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SHADOWING IN SET-VALUED SYSTEMS

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The study of the shadowing problem for classical dynamical systems (diffeomorphisms and flows) was originated by D. V. Anosov [1] and R. Bowen [2]; the modern state of the shadowing theory is reflected in the monographs [3, 4].

The shadowing property means that, near approximate trajectories (pseudotrajectories), there exist exact trajectories of the system.

Another type of shadowing properties (inverse shadowing properties) is related to the following problem: given a family of mappings that approximate the defining mapping for the dynamical system considered, can we find, for a chosen exact trajectory, a close pseudotrajectory generated by the given family? Such properties were considered by various authors (let us mention, for example, the papers [5, 6]).

In this talk, we describe several recent results on shadowing and inverse shadowing for set-valued dynamical systems obtained together with J. Rieger (Bielefeld University, Germany).

Let us pass to basic notation. Consider a metric space $(\mathcal{M}, \mathrm{dist})$ and denote by $\mathcal{C}(\mathcal{M})$ the set of closed subsets of $\mathcal{M}$.

A set-valued dynamical system on $\mathcal{M}$ is determined by a set-valued mapping $F : \mathcal{M} \to \mathcal{C}(\mathcal{M}) \setminus \{\emptyset\}$ and its iterates. In what follows, we identify the mapping F and the corresponding dynamical system.

A sequence $\eta = \{p_k\}$ is a trajectory of the system F if

$$p_{k+1} \in F(p_k) \text{ for any } k \in \mathbb{Z}. \tag{1}$$

A sequence $\xi = \{x_k\}$ is called a d-pseudotrajectory of F if an error of size $d > 0$ is allowed in every step, i.e., if

$$\mathrm{dist}(x_{k+1}, F(x_k)) \le d \quad \text{for any } k \in \mathbb{Z}. \tag{2}$$

We say that the system F has the shadowing property if given $\epsilon > 0$ there exists $d > 0$ such that for any d-pseudotrajectory $\xi = \{x_k\}$ of F there exists a trajectory $\eta = \{p_k\}$ with

$$\mathrm{dist}(x_k, p_k) \le \epsilon \quad \text{for any } k \in \mathbb{Z}.$$

The distance between two nonempty compact subsets A and B of $\mathbb{R}^m$ is measured by the deviation

$$\mathrm{dev}(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b|$$

or by the Hausdorff distance

$$\mathrm{dist}_H(A, B) = \max\{\mathrm{dev}(A, B), \mathrm{dev}(B, A)\}.$$

The collection of nonempty, compact, and convex subsets of $\mathbb{R}^m$ will be denoted by $\mathcal{CC}(\mathbb{R}^m)$. As usual, for a sequence $\eta = \{\eta_k\} \in (\mathbb{R}^m)^{\mathbb{Z}}$,

$$||\eta||_\infty = \sup_{k \in \mathbb{Z}} |\eta_k|.$$

Our main shadowing results for set-valued mappings are based on a new definition of hyperbolicity for such mappings. Previuosly, hyperbolicity conditions for set-valued mappings (or for systems defined by relations) were introduced by Akin [7] and Sander [8]. Let us note that our hyperbolicity condition is qualitatively different from those in [7, 8] since, under our condition, the shadowing trajectory is not necessarily unique (in contrast to Akin's and Sander's cases).

Consider a set-valued mapping F of the form

$$F(x) = L(x) + M(x), \tag{3}$$

where $L : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous single-valued mapping and $M : \mathbb{R}^m \to \mathcal{CC}(\mathbb{R}^m)$ is a set-valued mapping with compact and convex images. We say that the mapping (3) is hyperbolic in $\mathbb{R}^m$ if there exist constants $N \ge 1$, $a, \kappa, l > 0$, and $\lambda \in (0, 1)$ such that the following conditions hold:

(P1) For any point $x \in \mathbb{R}^m$ there exist linear subspaces $U(x), S(x) \subset \mathbb{R}^m$ such that

$$S(x) \oplus U(x) = \mathbb{R}^m,$$

and if $P(x)$ and $Q(x)$ are the corresponding complementary projections from $\mathbb{R}^m$ to $U(x)$ and $S(x)$, then

$$\|P(x)\|, \|Q(x)\| \le N. \tag{4}$$

(P2) If $x, y, v \in \mathbb{R}^m$ satisfy the inequalities $|v| \le a$ and

$$\operatorname{dist}(y, F(x)) \le a,$$

then we can represent $L(x+v)$ as

$$L(x+v) = L(x) + A(x)v + B(x,v), \tag{5}$$

where $A(x) : \mathbb{R}^m \to \mathbb{R}^m$ is a linear mapping that is continuous with respect to x and such that

$$|Q(y)A(x)v| \le \lambda |v| \text{ for } v \in S(x), \tag{6}$$

$$|Q(y)A(x)v| \le \kappa |v| \text{ for } v \in U(x), \tag{7}$$

$$|P(y)A(x)v| \le \kappa |v| \text{ for } v \in S(x), \tag{8}$$

and the restriction $P(y)A(x)|_{U(x)} : U(x) \to U(y)$ is a linear isomorphism satisfying

$$|P(y)A(x)v| \ge \frac{1}{\lambda}|v| \text{ for } v \in U(x). \tag{9}$$

Note that since $L(x)$ and $A(x)$ are assumed to be continuous, $B(x,v)$ is continuous for any x and v with $|v| \le a$.

(P3) If $v \in \mathbb{R}^m$ satisfies the inequality $|v| \le a$, then

$$|B(x,v)| \le l|v| \tag{10}$$

and

$$\operatorname{dist}_H(M(x), M(x+v)) \le l|v| \text{ for } x \in \mathbb{R}^m. \tag{11}$$

Note that condition (11) implies the continuity of M w.r.t. the Hausdorff distance.

A simple example of a set-valued hyperbolic mapping is as follows. Assume that A is an $m \times m$ hyperbolic matrix (this means that the eigenvalues λ_j of A satisfy the inequalities $|\lambda_j| \ne 1$). Let S and U be the invariant subspaces of A that correspond to the parts of its spectrum inside and outside the unit disk, respectively.

Let M be a fixed compact convex subset of $\mathbb{R}^m$. Then the set-valued mapping $F(x) = Ax + M$ is hyperbolic (in this case, the spaces $S(x)$ and $U(x)$ coincide with S and U, respectively, for any x).

Theorem 1 [9]. *Let F be a set-valued hyperbolic mapping as described above. If*

$$\lambda + \kappa + 4lN < 1, \tag{12}$$

then F has the Lipschitz shadowing property: there exists a constant $d_0 > 0$ such that if $\{x_k\}$ is a d-pseudotrajectory of F with $d \le d_0$, then there exists a trajectory $\{p_k\}$ of F such that

$$\|\{x_k\} - \{p_k\}\|_\infty \le \mathcal{L}d,$$

where

$$\mathcal{L}^{-1} = \frac{1}{2N}\left(1 - \lambda - \kappa - 4lN\right).$$

Our result on inverse shadowing is, in a sense, local (in contrast to the case of shadowing) – we consider a fixed trajectory of the set-valued mapping F and look for close trajectories of sequences of mappings that approximate F.

Thus, let us fix a sequence of points $p_k \in \mathbb{R}^m$ such that $p_{k+1} \in F(p_k)$.

We assume that the mapping F is hyperbolic at the trajectory $\eta = \{p_k\}$ in the following sense: there exist constants $N \ge 1$, $a, \kappa, l > 0$, and $\lambda \in (0,1)$ such that condition (P1) holds for points $x = p_k$, condition (P2) holds for points $x = p_k, y = p_{k+1}$, and vectors v with $|v| \le a$, and, finally, condition (P3) holds for points $x = p_k$ and vectors v with $|v| \le a$.

We also fix a number $d > 0$ and a sequence of mappings

$$\Phi = \{\Phi_k : \mathbb{R}^m \to CC(\mathbb{R}^m)\}$$

such that each Φ_k is continuous w.r.t. dist_H and

$$\text{dist}_H(F(p_k + v), \Phi_k(p_k + v)) \le d \text{ for } k \in \mathbb{Z} \text{ and } |v| \le a. \tag{13}$$

We say that a sequence of points $x_k \in \mathbb{R}^m$ is a trajectory of the sequence Φ if $x_{k+1} \in \Phi_k(x_k)$.

Theorem 2 [9]. *Assume that a trajectory $\eta = \{p_k\}$ of F is hyperbolic in the above sense. If condition (12) is satisfied, then F has the inverse Lipschitz shadowing property: there exists a constant $d_0 > 0$ such that if a family of mappings Φ satisfies inequalities (13), where $d < d_0$, then there exists a trajectory $\{x_k\}$ of Φ such that*

$$\|\{x_k\} - \{p_k\}\|_\infty \le \mathcal{L}d,$$

where $\mathcal{L}$ is the same as in Theorem 1.

Various authors studied shadowing properties of set-valued dynamical systems with contractive properties (see [10-12]).

A set-valued dynamical system on a metric space $(\mathcal{M}, \text{dist})$ determined by a set-valued mapping $F : \mathcal{M} \to \mathcal{C}(\mathcal{M}) \setminus \{\emptyset\}$ is called contractive if there exist constants

$a > 0$ and $l \in (0, 1)$ such that if $p, q \in \mathcal{M}$ and $\text{dist}(p, q) \le a$, then

$$\text{dist}_H(F(p), F(q)) \le l\text{dist}(p, q) \tag{14}$$

(we give here one of possible variants of the definition).

Shadowing results for contractive set-valued dynamical systems were established in [12] (for the case of mappings of $\mathbb{R}^m$ whose images are either convex or have "large continuous convex kernels") and in [10] (without the convexity assumption; unfortunately, the proof in [10] contains an error).

Let us note that in the case of $\mathcal{M} = \mathbb{R}^m$, a contractive set-valued dynamical system with convex and compact images $F(x)$ is a particular case of a system defined by a hyperbolic mapping (in the sense of the definition above). Indeed, in this case we may take any $\lambda \in (0, 1)$, $S(x) = \mathbb{R}^m, U(x) = \{0\}$, and $L(x) = 0$ (thus, $A = 0$) for any $x \in \mathbb{R}^m$. Then conditions (P1), (P2), and (10) hold with $N = 1$ and any $l, \kappa > 0$, while inequalities (11) are a reformulation of (14).

It was shown in [12] (see Remark 3) that there exist contractive set-valued dynamical systems such that, for pseudotrajectories with arbitrarily small errors, the shadowing trajectories are not necessarily unique. This means that our definition of hyperbolicity for set-valued dynamical systems does not imply the uniqueness of shadowing trajectories.

The proof of the following shadowing result in [9] is based on a fixed point theorem by Frigon and Granas [13].

Theorem 3. *Let* $(\mathcal{M}, \text{dist})$ *be a complete metric space. Assume that for a set-valued mapping* $F : \mathcal{M} \to \mathcal{C}(\mathcal{M}) \setminus \{\emptyset\}$ *there exist constants* $a > 0$ *and* $l \in (0, 1)$ *such that if* $p, q \in \mathcal{M}$ *and* $\text{dist}(p, q) \le a$, *then inequality (14) holds.*

Then F *has the Lipschitz shadowing property: there exists a constant* $d_0 > 0$ *such that if* $\{x_k\}$ *is a* d*-pseudotrajectory of* F *with* $d \le d_0$, *then there exists a trajectory* $\{p_k\}$ *of* F *such that*

$$\sup_k \text{dist}(x_k, p_k) \le \mathcal{L}d,$$

where $\mathcal{L}^{-1} = 1 - l$.

References

[1] . V. Anosov. On a class of invariant sets of smooth dynamical systems. *Trudy 5th Int. Conf. Nonlin. Oscill.*, Kiev, Vol. 2, pp. 39-45 (1970).

[2] . Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.* Lect. Notes Math., Vol. 470, Springer (1975).

[3] . Yu. Pilyugin. *Shadowing in Dynamical Systems.* Lect. Notes Math., Vol. 1706, Springer (1999).

[4] . Palmer. *Shadowing in Dynamical Systems. Theory and Applications.* Kluwer Acad. Publ. (2000).

[5] . M. Corless and S. Yu. Pilyugin. Approximate and real trajectories for generic dynamical systems. *J. Math. Anal. Appl.*, Vol. 189, pp. 409-423 (1995).

[6] . E. Kloeden and J. Ombach. Hyperbolic homeomorphisms and shadowing. *Ann. Polon. Math.*, Vol. 65, pp. 171-177 (1997).

[7] . Akin. *Simplicial Dynamical Systems.* Amer. Math. Soc. (1999).

[8] . Sander. Hyperbolic sets for noninvertible maps and relations. *Discrete Cont. Dyn. Syst.*, Vol. 8, pp. 339-357 (1999).

[9] . Yu. Pilyugin and J. Rieger. Shadowing and inverse shadowing in set-valued dynamical systems. Hyperbolic case. *Topol. Meth. Nonlin. Anal.* (accepted).

[10] . Glavan and V. Gutu. On the dynamics of contractive relations. *Analysis and Optimisation of Differential Systems*, pp. 179-188, Kluwer Acad. Publ. (2003).

[11] . Glavan and V. Gutu. Attractors and fixed points of weakly contractive relations. *Fixed Point Theory*, Vol. 5, pp. 265-284 (2003).

[12] . Yu. Pilyugin and J. Rieger. Shadowing and inverse shadowing in set-valued dynamical systems. Contractive case. *Topol. Meth. Nonlin. Anal.* (2008).

[13] . Frigon and A. Granas. Résultats du type de Leray-Schauder pour des contractions multivoques. *Topol. Meth. Nonlin. Anal.*, Vol. 4, pp. 197-208 (1994).

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