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MODEL SETS IN TERMS OF TWO ASSOCIATED DYNAMICAL SYSTEMS

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Abstract. Model sets, or cut and project sets, are an essential ingredient in the study of long-range aperiodic order and in both theoretical and experimental sides of quasicrystal theory. We introduce two very different topologies on point sets, one local and one global, and consider two associated dynamical systems. There is no reason to expect there to be any relationship between these two. However, in model sets these two dynamical systems are strongly interconnected and this connection is essentially definitive. We characterize the model set in terms of the two associated dynamical systems.

In the mid 1980's, there was a discovery of quasicrystals in material science. Quasicrystals are metallic alloys whose diffraction patterns consist purely of bright peaks, but show certain symmetries which are forbidden in crystal diffraction patterns.

Delone sets provide an important model class for the description of aperiodic order. In particular, they can be viewed as a mathematical abstraction of the set of atomic positions of a physical quasicrystal. A Delone set is the set of points in space which has a minimum separation between points (uniform discreteness) and no arbitrary big hole in the space without containing a point of the set (relative denseness). Delone sets contain the important class of model sets, which is the main focus of this talk.

Since the discovery of quasicrystals, model sets have been a particular focus of attention because they are, except under extreme conditions, pure point diffractive. However, it is not easy to characterize these sets. By definition a model set, say in real space $\mathbb{R}^n$, is the projection of part of a lattice from some 'super-space' $\mathbb{R}^n \times H$, where H is some locally compact Abelian group, the part to be projected being determined by some suitable compact set W in H. For a model set given to us in $\mathbb{R}^n$, all this additional baggage is hidden. The process of reconstructing it is part of the problem.

Model sets are in principle easy to understand, but the virtually unlimited scope of the defining window W produces all sorts of complications, and there are various definitions of model sets in the literature, depending on exactly which conditions are imposed on W. Here we introduce inter model sets which are a new notion of model sets. This provides for a smooth characterization of model sets.

In the study of aperiodic order, one needs to decide what it means for two aperiodic sets to look like each other. There are two very distinct ways in which this is commonly done. The first is to compare the two sets locally and see if, perhaps after a small translational shift, they agree on some large patch of space, say on a large ball around the origin. This leads to a topology on discrete sets which we call here the *local topology*. On the other hand one may take the view that it is not the local structure that is important, but rather the average structure. This leads us to determine that two discrete sets are close if, again perhaps after a small translational shift, the density of the symmetric difference of the two sets is small. This leads to the *autocorrelation topology*.

Starting with a discrete set $\mathbf{\Lambda}$ one forms the translational orbit $\mathbb{R}^n + \mathbf{\Lambda}$. The closures of this orbit in the local and autocorrelation topologies lead to two dynamical hulls, $\mathbb{X}(\mathbf{\Lambda})$ and $\mathbb{A}(\mathbf{\Lambda})$. There is no reason to expect there to be any relationship between these two hulls since there is no inherent relationship between the local and autocorrelation topologies. But for model sets there is. In fact there is a continuous surjective $\mathbb{R}^n$-map $\beta : \mathbb{X}(\mathbf{\Lambda}) \rightarrow \mathbb{A}(\mathbf{\Lambda})$, the so-called *torus parameterization*, [3]. This remarkable confluence of the two topologies seems to be a fundamental property of quasicrystal theory. Here we show that it characterizes the model sets.

In substitution point sets, the torus parameterization plays an essential role in showing equivalence on the properties of pure point diffractivity, inter model sets and algebraic coincidence [1].

The underlying context of the characterization is that of Meyer sets. A Meyer set is a Delone set in $\mathbb{R}^d$ for which all the translations of the points form a uniformly discrete set. All model sets are Meyer sets, but the reverse is far from true.

The setting is not just sets, but more generally multi-colour sets. These are sets in which points are coloured by some finite number m of colours, so that in effect we have m different types of points. This generalization is rather essential for any attempts to model physical structures by using points to designate atomic positions, and it is likewise essential in the study of aperiodic tilings, in which we represent tiles of different types by suitably coloured points.

References

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