Trends in Mathematics - New Series
Information Center for Mathematical Sciences
Volume 11, Number 1, 2009, pages 53–61

DISCRETENESS PROPERTIES OF TRANSLATION NUMBERS IN GARSIDE GROUPS: EXTENDED ABSTRACT

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Abstract. The translation number of an element in a combinatorial group is defined as the asymptotic word length of the element. The discreteness properties of translation numbers have been studied for geometric groups such as biautomatic groups and hyperbolic groups.

The Garside group is a lattice-theoretic generalization of braid groups and Artin groups of finite type. In this extended abstract, we show that the discreteness properties of translation numbers in Garside groups are as good as in hyperbolic groups: (i) translation numbers of elements in a Garside group are rational with uniformly bounded denominators; (ii) for every element g of a Garside group, some power g^m is conjugate to a periodically geodesic element.

1. Introduction

This note is a survey of our recent papers [14, 15, 16] on the translation numbers in Garside groups. It is based on the second author's talks at the conferences "The Many Strands of the Braid Groups" (Banff International Research Station, Banff, Canada, April 22–27, 2007) and "The 2008 spring meeting of the Korean Mathematical Society" (Keimyung University, Daegu, Korea, April 26, 2008). We focus on the motivation, main results and important ideas, hence it is not self-contained.

Date: March 3, 2009.

2000 *Mathematics Subject Classification.* 20F36, 20F10.

Key words and phrases. Garside group; braid group; translation number; periodically geodesic.

1.1. **Translation numbers.** The notion of translation numbers was first introduced by Gersten and Short [11]. This concept comes from the action of the fundamental group of a compact Riemannian manifold of non-positive curvature on the universal cover of this manifold.

Definition 1.1. For a finitely generated group G and a finite set X of semigroup generators for G, the *translation number* with respect to X of a non-torsion element $g \in G$ is defined by

$$t_{G,X}(g) = \lim_{n\to\infty} \frac{|g^n|_X}{n},$$

where $|\cdot|_X$ denotes the shortest word length in the alphabet X. If there is no confusion about the group G, we simply write $t_X(g)$ instead of $t_{G,X}(g)$. When A is a set of group generators, $|g|_A$ and $t_A(g)$ indicate $|g|_{A\cup A^{-1}}$ and $t_{A\cup A^{-1}}(g)$, respectively.

Definition 1.2. An element g is said to be *periodically geodesic* with respect to X if $|g^n|_X = |n| \cdot |g|_X$ for all integers n.

The notion of translation numbers is useful since it has both algebraic and geometric aspects. Kapovich [13] and Conner [6] suggested the following notions.

Definition 1.3. A finitely generated group G is said to be

(i) *translation separable* (or *translation proper*) if for some (and hence for any) finite set X of semigroup generators for G the translation numbers of non-torsion elements are strictly positive;
(ii) *translation discrete* if it is translation separable and for some (and hence for any) finite set X of semigroup generators for G the set $t_X(G)$ has 0 as an isolated point;
(iii) *strongly translation discrete* if it is translation separable and for some (and hence for any) finite set X of semigroup generators for G and for any real number r the number of conjugacy classes $[g] = \{h^{-1}gh : h \in G\}$ with $t_X(g) \leqslant r$ is finite. (The translation number is constant on each conjugacy class [11].)

Groups have nice properties if they have some level of translation discreteness. Gersten and Short [11] proved that every finitely generated nilpotent subgroup of a translation separable group is virtually abelian. Conner [6] proved that a translation separable solvable group of finite virtual cohomological dimension is metabelian by finite and that every solvable subgroup of finite virtual cohomological dimension in a translation discrete group is a finite extension of $\mathbb{Z}^m$. It is observed by Kapovich [13] that translation discrete groups cannot contain subgroups isomorphic to $\mathbb{Q}$ or $\mathbb{Q}_p = \{k/p^\ell \mid k \in \mathbb{Z},\ \ell \in \mathbb{N}\}$.

Therefore, it would be interesting to study translation discreteness of groups. Biautomatic groups are translation separable by Gersten and Short [11]. Word hyperbolic groups are strongly translation discrete. Moreover, the translation numbers in a word hyperbolic group are rational and have bounded denominators. It follows from a claim of Gromov [12, 5.2C] and was accurately proved by Swenson [20, Corollary of Theorem 13]. In fact, Swenson proved that for every element g of a word hyperbolic group, there exists a positive integer m such that g^m is conjugate to a periodically geodesic element h. (And the smallest such integer m is uniformly bounded.) The rationality of the translation numbers in a word hyperbolic group is an immediate consequence of this result because the translation numbers are constant on the conjugacy class and $t_X(h^n) = |n| t_X(h)$ for all elements h and integers n. Kapovich [13] proved that C(4)-T(4)-P, C(3)-T(6)-P and C(6)-P small cancellation groups are strongly translation discrete. (These groups are biautomatic but not necessarily word hyperbolic by Gersten and Short [10].) Alibegovic [1] showed that $\mathrm{Out}(F_n)$ is translation discrete.

1.2. **Translation discreteness of Garside groups.** The class of Garside groups provides a lattice-theoretic generalization of the braid groups and the Artin groups of finite type. Garside groups have been studied actively for the last ten years. Bestvina showed that Artin groups of finite type are translation discrete, and his result was generalized to Garside groups by Charney, Meier and Whittlesey.

Theorem 1.4 (Bestvina [3]). *Artin groups of finite type are translation discrete.*

Theorem 1.5 (Charney, Meier and Whittlesey [5]). *Garside groups with tame Garside element are translation discrete.*

The tameness for a Garside element is a mild condition. It is conjectured that every Garside group has a tame Garside element.

Recently, the authors have proved that the discreteness properties of translation numbers in Garside groups are as good as in hyperbolic groups.

Theorem 1.6 (S.J. Lee [14]). *All Garside groups are strongly translation discrete.*

Theorem 1.7 (E.-K.Lee and S.J. Lee [15]). *Translation numbers of elements in a Garside group are rational with uniformly bounded denominators.*

Theorem 1.8 (E.-K.Lee and S.J. Lee [16]). *For every element g of a Garside group, some power g^m is conjugate to a periodically geodesic element.*

The above theorems imply the following for Garside groups G.

(i) Every solvable subgroups of G are finitely generated and virtually abelian.
(ii) G cannot contain subgroups isomorphic to $\mathbb{Q}$ or the group of p-adic numbers $\mathbb{Q}_p = \{k/p^l \mid k \in \mathbb{Z},\ l \in \mathbb{N}\}$.
(iii) The following problems are solvable: root problem (given $g \in G$ and $n \geqslant 1$, find $h \in G$ such that $h^n = g$); power problem (given $g, h \in G$, find $n \in \mathbb{Z}$ such that. $h^n = g$); proper power problem (given $g \in G$, find $h \in G$ and $n \geqslant 2$ such that $h^n = g$). See [15].

Furthermore, we can easily prove the Algebraic Flat Torus Theorem for Garside groups, which states that every abelian subgroup of a Garside group is quasi-isometric to $\mathbb{Z}^n$ [17]. The results and techniques developed in this approach are useful in the study of the conjugacy problem and the reducibility problem in braid groups. For example, see [18].

2. Ideas of our approach

In this section, we describe main ideas in proving Theorems 1.6, 1.7 and 1.8. First, we briefly review basic properties of Garside groups. See [7, 8] for details.

2.1. **Garside groups.** Let M be a monoid. Let *atoms* be the elements $a \in M \setminus \{1\}$ such that $a = bc$ implies either $b = 1$ or $c = 1$. Let $\|a\|$ be the supremum of the lengths of all expressions of a in terms of atoms. The monoid M is said to be *atomic* if it is generated by its atoms and $\|a\| < \infty$ for any $a \in M$. In an atomic monoid M, there are partial orders $\leqslant_L$ and $\leqslant_R$: $a \leqslant_L b$ if $ac = b$ for some $c \in M$; $a \leqslant_R b$ if $ca = b$ for some $c \in M$.

Definition 2.1. An atomic monoid M is called a *Garside monoid* if

(i) M is finitely generated;
(ii) M is left and right cancellative;
(iii) $(M, \leqslant_L)$ and $(M, \leqslant_R)$ are lattices;
(iv) there exists an element Δ, called a *Garside element*, satisfying the following:
  (a) for each $a \in M$, $a \leqslant_L \Delta$ if and only if $a \leqslant_R \Delta$;
  (b) the set $\{a \in M : a \leqslant_L \Delta\}$ generates M.

An element a of M is called a *simple element* if $a \leqslant_L \Delta$. Let $\mathcal{D}$ denote the set of all simple elements. Let $\wedge_L$ and $\vee_L$ denote the gcd and lcm with respect to $\leqslant_L$.

Garside monoids satisfy Ore's conditions, and thus embed in their groups of fractions. A *Garside group* is defined as the group of fractions of a Garside monoid. When M is a Garside monoid and G the group of fractions of M, we identify the elements of M and their images in G and call them *positive elements* of G. M is called the *positive monoid* of G, often denoted by G^+.

Let $\tau\colon G \to G$ be the inner automorphism of G defined by $\tau(g) = \Delta^{-1}g\Delta$. It is known that $\tau(G^+) = G^+$, that is, the positive monoid is invariant under the conjugation by Δ.

The partial orders $\leqslant_L$ and $\leqslant_R$, and thus the lattice structures in the positive monoid G^+ can be extended to the Garside group G as follows: $g \leqslant_L h$ (resp. $g \leqslant_R h$) for $g, h \in G$ if $gc = h$ (resp. $cg = h$) for some $c \in G^+$.

For $g \in G$, there are integers r and s with $r \leqslant s$ such that $\Delta^r \leqslant_L g \leqslant_L \Delta^s$. Hence, the following invariants are well-defined.

$$\inf(g) = \max\{r \in \mathbb{Z} : \Delta^r \leqslant_L g\} \quad \text{and} \quad \sup(g) = \min\{s \in \mathbb{Z} : g \leqslant_L \Delta^s\}.$$

It is known that for $g \in G$ there is a unique expression

$$g = \Delta^r s_1 \cdots s_k,$$

where $s_1, \ldots, s_k \in \mathcal{D} \setminus \{1, \Delta\}$ and $(s_i s_{i+1} \cdots s_k) \wedge_L \Delta = s_i$ for $i = 1, \ldots, k$. In this case, $\inf(g) = r$ and $\sup(g) = r + k$. Such an expression is called the *normal form* of g.

For $g \in G$, we denote its conjugacy class $\{h^{-1}gh : h \in G\}$ by $[g]$. Define

$$\inf{}_s(g) = \max\{\inf(h) : h \in [g]\} \quad \text{and} \quad \sup{}_s(g) = \min\{\sup(h) : h \in [g]\}.$$

The *super summit set* $[g]^S$ and the *stable super summit set* $[g]^{St}$ are subsets of the conjugacy class of g defined as follows:

$$\begin{aligned} [g]^S &= \{h \in [g] : \inf(h) = \inf{}_s(g) \ \text{ and } \ \sup(h) = \sup{}_s(g)\}; \\ [g]^{St} &= \{h \in [g]^S : h^k \in [g^k]^S \ \text{ for all positive integers } k\}. \end{aligned}$$

Both of these sets are finite and nonempty [9, 17].

In the rest of the paper, if it is not specified, G is assumed to be a Garside group, whose positive monoid is G^+, with Garside element Δ and the set $\mathcal{D}$ of simple elements.

2.2. **It is sufficient to study $t_{\inf}(g)$.** It is observed by Charney [4] that the word length of an element $g \in G$ is determined by $\inf(g)$ and $\sup(g)$ as follows.

$$|g|_{\mathcal{D}} = \begin{cases} \sup(g) & \text{if } \inf(g) \geqslant 0, \\ -\inf(g) & \text{if } \sup(g) \leqslant 0, \\ \sup(g) - \inf(g) & \text{otherwise.} \end{cases} \tag{1}$$

This suggests us to define the following analogue of the translation number.

Definition 2.2. Define $t_{\inf}(g)$ and $t_{\sup}(g)$ by

$$t_{\inf}(g) = \lim_{n\to\infty} \frac{\inf(g^n)}{n}; \qquad t_{\sup}(g) = \lim_{n\to\infty} \frac{\sup(g^n)}{n}.$$

Using Eq. (1) and some properties of Garside groups, we have

$$t_{\mathcal{D}}(g) = \begin{cases} t_{\sup}(g) & \text{if } \inf_s(g) \geqslant 0, \\ -t_{\inf}(g) & \text{if } \sup_s(g) \leqslant 0, \\ t_{\sup}(g) - t_{\inf}(g) & \text{otherwise.} \end{cases} \tag{2}$$

Because $\sup(g) = -\inf(g^{-1})$, we have $t_{\sup}(g) = -t_{\inf}(g^{-1})$. From the above formulae, we can conclude that it is sufficient to consider $t_{\inf}(\cdot)$ to study the translation discreteness of Garside groups. The following are results on $t_{\inf}(\cdot)$, from which Theorems 1.6, 1.7 and 1.8 follow easily. See [14], [15] and [16] for (i), (ii) and (iii), respectively.

Theorem 2.3. *Let G be a Garside group with Garside element Δ, and let $g \in G$.*

(i) $\inf_s(g) \leqslant \inf_s(g^n)/n < \inf_s(g) + 1$.
(ii) $t_{\inf}(g)$ *is rational of the form p/q with $1 \leqslant q \leqslant \|\Delta\|$.*
(iii) $\inf_s(g) \leqslant t_{\inf}(g) < \inf_s(g) + 1$, *hence* $\inf_s(g) = \lfloor t_{\inf}(g) \rfloor$.

2.3. **Theorem 2.3 implies Theorems 1.6, 1.7 and 1.8.** From Theorem 2.3 (i), we have $\inf_s(g) \leqslant t_{\inf}(g) \leqslant \inf_s(g) + 1$, hence $|g|_{\mathcal{D}} - 2 \leqslant t_{\mathcal{D}}(g) \leqslant |g|_{\mathcal{D}}$ for a super summit element g. This implies Theorem 1.6 that Garside groups are strongly translation discrete.

Note that $t_{\sup}(g) = -t_{\inf}(g^{-1})$ and that $t_{\mathcal{D}}(g)$ is either $t_{\sup}(g) - t_{\inf}(g)$ or $t_{\sup}(g)$ or $-t_{\inf}(g)$. Therefore, Theorem 2.3 (ii) implies Theorem 1.7 that $t_{\mathcal{D}}(g)$ is rational.

Now assume that Theorem 2.3 (iii) holds. If $t_{\inf}(g)$ is an integer, then $\inf_s(g^k) = \lfloor t_{\inf}(g^k) \rfloor = \lfloor k t_{\inf}(g) \rfloor = k t_{\inf}(g) = k \inf_s(g)$, hence $\inf_s(g^k) = k \inf_s(g)$ for all $k \geqslant 1$. If both $t_{\inf}(g)$ and $t_{\sup}(g)$ are integers, then any element of $[g]^{St}$ is periodically geodesic because for $h \in [g]^{St}$

$$\inf(h^k) = k \inf(h) \quad \text{and} \quad \sup(h^k) = k \sup(h) \quad \text{for all } k \geqslant 1,$$

hence $|h^k|_{\mathcal{D}} = k\,|h|_{\mathcal{D}}$. Let $t_{\inf}(g) = p_1/q_1$ and $t_{\sup}(g) = p_2/q_2$. Let $m = \operatorname{lcm}(q_1, q_2)$. Then any element of $[g^m]^{St}$ is periodically geodesic. This implies Theorem 1.8.

2.4. **Ingredients for the proof of Theorem 2.3.** The following three facts are the main ingredients for the proof of Theorem 2.3. (i) is proved in [14] and [19]; (ii) is a well-known fact; (iii) is well-known as Schur's theorem.

(i) If M_1 and M_2 are Garside monoids, then $M_1 \ltimes M_2$ is a Garside monoid.
(ii) If $x \in \mathbb{R}$ is irrational, then $\{nx - \lfloor nx \rfloor : n \geqslant 1\}$ is dense in $[0, 1]$.
(iii) For each integer $k \geqslant 1$, there exists a number $S(k)$, called *Schur's number*, such that for every partition $\{1, 2, \dots, S(k)\} = T_1 \cup \cdots \cup T_k$, some T_i contains two integers n and m together with $n + m$.

2.5. **Idea of proof of Theorem 2.3 (i).** We explain how to prove Theorem 2.3 (i) that $\inf_s(g) \leqslant \inf_s(g^n)/n < \inf_s(g) + 1$. Let $\mathbb{Z} = \langle \delta \rangle$ act on G^n by $(g_1, \dots, g_n)^\delta = (g_n, g_1, \dots, g_{n-1})$. Then $G(n) = \mathbb{Z} \ltimes G^n$ is a Garside group with a Garside element $(\delta, (\Delta, \dots, \Delta))$.

Let $g \sim_c h$ mean that g is conjugate to h. The following hold in the group $G(n)$.

(i) $\inf(\delta^k, (g_1, \dots, g_n)) = \min\{k, \inf(g_1), \dots, \inf(g_n)\}$.
(ii) $\inf_s(\delta^k, (g, \dots, g)) = \inf_s(g)$ if $k \geqslant \inf_s(g)$.
(iii) Let $k \equiv 1 \bmod n$. Then $(\delta^k, (g_1, \dots, g_n)) \sim_c (\delta^k, (h_1, \dots, h_n))$ in $G(n)$ if and only if $g_1 \cdots g_n \sim_c h_1 \cdots h_n$ in G.

It is well-known that $n \inf_s(g) \leqslant \inf_s(g^n)$. Let $\inf_s(g) = r$, and assume $\inf_s(g^n) \geqslant n(r + 1)$. Choose an integer $k \gg 1$ such that $k \equiv 1 \bmod n$. Note that

$$\inf_s(\delta^k, (g, \dots, g)) = \inf_s(g) = r.$$

On the other hand, we have

$$\inf_s(\delta^k, (g, \dots, g)) \geqslant r + 1,$$

because

$$\begin{aligned}(\delta^k, (g, \dots, g)) &\sim_c (\delta^k, (g^n, 1, \dots, 1))\\ &\sim_c (\delta^k, (a\Delta^{n(r+1)}, 1, \dots, 1), \text{ for some } a \in G^+\\ &\sim_c (\delta^k, (a\Delta^{r+1}, \Delta^{r+1}, \dots, \Delta^{r+1})),\end{aligned}$$

which is a contradiction. Therefore $\inf_s(g^n) < n(r + 1) = n(\inf_s(g) + 1)$.

2.6. **Idea of proof of Theorem 2.3 (ii).** We explain how to prove Theorem 2.3 (ii) that $t_{\inf}(g)$ is rational with denominators $\leqslant \|\Delta\|$.

Lemma 2.4. *The fractional part of* $t_{\inf}(g)$ *cannot be contained in* $(0, 1/\|\Delta\|)$.

Sketchy proof. It is observed by K.H. Ko that if $\inf(g^k) = k\inf(g)$ for $1 \leqslant k \leqslant \|\Delta\|$, then $\inf(g^k) = k\inf(g)$ for all $k \geqslant 1$. Using the stable super summit set, we can generalize Ko's result to the conjugacy invariant $\inf_s(\cdot)$: if $\inf_s(g^k) = k\inf_s(g)$ for $1 \leqslant k \leqslant \|\Delta\|$, then $\inf_s(g^k) = k\inf_s(g)$ for all $k \geqslant 1$. This implies that if $0 < t_{\inf}(g) - \lfloor t_{\inf}(g) \rfloor < \|\Delta\|^{-1}$, then $\inf_s(g^k) = k\inf_s(g)$ for $1 \leqslant k \leqslant \|\Delta\|$, hence $t_{\inf}(g) = \inf_s(g)$ is an integer. □

If $t_{\inf}(g)$ is irrational, then $\{nt_{\inf}(g) - \lfloor nt_{\inf}(g) \rfloor : n \geqslant 1\}$ is dense in $[0,1]$, hence fractional part of $nt_{\inf}(g) = t_{\inf}(g^n)$ belongs to $(0, 1/\|\Delta\|)$ for some n, a contradiction.

2.7. **Idea of proof of Theorem 2.3 (iii).** We explain how to prove Theorem 2.3 (iii) that $\inf_s(g) \leqslant t_{\inf}(g) < \inf_s(g) + 1$. Because "$\inf_s(g) \leqslant \frac{\inf_s(g^n)}{n} < \inf_s(g) + 1$" by Theorem 2.3 (i), we have "$\inf_s(g) \leqslant t_{\inf}(g) \leqslant \inf_s(g)+1$". Therefore, it is enough to show that $t_{\inf}(g) \neq \inf_s(g) + 1$. Suppose on the contrary that $t_{\inf}(g) = \inf_s(g) + 1$. Then it is not difficult to see that for all $n, m \geqslant 1$,

$$\inf_s(g^{n+m}) = \inf_s(g^n) + \inf_s(g^m) + 1. \tag{3}$$

We may assume that g is a stable super summit element. Let

$$g^n = s_n a_n \Delta^{r_n},$$

where $r_n = \inf(g^n)$ and $s_n = \Delta \wedge_L (s_n a_n)$. Let $\{s^{(i)} : i = 1, \dots, |\mathcal{D}|\}$ be the set of simple elements. Define $T_i = \{n \in \mathbb{N} \mid s_n = s^{(i)}\}$ for $i = 1, \dots, |\mathcal{D}|$. Then, $\mathbb{N} = T_1 \cup T_2 \cup \cdots \cup T_{|\mathcal{D}|}$. By Schur's theorem, there exists T_i that contains n, m and $n+m$ for some $n, m \geqslant 1$. Therefore, there exists n, m such that

$$s_n = s_m = s_{n+m}.$$

Note that $g^{n+m} = (s_n a_n \Delta^{r_n})(s_m a_m \Delta^{r_m})$ is conjugate to $(a_n \Delta^{r_n} s_m)(a_m \Delta^{r_m} s_n)$. Then we can show that $s_n = s_{n+m}$ implies $\inf(a_n \Delta^{r_n} s_m) \geqslant r_n + 1$, and that $s_m = s_{n+m}$ implies $\inf(a_m \Delta^{r_m} s_n) \geqslant r_m + 1$. Therefore,

$$r_{n+m} = \inf(g^{n+m}) \geqslant (r_n + 1) + (r_m + 1) = r_n + r_m + 2.$$

It is a contradiction because $r_{n+m} = \inf_s(g^{n+m}) = \inf_s(g^n) + \inf_s(g^m) + 1 = r_n + r_m + 1$.

References

[1] E. Alibegovic, *Translation lengths in* $\mathrm{Out}(F_n)$, Geom. Dedicata **92** (2002) 87–93.

[2] G. Baumslag, S. Gersten, M. Shapiro and H. Short, *Automatic groups and amalgams*, J. Pure Appl. Algebra **76** (1991) 229–316.

[3] M. Bestvina, *Non-positively curved aspects of Artin groups of finite type*, Geometry and Topology **3** (1999) 269–302.

[4] R. Charney, *Geodesic automation and growth functions for Artin groups of finite type*, Math. Ann. **301** (1995) 307–324.

[5] R. Charney, J. Meier and K. Whittlesey, *Bestvina's normal form complex and the homology of Garside groups*, Geom. Dedicata **105** (2004) 171–188.

[6] G.R. Conner, *Discreteness properties of translation numbers in solvable groups*, J. Group Theory **3** (2000) 77–94.

[7] P. Dehornoy and L. Paris, *Gaussian groups and Garside groups, two generalisations of Artin groups*, Proc. Lond. Math. Soc. **79** (1999) 569–604.

[8] P. Dehornoy, *Groupes de Garside*, Ann. Scient. Ec. Norm. Sup. **35** (2002) 267–306.

[9] E.A. Elrifai and H.R. Morton, *Algorithms for positive braids*, Quart. J. Math. Oxford Ser. **45** (1994) 479–497.

[10] S.M. Gersten and H.B. Short, *Small cancellation theory and automatic groups*, Invent. Math. **102** (1990) 305–334.

[11] S.M. Gersten and H.B. Short, *Rational subgroups of biautomatic groups*, Ann. Math. (2) **134** (1991) 125–158.

[12] M. Gromov, *Hyperbolic groups*, in Essays in group theory, 75–263, Springer, 1987.

[13] I. Kapovich, *Small cancellation groups and translation numbers*, Trans. Amer. Math. Soc. **349** (1997) 1851–1875.

[14] S. J. Lee, *Garside groups are strongly translation discrete*, J. Algebra **309** (2007) 594–609.

[15] E.-K. Lee and S.J. Lee, *Translation numbers in a Garside group are rational with uniformly bounded denominators*, J. Pure Appl. Algebra **211** (2007) 732–743.

[16] E.-K. Lee and S.-J. Lee, *Some power of an element in a Garside group is conjugate to a periodically geodesic element*, to appear in Bull. Lond. Math. Soc.

[17] E.-K. Lee and S. J. Lee, *Abelian subgroups of Garside groups*, Comm. Algebra **36** (2008) 1121–1139.

[18] E.-K. Lee and S.-J. Lee, *A Garside-theoretic approach to the reducibility problem in braid groups*, J. Algebra **320** (2008) 783–820.

[19] M. Picantin, *The center of thin Gaussian groups*, J. Algebra **245** (2001) 92–122.

[20] E. Swenson, *Hyperbolic Elements in Negatively Curved Groups*, Geom. Dedicata **55** (1995) 443–451.

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