Trends in Mathematics - New Series
Information Center for Mathematical Sciences
Volume 11, Number 1, 2009, pages 63–70

LEFT-INVARIANT MINIMAL UNIT VECTOR FIELDS ON LIE GROUPS

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Abstract. We provide examples of left-invariant unit vector fields on Lie groups.

1. Introduction

A smooth unit vector field on a Riemannian manifold (M, g) is a cross section of its unit sphere bundle $T^1(M)$ and hence can be considered as a submanifold of $T^1(M)$. Let M be a compact manifold and if $T^1(M)$ is equipped with the Sasaki metric g_s, then the volume of the unit vector field is defined as the volume of this submanifold.

Gluck and and Ziller considered the problem of determining unit vector fields with minimal volume in [5] and they showed that the unit vector fields of minimal volume on S^3 are precisely the Hopf vector fields and no others. But in the higher dimensional spheres, $S^{2n+1}, k \geq 2$, this is not the case([4], [9], [12]).

Gil-Medriano and Llinares-Fuster showed that a unit vector field is a critical point of the volume functional if and only if the corresponding immersion in (T^1M, g_s) is minimal([3]). So such unit vector fields are called minimal even though the manifold is not compact.

The aim of this paper is to provide examples of left-invariant unit vector fields on the Lie groups.

2000 *Mathematics Subject Classification.* 53C20, 53C25, 53C42.
Key words and phrases. invariant minimal unit vector fields, Lie groups, semi-direct product.

In section 2, we give some basic notions and facts. In section 3, we provide examples of left-invariant unit vector fields on the Lie groups.

2. Minimal unit vector field

Let (M, g) be a smooth Riemannian manifold, ∇ be the Levi-Civita connection on (M, g) and R be the associated Riemannian curvature tensor with the sign convention $R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$.

We assume that the set $\chi^1(M)$ of unit vector fields on M is non-empty. For $V \in \chi^1(M)$ we define the positive definite symmetric tensor field L_V by

$$L_V := I + (\nabla V)^* \nabla V$$

where I is the identity map and $(\nabla V)^*$ is the adjoint. Put $f(V) = (\det L_V)^{\frac{1}{2}}$. Then the volume functional $F : \chi^1(M) \longrightarrow \mathbb{R}$ is defined by

$$\int_M f(V) dv$$

where dv is the volume form on (M, g).

Define a $(1,1)$-tensor field K_V and a 1-form ω_V associated to V as follows.

$$K_V = f(V) \cdot L_V^{-1} \circ (\nabla V)^*,$$
$$\omega_V(X) = (\mathrm{tr}(Z \longmapsto \nabla_Z K_V)(X)).$$

More precisely, for an orthonormal basis $\{E_1, E_2, \cdots, E_n\}$ of the tangent space, $\omega_V(X)$ is given by

$$\omega_V(X) = \sum_{i=1}^{n} g((\nabla_{E_i} K_V)(X), E_i).$$

In [3] it is proved that a unit vector field V is a critical point for the volume functional F if and only if the 1-form ω_V annihilates the distribution $\mathcal{H}^V$ consisting of tangent vectors orthogonal to V. Moreover it is shown that V is critical if and only if the map $V : M \longrightarrow (T^1M, g_s)$ is a minimal immersion, where (T^1M, g_s) is the unit tangent bundle T^1M equipped with the Sasaki metric g_s. Thus the following definition is natural.

Definition 2.1. *A unit vector field V on a Riemannian manifold (M, g) is called minimal if $\omega_V(X) = 0$ for all $X \in \mathcal{H}^V$.*

From now on we consider left-invariant unit vector fields on a Lie group. Let G be an n-dimensional connected Lie group equipped with a left-invariant metric and $\mathfrak{g}$ be its Lie algebra. Then the left-invariant metric on G determines the associated inner product $<,>$ on $\mathfrak{g}$. Furthermore let $\mathcal{S}$ be the unit sphere of $\mathfrak{g}$ with respect to $<,>$. For $V \in \mathcal{S}, \nabla V, L_V, K_V$ and ω_V are invariant by left translation, they can be viewed as tensors on $\mathfrak{g}$ which are determined by the Lie algebra structure and its inner product $<,>$ and f is viewed as a function on $\mathcal{S}$ defined by $V \in \mathcal{S} \longmapsto (\det L_V)^{\frac{1}{2}}$.

The distribution $\mathcal{H}^V$ is invariant by left translation and is identified with the orthogonal complement $V^\perp$ of V in $\mathfrak{g}$ and thus $V^\perp$ may be naturally identified with the tangent space $T_V\mathcal{S}$ of the unit sphere $\mathcal{S}$ at V. Thus a left invariant unit vector field V is minimal if and only if the 1-form ω_V on $\mathfrak{g}$ vanishes on $V^\perp \cong T_V\mathcal{S}$.

Proposition 2.1 ([15]; Proposition 2.1)**.** *For $X \in T_V\mathcal{S}$ we have*

$$\omega_V(X) = -df_V(X) - \operatorname{tr} ad_{K_V X}$$

and V is minimal if and only if

$$df_V(X) = -\operatorname{tr} ad_{K_V X}$$

for all $X \in T_V\mathcal{S}$.

Thus on a unimodular Lie group G, i.e., $\operatorname{tr} ad_X = 0$, for all $X \in \mathfrak{g}$, a left-invariant unit vector field V is minimal if and only if V is a critical point of the function f on $\mathcal{S}$.

Let G be a non-unimodular Lie group with a left-invariant metric. We denote by $\mathcal{U}$ its *unimodular kernel*, i.e.,

$$\mathcal{U} = \{X \in \mathfrak{g} | \operatorname{tr} ad_X = 0\}.$$

Then $\mathcal{U}$ is an ideal of codimension 1 since $\operatorname{tr} ad_X$ is a linear functional. We denote H a unit vector orthogonal to $\mathcal{U}$. Then the linear transformation ad_H restricted to $\mathcal{U}$ is a derivation of $\mathcal{U}$. This yields the following.

Proposition 2.2 ([15]; Proposition 2.5)**.** *Let $\mathcal{U}$ be the unimodular kernel of a non-unimodular Lie group such that $ad_H|_{\mathcal{U}}$ is a symmetric endomorphism of $\mathcal{U}$ with respect to $<,>$. Then a left-invariant unit vector field V is minimal if and only if it is a critical point of the function f on $\mathcal{S}$.*

3. Examples of left-invariant unit vector fields on Lie groups

3.1 Three dimensional Lie groups

In the first consider a three-dimensional unimodular Lie algebra with inner product $<,>$. Let e_1, e_2, e_3 be an orthonormal basis such that

$$[e_2, e_3] = \lambda_1 e_1, [e_3, e_1] = \lambda_2 e_2, [e_1, e_2] = \lambda_3 e_3. \tag{1}$$

According to the signs of $\lambda_1, \lambda_2, \lambda_3$, we have six kinds of Lie groups as follows([11]).

signs of $\lambda_1, \lambda_2, \lambda_3$	associated Lie groups
+, +, +	SU(2) or SO(3)
+, +, −	SL(2, R) or O(1,2)
+, +, 0	E(2)
+, −, 0	E(1,1)
+, 0, 0	Heisenberg group
0, 0, 0	R $\oplus$ R $\oplus$ R

Then we have the following.

Proposition 3.1 ([15]). *Let G be a three-dimensional unimodular Lie group with left-invariant metric and let $\{e_i | i = 1, 2, 3\}$ be an orthonormal basis of the Lie algebra satisfying* (1). *Moreover, assume that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Then the left-invariant minimal unit vector fields of G are given as follows:*

G	*condition for λ_i*	*the set of left-invariant minimal unit vector fields*
SU(2)	$\lambda_1 = \lambda_2 = \lambda_3$	$\mathcal{S}$
	$\lambda_1 > \lambda_2 = \lambda_3$	$\pm e_1, \mathcal{S} \cap \{e_2, e_3\}_R$
	$\lambda_1 = \lambda_2 > \lambda_3$	$\pm e_3, \mathcal{S} \cap \{e_1, e_2\}_R$
	$\lambda_1 > \lambda_2 > \lambda_3$	$\pm e_1, \pm e_2, \pm e_3$
SL(2, R)	$\lambda_1 = \lambda_2$	$\pm e_3, \mathcal{S} \cap \{e_1, e_2\}_R$
	$\lambda_1 > \lambda_2$	$\pm e_1, \pm e_2, \pm e_3$
E(2)		$\pm e_3, \mathcal{S} \cap \{e_1, e_2\}_R$
E(1, 1)		$\pm e_2, \mathcal{S} \cap \{e_1, e_3\}_R$
Heisenberg group		$\mathcal{S}$
$R \oplus R \oplus R$		$\mathcal{S}$

where $\{e_i, e_j\}_R$ denotes the plane spanned by e_i and e_j.

Now consider the non-unimodular Lie groups. Let $\mathfrak{g}$ be a three-dimensional Lie algebra with inner product $<,>$. Let $\mathcal{U}$ be the unimodular kernel, e_1 be a unit vector orthogonal to $\mathcal{U}$ and $\{e_2, e_3\}$ be an orthonormal basis of $\mathcal{U}$ which diagonalizes the symmetric part of $ad_{e_1|\mathcal{U}}$. The bracket operation can be expressed as follows([11]).

$$
\begin{aligned}
[e_1, e_2] &= \alpha e_2 + \beta e_3, \\
[e_1, e_3] &= -\beta e_2 + \delta e_3, \\
[e_2, e_3] &= 0.
\end{aligned}
$$

Then we have the following.

Proposition 3.2 ([15]). *Let G be a three-dimensional non-unimodular Lie group with left-variant metric and let $\{e_i | i = 1, 2, 3\}$ be an orthonormal basis of the Lie group. Then the left-invariant minimal unit vector fields of G are given as follows:*

conditions for α and δ	*conditions for β*	*the sets of left-invariant minimal unit vector fields*
$\alpha = \delta$	$\beta = 0$	$\pm e_1, \mathcal{S} \cap \{e_2, e_3\}_R$
	$\beta \neq 0$	$\pm e_1$
$\alpha > \delta$	$\beta^2 + \alpha\delta = 0$	$\mathcal{A}, \mathcal{B}$
	$\beta^2 + \alpha\delta \neq 0$	$\pm e_1, \mathcal{B}$

where $\mathcal{A}$ is the set of unit vector fields of the plane determined by $\beta(1+\alpha^2)x_2 - \alpha(1+\delta^2)x_3 = 0$ and $\mathcal{B}$ is the set of unit vectors $x_2e_2 + x_3e_3$ where x_2 and x_3 satisfy $\beta(1+\alpha^2){x_2}^2 - (1+\beta^2)(\alpha-\delta)x_2x_3 + \beta(1+\delta^2){x_3}^2 = 0$.

3.2 Lie group of constant negative sectional curvature

For an integer $n > 1$, define a Lie group G_n as follows.

$$
G_n := \left\{ \begin{pmatrix} 1 & 0 \\ v & sI_{n-1} \end{pmatrix} \in GL(n, \mathbb{R}) \,|\, v \in \mathbb{R}^{n-1}, s > 0 \right\},
$$

where $v \in \mathbb{R}^{n-1}$ and I_{n-1} is the $(n-1) \times (n-1)$ identity matrix. Then the Lie algebra $\mathfrak{g}_n$ of G_n consists of the $n \times n$ matrices of the form

$$
\begin{pmatrix} 0 & 0 \\ v & sI_{n-1} \end{pmatrix}, v \in \mathbb{R}^{n-1}, s > 0.
$$

Let $\{e_1, e_2, \cdots, e_n\}$ be the usual orthonormal basis for $\mathbb{R}^{n-1}$. Put

$$
E_i = \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, \, i < n, \qquad E_n = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix}.
$$

Then we have $[E_i, E_j] = 0$, for $1 \leq i, j < n$, and $[E_n, E_k] = E_k$, for $1 \leq k \leq n$.

Let G_n be equipped with a left-invariant metric such that $\{E_1, E_2, \cdots, E_n\}$ is an orthonormal basis for $\mathfrak{g}_n$.

Let ∇ be the Levi-Civita connection of G_n. Then for $X, Y, Z \in \mathfrak{g}$ it satisfies the following identity([11]).

$$< \nabla_X Y, Z >= \frac{1}{2}\{< [X,Y], Z > - < [Y,Z], X > + < [Z,X], Y >\}. \tag{2}$$

For $1 \le i, j < n,\ 1 \le k \le n$, it is easy to show that

$$\nabla_{E_i} E_j = \delta_{ij} E_n, \quad \nabla_{E_i} E_n = -E_i, \quad \nabla_{E_n} E_k = 0. \tag{3}$$

Thus $K_{E_i \wedge E_j} =< R(E_i, E_j)E_j, E_i >= -1$ and the Lie group $(G_n, <,>)$ has constant negative sectional curvature -1. In fact it is simply connected and complete. Thus the Lie group $(G_n, <,>)$ is isometric to the hyperbolic space H^n([13]).

Theorem 3.3 ([16]). *For the Lie group G_2 every unit vector field is minimal. For the Lie group $G_n, n > 2$, the set of left-invariant minimal unit vector fields is $\{\pm E_n\} \cup (\mathcal{S} \cap E_n^{\perp})$.*

3.3 The semi-direct product

Let $\mathfrak{a}$ and $\mathfrak{r}$ be an abelian Lie algebra of dimension n and 1, respectively and let

$$P = (p_{ij}) \in \mathfrak{gl}(n, \mathbb{R})$$

be any real $(n \times n)$-matrix. A homomorphism $\varphi : \mathfrak{r} \to \mathrm{Endo}(\mathfrak{a})$ can be defined by

$$\varphi(\alpha)(x) = \alpha P x$$

for $\alpha \in \mathfrak{r}$ and $x \in \mathfrak{a}$.

One can form a semi-direct product of the Lie algebra $\mathfrak{a}$ by $\mathfrak{r}$ as follows: The underlying linear space is the direct sum $\mathfrak{a} \oplus \mathfrak{r}$, and the bracket operation is given by

$$[(a, \alpha), (b, \beta)] = (\varphi(\alpha)b - \varphi(\beta)a, [\alpha, \beta]) = (\varphi(\alpha)b - \varphi(\beta)a, 0).$$

We denote this new Lie algebra by $\mathfrak{a} \oplus_P \mathfrak{r}$.

Put $E_i = (0, \cdots, 1, \cdots, 0) \in \mathbb{R}^{n+1}$ and let $\{E_1, \cdots, E_{n+1}\}$ be orthonormal basis for $\mathfrak{g}$ and equip the left invariant metric on the associated Lie group with the Lie algebra $\mathfrak{g}$. Then for $1 \le i, j \le n$ we have the following.

$$[E_i, E_j] = 0, \quad [E_{n+1}, E_i] = \sum_{j=1}^{n} p_{ji} E_j, \quad [E_{n+1}, E_{n+1}] = 0.$$

Let α_{ijk} be defined by $[E_i, E_j] = \sum_{k=1}^{n+1} \alpha_{ijk} E_k$ then for $1 \le i, j, k \le n$ we have

$$\alpha_{ijk} = 0, \quad \alpha_{(n+1)jk} = -\alpha_{j(n+1)k} = p_{kj}, \quad \alpha_{(n+1)j(n+1)} = \alpha_{(n+1)(n+1)k} = 0.$$

By the equation

$$\nabla_{E_i} E_j = \sum_{k=1}^{n+1} \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) E_k,$$

for $1 \le i, j \le n$ we have the following:

$$\nabla_{E_i} E_j = \frac{1}{2}(p_{ij} + p_{ji}) E_{n+1}, \quad \nabla_{E_i} E_{n+1} = -\frac{1}{2} \sum_{k=1}^{n} (p_{ki} + p_{ik}) E_k$$

$$\nabla_{E_{n+1}} E_i = \frac{1}{2} \sum_{k=1}^{n} (p_{ki} - p_{ik}) E_k, \quad \nabla_{E_{n+1}} E_{n+1} = 0.$$

Thus we have the followings.

$$\begin{aligned} \nabla E_i \qquad &= \frac{1}{2} \sum_{j=1}^{n} (p_{ij} + p_{ji}) E_{n+1} \otimes \theta_j + \frac{1}{2} \sum_{j=1}^{n} (p_{ji} - p_{ij}) E_j \otimes \theta_{n+1} \\ \nabla E_{n+1} \qquad &= -\frac{1}{2} \sum_{j=1}^{n} \left(\sum_{i=1}^{n} (p_{ij} + p_{ji}) E_i \right) \otimes \theta_j \end{aligned}$$

Now we have the following.

Proposition 3.4 ([6]). *$V = E_{n+1}$ is a minimal unit vector field of $\mathfrak{a} \oplus_P \mathfrak{r}$.*

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