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CONTROLLABILITY FOR PARTIAL NEUTRAL FUNCTIONAL STOCHASTIC INTEGRODIFFERENTIAL INCLUSIONS WITH UNBOUNDED DELAY

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ABSTRACT. In this paper, the sufficient conditions are established for the controllability of partial neutral functional stochastic integrodifferential inclusions with unbounded delay in abstract phase space. The result is obtained using analytic semigroup, fractional power operators and fixed point theorem of Leray-Schauder alternative. An example is illustrated for the obtained result.

1. INTRODUCTION

Random neutral differential and integral inclusions play an important role in characterizing many social, physical, biological and engineering problems. The theory for differential and integral inclusions in deterministic cases may be found in several papers and monographs (see for example [3], [9], [16], [21]). In particular Benchohra and Ntouyas [7] and Liu [19] respectively studied existence and controllability results for neutral functional differential inclusions. The existence results of differential inclusions have been generalized to stochastic differential inclusions (see [1], [17]) and for functional differential inclusions (see [4], [6]) by using the fixed point argument.

Semigroup theory gives a unified treatment of a wide class of stochastic parabolic, hyperbolic and functional differential equations, and much effort has been devoted to the study of controllability results for such evolution equations. Stochastic control theory is a stochastic generalization of classical control theory. Controllability of linear stochastic systems is a well-known problem discussed in the literature ([2], [10], [20], [23]). Controllability of a linear stochastic system in Hilbert space recently has been extended to semilinear stochastic delay evolution equations by Balasubramaniam and Dauer [5] using Carathéodory successive approximations.

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In this paper, we are interested in the controllability of the following nonlinear neutral stochastic functional differential inclusions in a Hilbert space,

$$(1)\quad \begin{aligned} d[x(t)-f(t,x_t)] &\in [Ax(t)+Bu(t)]dt+\textstyle\int_0^t G(s,x_s)dw(s), \quad t\in J:=[0,b],\\ x(t) &= \phi(t)\in L_2(\Omega,\mathfrak{B}), \quad \text{for a .e} \quad t\in J_0:=(-\infty,0], \end{aligned}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $T(t), t\geq 0$, on a separable Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let K be another separable Hilbert space with inner product $(\cdot,\cdot)_K$ and norm $\|\cdot\|_K$. Suppose $\{w(t)\}_{t\geq 0}$ is a given $K-$valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q\geq 0$. We are also employing the same notation $\|\cdot\|$ for the norm $L(K,H)$, where $L(K,H)$ denotes the space of all bounded linear operators from K into H. For the infinite delays (see [14]), the histories x_t belongs to some abstract phase space $\mathfrak{B}$ defined axiomatically (see Section 2); $f: J\times\mathfrak{B}\to H$ and $G: J\times\mathfrak{B}\to\mathcal{P}(L_Q(K,H))$ ($\mathcal{P}(L_Q(K,H))$ is the family of all nonempty subsets of $L_Q(K,H)$) are the measurable mappings and multivalued measurable mapping such that $f(t,0)$ and $g(t,0)\in G(t,x_t)$ are locally bounded $H-$norm and $L_Q(K,H)-$norm, respectively. Here $L_Q(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators from K into H. The control function $u(\cdot)$ taking values in $L_2(J,U)$ of admissible control functions for a separable Hilbert space U and B is a bounded linear operator from U into H.

The paper is organized as follows. In Section 2, we recall some necessary preliminaries. In section 3, we prove our main result on controllability. Finally in Section 4, an example is presented which illustrates the main theorem.

2. Preliminaries

Let $(\Omega,\mathfrak{F},P)$ be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\mathfrak{F}_t, t\in J\}$ satisfying $\mathfrak{F}_t\subset\mathfrak{F}$.

An *H-valued random variable* is an $\mathfrak{F}$-measurable function $x(t):\Omega\to H$. Usually we suppress the dependence on $w\in\Omega$ in the *stochastic process* $S=\{x(t,w):\Omega\to H|t\in J\}$ and write $x(t)$ instead of $x(t,w)$ and $x(t): J\to H$ in the place of S. We suppose that $0\in\rho(A)$ and that the semigroup $T(\cdot)$ is uniformly bounded, that is to say, $\|T(t)\|\leq\bar{M}_1$, for some constant $\bar{M}_1\geq 1$ and every $t\geq 0$. For $0<\alpha\leq 1$, it is possible to define the fractional power operator $(-A)^\alpha$, as a closed linear operator on its domain $D\big((-A)^\alpha\big)$. Furthermore, the subspace $D\big((-A)^\alpha\big)$ is dense in H and the expression $\|x\|_\alpha=\|(-A)^\alpha x\|, \quad x\in D\big((-A)^\alpha\big)$, defines a norm on $D\big((-A)^\alpha\big)$. Hereafter we represent by H_α the space $D\big((-A)^\alpha\big)$ endowed with the norm $\|\cdot\|_\alpha$. Then the following properties are well known ([22]).

Lemma 2.1. *Suppose that the preceding conditions are satisfied.*

(a) *Let $0<\alpha\leq 1$. Then H_α is a Banach space.*
(b) *If $0<\beta\leq\alpha$ then $H_\alpha\hookrightarrow H_\beta$, the imbedding is continuous.*

(c) *For every* $0 < \alpha \leq 1$, *there exists a positive constant* M_α *such that*

$$\|(-A)^\alpha T(t)\| \leq \frac{M_\alpha}{t^\alpha}, \quad 0 < t \leq b.$$

In this work we will employ an axiomatic definition of the phase space $\mathfrak{B}$ introduced by Hale and Kato [13]. The axioms of the space $\mathfrak{B}$ are established for $\mathcal{F}_0$-measurable functions from J_0 into H, endowed with a seminorm $\|\cdot\|_{\mathfrak{B}}$. We will assume that $\mathfrak{B}$ satisfies the following axioms:

(ai) If $x : (-\infty, a) \to H,\ a > 0$, is continuous on $[0, a)$ and x_0 in $\mathfrak{B}$, then for every $t \in [0, a)$ the following conditions hold:
1. x_t is in $\mathfrak{B}$,
2. $\|x(t)\| \leq L\|x_t\|_{\mathfrak{B}}$,
3. $\|x_t\|_{\mathfrak{B}} \leq K(t)\sup\{\|x(s)\| : 0 \leq s \leq t\} + N(t)\|x_0\|_{\mathfrak{B}}$, where $L > 0$ is a constant; $K, N : [0, \infty) \to [0, \infty)$, K is continuous, N is locally bounded and L, K, N are independent of $x(\cdot)$.

(aii) For the function $x(\cdot)$ in (ai), x_t is a $\mathfrak{B}$-valued function $[0, a)$.

(aiii) The space $\mathfrak{B}$ is complete.

The $\mathfrak{B}$-valued stochastic process $x_t : \Omega \to \mathfrak{B},\ t \geq 0$ be defined by setting $x_t = \{x(t+s)(w) : s \in (-\infty, 0]\}$. The collection of all strongly-measurable, square-integrable H-valued random variables, denoted by $L_2(\Omega, \mathfrak{F}, P; H) \equiv L_2(\Omega; H)$, is a Banach space equipped with norm $\|x(\cdot)\|_{L_2} = \big(E\|x(\cdot; w)\|_H^2\big)^{\frac{1}{2}}$, where the expectation, E is defined by $E(h) = \int_\Omega h(w)dP$. Let $J_1 = (-\infty, b]$ and $C(J_1, L_2(\Omega; H))$ be the Banach space of all continuous maps from J_1 into $L_2(\Omega; H)$ satisfying the condition $\sup_{t\in J_1} E\|x(t)\|^2 < \infty$.

Let Z be the closed subspace of all continuous process x that belong to the space $C\big(J_1, L_2^{\mathfrak{F}}(\Omega; \mathfrak{B})\big)$ consisting of measurable and $\mathfrak{F}_t$-adapted processes such that $\phi \in \mathfrak{B}$ and the restriction $x : J \to L_2^{\mathfrak{F}}(\Omega; \mathfrak{B})$ is continuous. Let $\|\cdot\|_Z$ be a seminorm in Z defined by

$$\|x\|_Z = \big(\sup_{t\in J} \|x_t\|_{\mathfrak{B}}^2\big)^{\frac{1}{2}}$$

where

$$\|x_t\|_{\mathfrak{B}} \leq \bar{N} E\|\phi\|_{\mathfrak{B}} + \bar{K}\sup\{E\|x(s)\| : 0 \leq s \leq b\},$$

$\bar{N} = \sup_{t\in J}\{N(t)\}, \bar{K} = \sup_{t\in J}\{K(t)\}$. It is easy to verify that Z furnished with the norm topology as defined above, is a Banach space.

In a Hilbert space H, a multivalued map $M : H \to \mathcal{P}(H)$ is convex (closed) valued, if $M(x)$ is convex (closed) for all $x \in H$. M is bounded on bounded sets if $M(V) = \bigcup_{x\in V} M(x)$ is bounded in H, for any bounded set V of H (i.e., $\sup_{x\in V}\big\{\sup\{\|y\| : y \in M(x)\}\big\} < \infty$).

M is called upper semicontinuous (u.s.c.) on H, if for each $x_* \in H$, the set $M(x_*)$ is a nonempty, closed subset of H, and if for each open set V of H containing $M(x_*)$, there exists an open neighborhood N of x_* such that $M(N) \subseteq V$.

M is said to be completely continuous if $M(V)$ is relatively compact, for every bounded subset $V \subseteq H$.

If the multivalued map M is completely continuous with nonempty compact values, then M is u.s.c. if and only if M has a closed graph (i.e., $x_n \to x_*, y_n \to y_*, y_n \in Mx_n$ imply $y_* \in Mx_*$).

M has a fixed point if there is $x \in H$ such that $x \in Mx$.

In the following by $\mathcal{P}_{b,cl,cv}(H)$ denotes the set of all nonempty bounded, closed and convex subsets of H.

A multivalued map $M : J \to \mathcal{P}_{b,cl,cv}(H)$ is said to be measurable if for each $x \in H$ the mean-square distance between x and $M(t)$ is measurable function on J. For more details on multivalued maps see ([9], [15]).

For each $x \in L_2(L_Q(K,H))$ define the set of selections of G by

$$g \in N_{G,x} = \{g \in L_2(L_Q(K,H)) : g(t) \in G(t,x_t) \text{ for a.e. } \ t \in J\}.$$

The consideration of this paper is based in the following alternative ([11]).

Theorem 2.2. *(Nonlinear alternative for Kakutani maps). Let Y be a Hilbert space, C a closed convex subset of Y, $\mho$ an open subset of C and $0 \in \mho$. Suppose that $F : \overline{\mho} \to \mathcal{P}_{c,cv}(C)$ is a upper semicontinuous compact map; here $\mathcal{P}_{c,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either (i) F has a fixed point in $\overline{\mho}$, or (ii) there is a $v \in \partial\mho$ and $\lambda \in (0,1)$ with $v \in \lambda F(v)$.*

Definition 2.3. *The multivalued map $F : J \times \mathfrak{B} \to \mathcal{P}(H)$ is said to be L_2-Carathéodory if:*

(i) $t \longmapsto F(t,v)$ *is measurable for each* $v \in \mathfrak{B}$;
(ii) $v \longmapsto F(t,v)$ *is upper semicontinuous for almost all* $t \in J$;
(iii) *For each $q > 0$, there exists $h_q \in L_1(J,\mathbb{R}_+)$ such that*

$$\|F(t,v)\|^2 := \sup\{E\|g\|^2 : g \in F(t,v)\} \le h_q(t) \quad \text{for all } \|v\|^2_{\mathfrak{B}} \le q \quad \text{and for a.e. } \ t \in J.$$

The following lemma is crucial in the proof of our main result.

Lemma 2.4. [18] *Let I be a compact interval and Y be a Hilbert space. Let G be an L_2-Carathéodory multivalued map with $N_{G,x} \neq \emptyset$ and let Γ be a linear continuous mapping from $L_2(I,Y)$ to $C(I,Y)$. Then the operator*

$$\Gamma \circ N_G : C(I,Y) \to \mathcal{P}_{b,cl,cv}(C(I,Y)), \quad x \to (\Gamma \circ N_G)(x) = \Gamma(N_{G,x})$$

is a closed graph operator in $C(I,Y) \times C(I,Y)$.

3. Main Result

Before stating and proving our main result, we give first the definition of the mild solution.

Definition 3.1. *An $\mathfrak{F}_t$-adapted stochastic process $x(t) : J_1 \to H$ is a mild solution of the abstract Cauchy problem (1) if $x_0 = \phi \in \mathfrak{B}$ on J_0 satisfying $\|\phi\|^2_{\mathfrak{B}} < \infty$; the restriction of $x(\cdot)$ to the interval $[0,b)$ is continuous stochastic processes, for each*

$s \in [0,t)$ the function $AT(t-s)f(s,x_s)$ is integrable and $g \in N_{G,x}$ is a selection of $G\big(t,x_t\big)$ such that

(2)

$$\begin{aligned} x(t) &= T(t)[\phi(0) - f(0,\phi)] + f(t,x_t) + \textstyle\int_0^t AT(t-s)f(s,x_s)ds \\ &+ \textstyle\int_0^t T(t-s)Bu(s)ds + \int_0^t T(t-s)\big(\int_0^s g(\tau)dw(\tau)\big)ds, \text{ for a. e. } t \in J. \end{aligned}$$

Definition 3.2. *The nonlinear partial neutral stochastic differential inclusion (1) is said to be controllable on the interval J_1, if for every continuous initial stochastic process $\phi \in \mathfrak{B}$ defined on J_0, there exists a stochastic control $u \in L_2(J,U)$ which is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that the solution $x(\cdot)$ of (1) satisfies $x(b) = x_1$, where x_1 and b are preassigned terminal state and time respectively.*

Theorem 3.3. *Assume that:*

(H1) *the semigroup $T(t)$ is compact for $t > 0$, and there exists $M_1 \geq 1$ such that*

$$\|T(t)\|^2 \leq M_1, \quad \textit{for all } t \geq 0;$$

(H2) *the linear operator W from $L_2^{\mathfrak{F}}(J,U)$ into $L_2(\Omega;H)$, defined by*

$$Wu = \int_0^b T(b-s)Bu(s)ds$$

has an induced inverse operator W^{-1} which takes values in $L_2^{\mathfrak{F}}(J,U)\backslash KerW$ (see [8]) and there exist positive constants M_3, M_4 such that

$$\|B\|^2 \leq M_3 \ \textit{and} \ \|W^{-1}\|^2 \leq M_4.$$

(H3) *(i) the function f is x_β-valued, $(-A)^\beta f : J \times \mathfrak{B} \to H$ is completely continuous and such that the operator $f_1 : \mathfrak{B} \to \mathfrak{B}$ defined by $(f_1\phi)(t) = f(t,\phi)$ is compact;*
(ii) there exist constants $0 < \beta < 1, c_1, c_2$ such that

$$\|(-A)^\beta f(t,v)\|^2 \leq c_1||v||_{\mathfrak{B}}^2 + c_2, \ \textit{for every } (t,v) \in J \times \mathfrak{B};$$

(H4) *$G : J \times \mathfrak{B} \to \mathcal{P}(L_Q(K,H))$ is a L_2-Carathéodory function;*

(H5) *there exist a continuous nondecreasing function $\psi : \mathbb{R}_+ \to (0,\infty)$, $P \in L_1(J,\mathbb{R}_+)$ such that*

$$\|G(t,v)\|_Q^2 = \sup\{\|g\|_Q^2 : g \in G(t,v)\} \leq P(t)\psi(\|v\|_{\mathfrak{B}}^2)$$

for almost all $t \in J$ and $v \in \mathfrak{B}$, and there exists a constant M_* with

$$\frac{\Big[1 - (N_2+N_4)\frac{b^{2\beta-1}}{2\beta-1}\Big]M_*}{N_1 + (N_3+N_5)\psi(M_*)\int_0^b P(s)ds} > 1,$$

where

$$\begin{aligned}
\overline{M} &= 9M_1\|\phi\|^2_{\mathfrak{B}}\{1+c_1M_0\}+9c_2M_0\{M_1+1\}+\frac{9M^2_{1-\beta}c_2b^{2\beta}}{2\beta-1}+81M_1M_3M_4b^2\\
&\quad\times\Big\{||x_1||^2+M_1||\phi||^2_{\mathfrak{B}}+M_1M_0[c_1||\phi||^2_{\mathfrak{B}}+c_2]+M_0c_2+\frac{M^2_{1-\beta}c_2b^{2\beta}}{2\beta-1}\Big\},\\
M_0 &= \|(-A)^{-\beta}\|^2,\\
N_0 &= 1-18\bar{K}c_1M_0(1+9M_1M_3M_4b^2)>0,\\
N_1 &= \frac{2(\bar{N}\|\phi\|^2_{\mathfrak{B}}+\bar{K}\bar{M})}{N_0},\\
N_2 &= \frac{162\bar{K}M_1M_3M_4b^3M^2_{1-\beta}c_1}{N_0},\\
N_3 &= \frac{162\bar{K}M_1^2TrQM_3M_4b^3}{N_0},\\
N_4 &= \frac{18\bar{K}bM^2_{1-\beta}c_1}{N_0},\\
N_5 &= \frac{18\bar{K}M_1TrQ.b}{N_0}
\end{aligned}$$

with $(N_2+N_4)\frac{b^{2\beta-1}}{2\beta-1}<1.$

Then the system (1) is controllable on J_1*.*

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