easy to check that Φ is expansive.

We show that Φ is not $\mathcal{T}_h$-inverse shadowing. Let $\varepsilon < 1$. For any positive $\delta < \varepsilon$, choose $k \in \mathbb{N}$ such that $\frac{1}{2^{k-1}} < \delta$.

For each $i \in I$, consider a homeomorphism $f_i : M \to M$ given by

$$f_i(x)_n = \begin{cases} x_{n+e_i} & \text{if } -k \leq n_i \leq k-1 \\ x_{n-2ke_i} & \text{if } n_i = k \\ x_n & \text{if } |n_i| > k \end{cases}$$

for all $n = (n_1, n_2) \in \mathbb{Z}^2$ and $x \in M$.

Define a map $\Psi : \mathbb{Z}^2 \times M \to M$ by

$$\Psi((n_1,n_2),x) = ({f_1}^{n_1} \circ {f_2}^{n_2})(x)$$

for all $(n_1, n_2) \in \mathbb{Z}^2$ and $x \in M$. Then Ψ is a $\mathbb{Z}^2$-action on M. Since $\Phi(e_i, x)_n = \Psi(e_i, x)_n$ for all $x \in M$ and $|n_i| \leq k-1$, we have $D_0(\Phi, \Psi) < \frac{1}{2^{k-1}} < \delta$. To show that the inequality (4) does not hold, it is enough to prove that there exists $x \in M$ such that for all $y \in M$ there is $n \in \mathbb{Z}^2$ satisfying

$$d(\Phi(n,x), \Psi(n,y)) \geq \varepsilon.$$

Take a point $x \in \mathbb{Z}^2$ whose only non-zero component is $x_{\mathbf{0}}$. If $y \in M$ with $y_{\mathbf{0}} \neq x_{\mathbf{0}}$, then $d(x,y) = 1 > \varepsilon$. Let $y_{\mathbf{0}} = 1$ and let $n = (2k+1, 2k+1)$. Then $\Psi(n,y)_{\mathbf{0}} = 1$ and $\Phi(n,x)_{\mathbf{0}} = 0$. Thus $d(\Phi(n,x), \Psi(n,y)) = 1 > \varepsilon$. The proof is completed. □

3. Structural Stability

In this section, M is assumed as a closed manifold (compact smooth manifold without boundary). Let $Z_d^2(M)$ be the set of all continuous $\mathbb{Z}^2$-actions Φ on M such that $\Phi(e_i, \cdot) \in \mathrm{Diff}(M)$ for each $i \in I$ where $\mathrm{Diff}(M)$ is the set of all C^1-diffeomorphisms on M endowed with topology induced by the C^1-metric d_1.

Define a metric $D_1 : Z_d^2(M) \times Z_d^2(M) \to [0, \infty)$ by

$$D_1(\Phi, \Psi) = D_0(\Phi, \Psi) + \sup_{p \in M} \{||Df_i(p) - Dg_i(p)|| : i \in I\}$$

where $f_i = \Phi(e_i, \cdot)$ and $g_i = \Psi(e_i, \cdot)$ for each $i \in I$.

Definition 3.1. We say that $\Phi \in Z_d^2(M)$ is *structurally stable* if for any $\varepsilon > 0$, there is $\delta > 0$ such that for any $\Psi \in Z_d^2(M)$ with $D_1(\Phi, \Psi) < \delta$, there exists a homeomorphism $h : M \to M$ such that

(a) $\Phi(n, h(x)) = h(\Psi(n,x))$
(b) $d(x, h(x)) < \varepsilon$

for all $x \in M$ and $n \in \mathbb{Z}^2$.

We say that Φ and Ψ in $Z^2(M)$ are *topologically conjugate* if there exists a homeomorphism h with the property (a).

For any $\delta > 0$ and $\Phi \in Z_d^2(M)$, every $\Psi \in Z_d^2(M)$ with $D_1(\Phi, \Psi) < \delta$ induces a continuous δ-method $\varphi_\Psi : M \to M^{\mathbb{Z}^2}$ for Φ defined by $\varphi_\Psi(x) = \{\Psi(n,x) : n \in \mathbb{Z}^2\}$ for each $x \in M$. Let $\mathcal{T}_d(\Phi, \delta)$ be the set of all continuous δ-methods φ_Ψ for Φ which are induced by $\Psi \in Z_d^2(M)$ with $D_1(\Phi, \Psi) < \delta$. Put

$$\mathcal{P}_d(\Phi, \delta) = \bigcup_{\varphi \in \mathcal{T}_d(\Phi, \delta)} \varphi(M).$$

Definition 3.2. A $\mathbb{Z}^2$-action $\Phi \in Z_d^2(M)$ is said to be $\mathcal{T}_d$*-shadowing* ($\mathcal{T}_d$-S) provided that for any $\varepsilon > 0$ there are $\delta > 0$ and a map $r : \mathcal{P}_d(\Phi, \delta) \to M$ such that

$$d(\Phi(n, r(\mathbf{x})), x_n) < \varepsilon$$

for all $\mathbf{x} = \{x_n : n \in \mathbb{Z}^2\} \in \mathcal{P}_d(\Phi, \delta)$ and $n \in \mathbb{Z}^2$.

We say that Φ is $\mathcal{T}_d$*-continuous shadowing* if the map r is continuous.

Similarly, we may introduce the notions of $\mathcal{T}_d$*-inverse shadowing* and $\mathcal{T}_d$*-continuous inverse shadowing* as in Section 1.

Theorem 3.1. *Suppose that Φ and Ψ in $Z^2(M)$ are topologically conjugate. Then for each $\alpha = 0, c, h, d$,*

(a) *if Φ is $\mathcal{T}_\alpha$-inverse shadowing, then so is Ψ;*
(b) *if Φ is $\mathcal{T}_\alpha$-continuous inverse shadowing, then so is Ψ;*
(c) *if Φ is $\mathcal{T}_\alpha$-shadowing, then so is Ψ;*
(d) *if Φ is $\mathcal{T}_\alpha$-continuous shadowing, then so is Ψ.*

Theorem 3.2. *If $\Phi \in Z^2_d(M)$ is structurally stable, then it is $\mathcal{T}_d$-continuous inverse shadowing.*

Theorem 3.3. *If $\Phi \in Z^2_d(M)$ is expansive and $\mathcal{T}_d$-continuous inverse shadowing, then it is structurally stable.*

The next result can be proved by the same way with Theorem 2.1.

Theorem 3.4. *If $\Phi \in Z^2_d(M)$ is $\mathcal{T}_d$-continuous inverse shadowing, then it is $\mathcal{T}_d$-shadowing.*

Therefore, we get the following theorem.

Theorem 3.5. *Suppose that $\Phi \in Z^2_d(M)$ is expansive. Then the following are equivalent :*

(a) *Φ is structurally stable;*
(b) *Φ is $\mathcal{T}_d$-continuous inverse shadowing.*

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