SHADOWING IN FLOWS: INFLUENCE OF REPARAMETRIZATIONS

SERGEI YU. PILYUGIN

Let M be a smooth closed manifold with Riemannian metric dist. We consider the space of tangent vector fields on M of class C^1 with the C^1 topology; denote by $\phi(t,x)$ the trajectory of a field X such that $\phi(0,x) = x$.

Fix a number $d > 0$. We say that a mapping $g : \mathbb{R} \to M$ is a *d-pseudotrajectory* of a flow ϕ (and of the corresponding field X) if the inequalities

$$\mathrm{dist}(\phi(t, g(\tau)), g(\tau + t)) < d, \quad |t| \le 1,$$

hold for any $\tau \in \mathbb{R}$.

The shadowing problem is related to the following question: under which condition on a field X, for any pseudotrajectory there exists a close trajectory? The study of this problem was originated by D. V. Anosov [1] and R. Bowen [2]; the modern state of the shadowing theory is reflected in the monographs [3, 4].

The main difference between the shadowing problem for flows and the similar problem for discrete dynamical systems generated by diffeomorphisms is related to the necessity of reparametrization of shadowing trajectories in the former case.

In this talk, we describe the structure of C^1-interiors of sets of vector fields with various shadowing properties.

A monotonically increasing homeomorphism h of the line $\mathbb{R}$ such that $h(0) = 0$ is called a *reparametrization*.

Let $a > 0$; denote by $\mathrm{Rep}(a)$ the set of reparametrizations h such that

$$\left| \frac{h(t_1) - h(t_2)}{t_1 - t_2} - 1 \right| < a$$

for any different $t_1, t_2 \in \mathbb{R}$.

Let us define the main shadowing properties which we study.

We say that a field X has the *regular shadowing property* if for any $\epsilon > 0$ there exists a number $d > 0$ with the following property: for any d-pseudotrajectory g of the field X there exists a point $p \in M$ and a reparametrization $h \in \mathrm{Rep}(\epsilon)$ such that

$$\mathrm{dist}(\phi(h(t), p), g(t)) < \epsilon, \quad t \in \mathbb{R}. \tag{1}$$

Denote by RegSh the set of vector fields having the regular shadowing property.

We say that a field X has the *Lipschitz shadowing property* if there exist numbers $d_0, \mathcal{L} > 0$ having the following property: for any d-pseudotrajectory g of the field X with $d \leq d_0$ there exists a point $p \in M$ and a reparametrization $h \in \mathrm{Rep}(\mathcal{L}d)$ such that inequalities (1) hold with $\epsilon = \mathcal{L}d$. Denote by LipSh the set of vector fields having the Lipschitz shadowing property.

We say that a field X has the *oriented shadowing property* if for any $\epsilon > 0$ there exists a number $d > 0$ having the following property: for any d-pseudotrajectory g of the field X there exists a point $p \in M$ and a reparametrization h such that inequalities (1) hold (thus, we do not require the reparametrization h to be close to the identity). Denote by OrientSh the set of vector fields having the Lipschitz shadowing property.

Finally, we say that a field X has the *orbital shadowing property* if for any $\epsilon > 0$ there exists a number $d > 0$ having the following property: for any d-pseudotrajectory g of the field X there exists a point $p \in M$ such that

$$\mathrm{dist}_H(\mathrm{Cl}\{\phi(t, p) : t \in \mathbb{R}\}, \mathrm{Cl}\{g(t) : t \in \mathbb{R}\}) < \epsilon,$$

where $\mathrm{Cl}A$ is the closure of a set A, and dist_H is the Hausdorff distance. Denote by OrbitSh the set of vector fields having the orbital shadowing property.

Clearly, the following inclusions hold:

$$\mathrm{LipSh} \subset \mathrm{RegSh} \subset \mathrm{OrientSh} \subset \mathrm{OrbitSh}.$$

We introduce the following notation: $\mathcal{S}$ denotes the set of structurally stable vector fields, and $\mathcal{N}$ denotes the set of nonsingular vector fields. If P is a set of vector fields, we denote by $\mathrm{Int}^1(P)$ the interior of the set P with respect to the C^1 topology.

It was shown in [5] that $\mathcal{S} \subset \mathrm{LipSh}$.

We define the following class of vector fields which is important for us. We say the a field X belongs to the class $\mathcal{B}$ if it has hyperbolic rest points p and q (not necessarily distinct) such that

(1) the Jacobi matrix $DX(p)$ has a pair of complex conjugate eigenvalues $a_1 \pm b_1 i$ with $a_1 < 0$ of multiplicity 1, and if $c_1 + d_1 i$ is an eigenvalue different from $a_1 \pm b_1 i$ and such that $c_1 < 0$, then $c_1 < a_1$;

(2) the Jacobi matrix $DX(q)$ has a pair of complex conjugate eigenvalues $a_2 \pm b_2 i$ with $a_2 > 0$ of multiplicity 1, and if $c_2 + d_2 i$ is an eigenvalue different from $a_2 \pm b_2 i$ and such that $c_2 > 0$, then $c_2 > a_2$;

(3) the unstable manifold $W^u(p)$ and the stable manifold $W^s(q)$ have a trajectory of nontransverse intersection.

Our main results (announced in [6]) are as follows.

Theorem 1. $\mathrm{Int}^1(\mathrm{OrbitSh}) \cap \mathcal{N} \subset \mathcal{S}$.

This theorem generalizes the main result of the recent paper [7], where it was shown that $\mathrm{Int}^1(\mathrm{RegSh}) \cap \mathcal{N} \subset \mathcal{S}$. For discrete dynamical systems generated by diffeomorphisms, an analog of Theorem 1 was obtained in [8].

Theorem 2. $\mathrm{Int}^1(\mathrm{OrientSh} \setminus \mathcal{B}) \subset \mathcal{S}$.

Theorem 3. $\mathrm{Int}^1(\mathrm{LipSh}) \subset \mathcal{S}$.

The above-mentioned result of [5] and Theorem 3 imply that

$$\mathrm{Int}^1(\mathrm{LipSh}) = \mathcal{S}.$$

Theorem 4.   *If* $\dim M \leq 3$, *then* $\mathrm{Int}^1(\mathrm{OrientSh}) \subset \mathcal{S}$.

Thus, if $\dim M \leq 3$, then

$$\mathrm{Int}^1(\mathrm{OrientSh}) = \mathcal{S}.$$

These results were obtained together with S. B. Tikhomirov.

References

[1] D. V. Anosov. On a class of invariant sets of smooth dynamical systems. *Trudy 5th Int. Conf. Nonlin. Oscill.*, Kiev, Vol. 2, pp. 39-45 (1970).

[2] . Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.* Lect. Notes Math., Vol. 470, Springer (1975).

[3] . Yu. Pilyugin. *Shadowing in Dynamical Systems.* Lect. Notes Math., Vol. 1706, Springer (1999).

[4] . Palmer. *Shadowing in Dynamical Systems. Theory and Applications.* Kluwer Acad. Publ. (2000).

[5] . Yu. Pilyugin. Shadowing in structurally stable flows. *J. Differ. Equations*, Vol. 140, pp. 238-265 (1997).

[6] . Yu. Pilyugin and S. B. Tikhomirov. Sets of vector fields with various properties of shadowing of pseudotrajectories. *Dokl. Ros. Akad. Nauk* (2008).

[7] . Lee and K. Sakai. Structural stability of vector fields with shadowing. *J. Differ. Equations*, Vol. 232, pp. 303-313 (2007).

[8] . Yu. Pilyugin, A. A. Rodionova, and K. Sakai. Orbital and weak shadowing properties. *Discrete Contin. Dyn. Syst.*, Vol. 9, pp. 287-308 (2003).

ST.PETERSBURG STATE UNIVERSITY. RUSSIA, 198504, ST.PETERSBURG, UNIVERSITY AV., 28,

E-mail address: sp@sp1196.spb.edu