LEFT-IN Variant MINIMAL UNIT VECTOR FIELDS ON LIE GROUPS

SEUNGHUN YI

Abstract. We provide examples of left-invariant unit vector fields on Lie groups.

1. Introduction

A smooth unit vector field on a Riemannian manifold \((M,g)\) is a cross section of its unit sphere bundle \(T^1(M)\) and hence can be considered as a submanifold of \(T^1(M)\). Let \(M\) be a compact manifold and if \(T^1(M)\) is equipped with the Sasaki metric \(g_s\), then the volume of the unit vector field is defined as the volume of this submanifold.

Gluck and Ziller considered the problem of determining unit vector fields with minimal volume in [5] and they showed that the unit vector fields of minimal volume on \(S^3\) are precisely the Hopf vector fields and no others. But in the higher dimensional spheres, \(S^{2n+1}, k \geq 2\), this is not the case([4], [9], [12]).

Gil-Medriano and Linares-Fuster showed that a unit vector field is a critical point of the volume functional if and only if the corresponding immersion in \((T^1M, g_s)\) is minimal([3]). So such unit vector fields are called minimal even though the manifold is not compact.

The aim of this paper is to provide examples of left-invariant unit vector fields on the Lie groups.

2000 Mathematics Subject Classification. 53C20, 53C25, 53C42.
Key words and phrases. invariant minimal unit vector fields, Lie groups, semi-direct product.
In section 2, we give some basic notions and facts. In section 3, we provide examples of left-invariant unit vector fields on the Lie groups.

2. Minimal unit vector field

Let \((M, g)\) be a smooth Riemannian manifold, \(\nabla\) be the Levi-Civita connection on \((M, g)\) and \(R\) be the associated Riemannian curvature tensor with the sign convention \(R_{XY} = \nabla_[X,Y] - [\nabla_X, \nabla_Y]\).

We assume that the set \(\chi^1(M)\) of unit vector fields on \(M\) is non-empty. For \(V \in \chi^1(M)\) we define the positive definite symmetric tensor field \(L_V\) by

\[
L_V := I + (\nabla V)^* \nabla V
\]

where \(I\) is the identity map and \((\nabla V)^*\) is the adjoint. Put \(f(V) = (\det L_V)^{\frac{1}{2}}\).

Then the volume functional \(F : \chi^1(M) \to \mathbb{R}\) is defined by

\[
\int_M f(V) dv
\]

where \(dv\) is the volume form on \((M, g)\).

Define a \((1, 1)\)-tensor field \(K_V\) and a 1-form \(\omega_V\) associated to \(V\) as follows.

\[
K_V = f(V) \cdot L_V^{-1} \circ (\nabla V)^*,
\]

\[
\omega_V(X) = (\text{tr}(Z \mapsto \nabla_Z K_V)(X)).
\]

More precisely, for an orthonormal basis \(\{E_1, E_2, \cdots, E_n\}\) of the tangent space, \(\omega_V(X)\) is given by

\[
\omega_V(X) = \sum_{i=1}^n g((\nabla_{E_i} K_V)(X), E_i).
\]

In [3] it is proved that a unit vector field \(V\) is a critical point for the volume functional \(F\) if and only if the 1-form \(\omega_V\) annihilates the distribution \(H_V\) consisting of tangent vectors orthogonal to \(V\). Moreover it is shown that \(V\) is critical if and only if the map \(V : M \to (T^1M, g_s)\) is a minimal immersion, where \((T^1M, g_s)\) is the unit tangent bundle \(T^1M\) equipped with the Sasaki metric \(g_s\). Thus the following definition is natural.

**Definition 2.1.** A unit vector field \(V\) on a Riemannian manifold \((M, g)\) is called minimal if \(\omega_V(X) = 0\) for all \(X \in H^V\).

From now on we consider left-invariant unit vector fields on a Lie group. Let \(G\) be an \(n\)-dimensional connected Lie group equipped with a left-invariant metric and \(\mathfrak{g}\) be its Lie algebra. Then the left-invariant metric on \(G\) determines the associated inner
product $<,>$ on $\mathfrak{g}$. Furthermore let $\mathcal{S}$ be the unit sphere of $\mathfrak{g}$ with respect to $<,>$. For $V \in \mathcal{S}$, $\nabla V, L_V, K_V$ and $\omega_V$ are invariant by left translation, they can be viewed as tensors on $\mathfrak{g}$ which are determined by the Lie algebra structure and its inner product $<,>$ and $f$ is viewed as a function on $\mathcal{S}$ defined by $V \in \mathcal{S} \mapsto (\det L_V)^{\frac{1}{2}}$.

The distribution $\mathcal{H}^V$ is invariant by left translation and is identified with the orthogonal complement $V^\perp$ of $V$ in $\mathfrak{g}$ and thus $V^\perp$ may be naturally identified with the tangent space $T_V \mathcal{S}$ of the unit sphere $\mathcal{S}$ at $V$. Thus a left invariant unit vector field $V$ is minimal if and only if the 1-form $\omega_V$ on $\mathfrak{g}$ vanishes on $V^\perp \cong T_V \mathcal{S}$.

**Proposition 2.1** ([15]; Proposition 2.1). For $X \in T_V \mathcal{S}$ we have

$$\omega_V(X) = -df_V(X) - \text{tr} \text{ad}_{K_V} X$$

and $V$ is minimal if and only if

$$df_V(X) = -\text{tr} \text{ad}_{K_V} X$$

for all $X \in T_V \mathcal{S}$.

Thus on a unimodular Lie group $G$, i.e., $\text{tr} \text{ad}_X = 0$, for all $X \in \mathfrak{g}$, a left-invariant unit vector field $V$ is minimal if and only if $V$ is a critical point of the function $f$ on $\mathcal{S}$.

Let $G$ be a non-unimodular Lie group with a left-invariant metric. We denote by $\mathcal{U}$ its unimodular kernel, i.e.,

$$\mathcal{U} = \{X \in \mathfrak{g} | \text{tr} \text{ad}_X = 0\}.$$ 

Then $\mathcal{U}$ is an ideal of codimension 1 since $\text{tr} \text{ad}_X$ is a linear functional. We denote $H$ a unit vector orthogonal to $\mathcal{U}$. Then the linear transformation $\text{ad}_H$ restricted to $\mathcal{U}$ is a derivation of $\mathcal{U}$. This yields the following.

**Proposition 2.2** ([15]; Proposition 2.5). Let $\mathcal{U}$ be the unimodular kernel of a non-unimodular Lie group such that $\text{ad}_H|_{\mathcal{U}}$ is a symmetric endomorphism of $\mathcal{U}$ with respect to $<,>$. Then a left-invariant unit vector field $V$ is minimal if and only if it is a critical point of the function $f$ on $\mathcal{S}$.

3. Examples of left-invariant unit vector fields on Lie groups

3.1 Three dimensional Lie groups
In the first consider a three-dimensional unimodular Lie algebra with inner product $<,>$. Let $e_1, e_2, e_3$ be an orthonormal basis such that

$[e_2, e_3] = \lambda_1 e_1, [e_3, e_1] = \lambda_2 e_2, [e_1, e_2] = \lambda_3 e_3.$

According to the signs of $\lambda_1, \lambda_2, \lambda_3$, we have six kinds of Lie groups as follows([11]).

<table>
<thead>
<tr>
<th>signs of $\lambda_1, \lambda_2, \lambda_3$</th>
<th>associated Lie groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>+, +, +</td>
<td>SU(2) or SO(3)</td>
</tr>
<tr>
<td>+, +, -</td>
<td>SL(2, R) or O(1,2)</td>
</tr>
<tr>
<td>+, +, 0</td>
<td>E(2)</td>
</tr>
<tr>
<td>+, −, 0</td>
<td>E(1,1)</td>
</tr>
<tr>
<td>+, 0, 0</td>
<td>Heisenberg group</td>
</tr>
<tr>
<td>0, 0, 0</td>
<td>$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$</td>
</tr>
</tbody>
</table>

Then we have the following.

**Proposition 3.1** ([15]). Let $G$ be a three-dimensional unimodular Lie group with left-invariant metric and let $\{e_i|i = 1,2,3\}$ be an orthonormal basis of the Lie algebra satisfying (1). Moreover, assume that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Then the left-invariant minimal unit vector fields of $G$ are given as follows:

<table>
<thead>
<tr>
<th>$G$</th>
<th>condition for $\lambda_i$</th>
<th>the set of left-invariant minimal unit vector fields</th>
</tr>
</thead>
<tbody>
<tr>
<td>SU(2)</td>
<td>$\lambda_1 = \lambda_2 = \lambda_3$</td>
<td>$S$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_1 &gt; \lambda_2 = \lambda_3$</td>
<td>$\pm e_1, S \cap {e_2, e_3}_R$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_1 = \lambda_2 &gt; \lambda_3$</td>
<td>$\pm e_3, S \cap {e_1, e_2}_R$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_1 &gt; \lambda_2 &gt; \lambda_3$</td>
<td>$\pm e_1, \pm e_2, \pm e_3$</td>
</tr>
<tr>
<td>SL(2, R)</td>
<td>$\lambda_1 = \lambda_2$</td>
<td>$\pm e_3, S \cap {e_1, e_2}_R$</td>
</tr>
<tr>
<td></td>
<td>$\lambda_1 &gt; \lambda_2$</td>
<td>$\pm e_1, \pm e_2, \pm e_3$</td>
</tr>
<tr>
<td>E(2)</td>
<td></td>
<td>$\pm e_3, S \cap {e_1, e_2}_R$</td>
</tr>
<tr>
<td>E(1, 1)</td>
<td></td>
<td>$\pm e_2, S \cap {e_1, e_3}_R$</td>
</tr>
<tr>
<td>Heisenberg group</td>
<td></td>
<td>$S$</td>
</tr>
<tr>
<td>$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$</td>
<td></td>
<td>$S$</td>
</tr>
</tbody>
</table>

where $\{e_i, e_j\}_R$ denotes the plane spanned by $e_i$ and $e_j$.

Now consider the non-unimodular Lie groups. Let $\mathfrak{g}$ be a three-dimensional Lie algebra with inner product $<,>$. Let $\mathcal{U}$ be the unimodular kernel, $e_1$ be a unit vector orthogonal to $\mathcal{U}$ and $\{e_2, e_3\}$ be an orthonormal basis of $\mathcal{U}$ which diagonalizes the symmetric part of $ad_{e_1}|\mathcal{U}$. The bracket operation can be expressed as follows([11]).
\[ [e_1, e_2] = \alpha e_2 + \beta e_3, \]
\[ [e_1, e_3] = -\beta e_2 + \delta e_3, \]
\[ [e_2, e_3] = 0. \]

Then we have the following.

**Proposition 3.2 ([15]).** Let \( G \) be a three-dimensional non-unimodular Lie group with left-variant metric and let \( \{e_i \mid i = 1, 2, 3\} \) be an orthonormal basis of the Lie group. Then the left-invariant minimal unit vector fields of \( G \) are given as follows:

<table>
<thead>
<tr>
<th>conditions for ( \alpha ) and ( \delta )</th>
<th>conditions for ( \beta )</th>
<th>the sets of left-invariant minimal unit vector fields</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = \delta )</td>
<td>( \beta = 0 )</td>
<td>( \pm e_1, S \cap {e_2, e_3}_R )</td>
</tr>
<tr>
<td>( \alpha &gt; \delta )</td>
<td>( \beta^2 + \alpha \delta = 0 )</td>
<td>( A, B )</td>
</tr>
<tr>
<td></td>
<td>( \beta^2 + \alpha \delta \neq 0 )</td>
<td>( \pm e_1, B )</td>
</tr>
</tbody>
</table>

where \( A \) is the set of unit vector fields of the plane determined by \( \beta(1 + \alpha^2)x_2 - \alpha(1 + \delta^2)x_3 = 0 \) and \( B \) is the set of unit vectors \( x_2 e_2 + x_3 e_3 \) where \( x_2 \) and \( x_3 \) satisfy \( \beta(1 + \alpha^2)x_2^2 - (1 + \beta^2)(\alpha - \delta)x_2x_3 + \beta(1 + \delta^2)x_3^2 = 0 \).

### 3.2 Lie group of constant negative sectional curvature

For an integer \( n > 1 \), define a Lie group \( G_n \) as follows.

\[
G_n := \left\{ \begin{pmatrix} 1 & 0 \\ v & sI_{n-1} \end{pmatrix} \in GL(n, \mathbb{R}) \mid v \in \mathbb{R}^{n-1}, s > 0 \right\},
\]

where \( v \in \mathbb{R}^{n-1} \) and \( I_{n-1} \) is the \((n - 1) \times (n - 1)\) identity matrix. Then the Lie algebra \( \mathfrak{g}_n \) of \( G_n \) consists of the \( n \times n \) matrices of the form

\[
\begin{pmatrix} 0 & 0 \\ v & sI_{n-1} \end{pmatrix}, v \in \mathbb{R}^{n-1}, s > 0.
\]

Let \( \{e_1, e_2, \cdots, e_n\} \) be the usual orthonormal basis for \( \mathbb{R}^{n-1} \). Put

\[
E_i = \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, \quad i < n, \quad E_n = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix}.
\]

Then we have \([E_i, E_j] = 0\), for \( 1 \leq i, j < n \), and \([E_n, E_k] = E_k\), for \( 1 \leq k \leq n \).
Let $G_n$ be equipped with a left-invariant metric such that \{\(E_1, E_2, \cdots, E_n\)\} is an orthonormal basis for \(\mathfrak{g}_n\).

Let $\nabla$ be the Levi-Civita connection of $G_n$. Then for $X,Y,Z \in \mathfrak{g}$ it satisfies the following identity([11]).

\[
(2) \quad <\nabla X Y, Z > = \frac{1}{2} \{ <[X,Y], Z > - <[Y,Z], X > + <[Z,X], Y > \}.
\]

For $1 \leq i, j < n$, $1 \leq k \leq n$, it is easy to show that

\[
(3) \quad \nabla E_i E_j = \delta_{ij} E_n, \quad \nabla E_i E_n = -E_i, \quad \nabla E_n E_k = 0.
\]

Thus $K_{E_i \wedge E_j} = < R(E_i, E_j)E_j, E_i > = -1$ and the Lie group $(G_n, <, >)$ has constant negative sectional curvature $-1$. In fact it is simply connected and complete. Thus the Lie group $(G_n, <, >)$ is isometric to the hyperbolic space $H^n([13])$.

**Theorem 3.3** ([16]). For the Lie group $G_2$ every unit vector field is minimal. For the Lie group $G_n$, $n > 2$, the set of left-invariant minimal unit vector fields is $\{ \pm E_n \} \cup (\mathcal{S} \cap E_+^{n})$.

### 3.3 The semi-direct product

Let $\mathfrak{a}$ and $\mathfrak{r}$ be an abelian Lie algebra of dimension $n$ and 1, respectively and let

\[
P = (p_{ij}) \in \mathfrak{gl}(n, \mathbb{R})
\]

be any real $(n \times n)$-matrix. A homomorphism $\varphi : \mathfrak{r} \to \text{Endo}(\mathfrak{a})$ can be defined by

\[
\varphi(\alpha)(x) = \alpha Px
\]

for $\alpha \in \mathfrak{r}$ and $x \in \mathfrak{a}$.

One can form a semi-direct product of the Lie algebra $\mathfrak{a}$ by $\mathfrak{r}$ as follows: The underlying linear space is the direct sum $\mathfrak{a} \oplus \mathfrak{r}$, and the bracket operation is given by

\[
[(a, \alpha), (b, \beta)] = (\varphi(\alpha)b - \varphi(\beta)a, [\alpha, \beta]) = (\varphi(\alpha)b - \varphi(\beta)a, 0).
\]

We denote this new Lie algebra by $\mathfrak{a} \oplus_{\varphi} \mathfrak{r}$.

Put $E_i = (0, \cdots, 1, \cdots, 0) \in \mathbb{R}^{n+1}$ and let $\{E_1, \cdots, E_{n+1}\}$ be orthonormal basis for $\mathfrak{g}$ and equip the left invariant metric on the associated Lie group with the Lie algebra $\mathfrak{g}$. Then for $1 \leq i, j \leq n$ we have the following.

\[
[E_i, E_j] = 0, \quad [E_{n+1}, E_i] = \sum_{j=1}^{n} p_{ij} E_j, \quad [E_{n+1}, E_{n+1}] = 0.
\]
Let $\alpha_{ijk}$ be defined by $[E_i, E_j] = \sum_{k=1}^{n+1} \alpha_{ijk} E_k$ then for $1 \leq i, j, k \leq n$ we have $\alpha_{ijk} = 0$, $\alpha_{(n+1)jk} = -\alpha_{j(n+1)k}$, $\alpha_{(n+1)j(n+1)} = \alpha_{(n+1)(n+1)k} = 0$.

By the equation

$$\nabla_{E_i} E_j = \sum_{k=1}^{n+1} \frac{1}{2} (\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) E_k,$$

for $1 \leq i, j \leq n$ we have the following:

$$\nabla_{E_i} E_j = \frac{1}{2} (p_{ij} + p_{ji}) E_{n+1}, \quad \nabla_{E_i} E_{n+1} = -\frac{1}{2} \sum_{k=1}^{n} (p_{ki} + p_{ik}) E_k,$$

$$\nabla_{E_{n+1}} E_i = \frac{1}{2} \sum_{k=1}^{n} (p_{ki} - p_{ik}) E_k, \quad \nabla_{E_{n+1}} E_{n+1} = 0.$$

Thus we have the followings.

$$\nabla E_i = \frac{1}{2} \sum_{j=1}^{n} (p_{ij} + p_{ji}) E_{n+1} \otimes \theta_j + \frac{1}{2} \sum_{j=1}^{n} (p_{ji} - p_{ij}) E_j \otimes \theta_{n+1},$$

$$\nabla E_{n+1} = -\frac{1}{2} \sum_{j=1}^{n} \left( \sum_{i=1}^{n} (p_{ij} + p_{ji}) E_{i} \right) \otimes \theta_j.$$

Now we have the following.

**Proposition 3.4 ([6])**. $V = E_{n+1}$ is a minimal unit vector field of $a \oplus_p v$.

**References**


Scientific and Liberal Arts (Mathematics), Youngdong University, Youngdong, Chungbuk, 370-701, Korea

E-mail address: seunghun@youngdong.ac.kr