

LECTURES ON COHOMOLOGY OF TORIC MANIFOLDS

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ABSTRACT. These are the notes for the lectures given at 2012 SNU/KIAS Topology Winter School.

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1. INTRODUCTION

In the first chapter, we briefly collect the basics on equivariant cohomology of torus actions on manifolds and explain the relation between equivariant cohomology and ordinary cohomology. It will be also mentioned how we should think about the (equivariant) cohomology of orbifolds with integer coefficients.

In the second chapter, we compute the equivariant cohomology and the ordinary cohomology of toric manifolds. The symplectic toric manifold for a Delzant polytope is constructed by symplectic reduction of \mathbb{C}^m by a subgroup of $U(1)^m$ that canonically acts on \mathbb{C}^m . The level set of the moment map at the regular value can be $U(1)^m$ -equivariantly identified with the moment angle manifold originally introduced by Davis-Januszkiewicz [13]. We compute the equivariant cohomology of toric manifolds as the Stanley-Reisner ring of the simplicial complex associated to the polytope, by using the disk-circle decomposition of the moment angle manifold introduced by Panov-Buchstaber [10]. To identify the ordinary cohomology of toric manifolds as the quotient of the Stanley-Reisner ring by the linear terms, we introduce the even-dimensional cell-decomposition following Davis-Januszkiewicz [13, p.431]. We also discuss what happens to the cohomology of toric orbifolds with integer coefficients in the perspective of the theorem that the equivariant cohomology of a moment angle complex is written as the Tor module of the Stanley-Reisner ring [31].

In the third chapter, the GKM theory for Hamiltonian torus actions on symplectic manifolds will be explained. As an application, we will identify, as an element of Stanley-Reisner ring, the equivariant cohomology classes of torus invariant submanifolds of the toric manifold. We also find the equivariant cohomology class of the symplectic structure (together with a moment map) in terms of the Stanley-Reisner ring, using the Cartan model of the equivariant cohomology.

In the fourth chapter, the localization theorem for a torus action on a compact manifold (orbifold) will be explained. We will redo the proof in [3] (we also follow [28]) for a torus action on an orbifold which is a global quotient by an abelian group and derive a special case of the localization theorem for a torus action on an orbifold. The integration formula for manifolds/orbifolds are also mentioned.

2. BASICS ON EQUIVARIANT COHOMOLOGY

In this chapter, we will collect some basic notions for equivariant cohomology. We will also discuss briefly about how we think about the (equivariant) cohomology of orbifolds. Then we will study the relation between the ordinary cohomology and the torus equivariant cohomology. Namely it will be shown that the vanishing of odd degrees in ordinary cohomology implies that the ordinary cohomology is the quotient of the equivariant cohomology by the ideal generated by the equivariant parameters.

2.1. Universal bundle and classifying space.

Definition 2.1 (c.f. Section 6.3 [34]). A “universal bundle” $EG \rightarrow BG$ for a topological group G is a principal G -bundle such that the total space EG is weakly contractible, i.e. $\pi_i(EG) = 0$ for all i . The base space BG is called the “classifying space”.

Theorem 2.2 (c.f. Cor 6.35 [34]). *The isomorphism classes of principal G bundles over a space X is in 1:1 correspondence with the homotopy equivalence classes of maps $X \rightarrow BG$.*

Theorem 2.3. *For any compact Lie group G , there is a contractible space EG with a free G -action and $EG \rightarrow BG := EG/G$ is a principal G -bundle ¹.*

Proof. Any compact Lie group is a closed subgroup of $GL_n(\mathbb{C})$ for sufficiently large n (see for example [14, Cor 4.6.5]). Thus, $EGL_n(\mathbb{C})$ can play the role of EG . The universal bundle $EGL_n(\mathbb{C})$ is the space of all basis of \mathbb{C}^∞ and the classifying space $BGL_n(\mathbb{C})$ is the infinite Grassmannian $Gr_n(\mathbb{C}^\infty)$ (c.f. [9, Section 1]).

Alternatively, by the fact that the maximal compact subgroup of a connected Lie group is unique up to conjugation (c.f. [24, Thm 2.2 (ii)]) and that the maximal compact subgroup of $GL_n(\mathbb{C})$ is $U(n)$, any compact Lie group is a closed subgroup of $U(n)$. Therefore the universal bundle $EU(n)$ works also for EG . The space $EU(n)$ is the space of all orthonormal basis of \mathbb{C}^∞ and $BU(n)$ is also $Gr_n(\mathbb{C}^\infty)$. It is not hard to prove that $EU(n)$ is weakly contractible. Since $EU(n)$ is naturally a CW complex, we can apply Whitehead theorem to conclude that $EU(n)$ is contractible. \square

Example 2.4. $EU(1) \cong S^\infty$ and $BU(1) \cong \mathbb{C}P^\infty$. $EC^\times \cong \mathbb{C}^\infty - \vec{0}$ and $BC^\times \cong \mathbb{C}P^\infty$. $E(\mathbb{Z}_2) \cong S^\infty$ and $B\mathbb{Z}_2 \cong \mathbb{R}P^\infty$.

Let $T \cong U(1)^m$ be an m -dimensional torus and let \mathfrak{t} be its Lie algebra. Let $\exp : \mathfrak{t} \rightarrow T$ be the exponential map and let $t_{\mathbb{Z}}$ be its kernel so that $T \cong \mathfrak{t}/t_{\mathbb{Z}}$. We have $ET \cong (S^\infty)^m$ and $BT \cong (\mathbb{C}P^\infty)^m$. The choices of \mathbb{Z} -basis of $t_{\mathbb{Z}}$ correspond bijectively to the choices of the isomorphisms $T \cong U(1)^m$.

Lemma 2.5. *There is a natural isomorphism $t_{\mathbb{Z}}^* \cong \text{Hom}(T, U(1))$ of groups.*

Proof. Let $x \in \text{Hom}(T, U(1))$, then it induces $x' : \mathfrak{t} \rightarrow \mathbb{R}$ such that there is a commutative diagram:

$$\begin{array}{ccc} T & \xrightarrow{x} & U(1) \\ \exp \uparrow & & \uparrow \exp \\ \mathfrak{t} & \xrightarrow{x'} & \mathbb{R} \end{array}$$

Thus the preimage of \mathbb{Z} by x' contains $t_{\mathbb{Z}}$ so that $x' \in t_{\mathbb{Z}}^*$. On the other hand, if $x' \in t_{\mathbb{Z}}^*$, then it induces $x : \mathfrak{t}/t_{\mathbb{Z}} \rightarrow \mathbb{R}/\mathbb{Z}$. \square

Example 2.6. The tautological line bundle on $\mathbb{C}P^\infty$ is defined by

$$L := \{([\vec{z}], a\vec{z}) \mid [\vec{z}] \in \mathbb{C}P^\infty, a \in \mathbb{C}\} \subset \mathbb{C}P^\infty \times \mathbb{C}^\infty.$$

It is the quotient of

$$\tilde{L} := \{(\vec{z}, a\vec{z}) \mid \vec{z} \in S^\infty, a \in \mathbb{C}\} \subset S^\infty \times \mathbb{C}^\infty.$$

¹A universal bundle actually exists for any topological group. See [36] or [34, p. 211].

by the action of $t \cdot (\vec{z}, a\vec{z}) \mapsto (t\vec{z}, a\vec{z})$. Then we have $H^*(\mathbb{C}\mathbb{P}^\infty; \mathbb{Z}) = \mathbb{Z}[c_1(L)]$ (c.f. [37, Thm 14.5]). Let $x \in \text{Lie}(U(1))_{\mathbb{Z}}^*$ be the canonical generator which corresponds to the identity map in $\text{Hom}(U(1), U(1))$ and consider the homomorphism

$$\text{Sym}(\text{Lie}(U(1))_{\mathbb{Z}}^*) = \mathbb{Z}[x] \rightarrow H^*(\mathbb{C}\mathbb{P}^\infty; \mathbb{Z}) = \mathbb{Z}[c_1(L)]$$

defined by, for $\lambda \in \text{Lie}(U(1))_{\mathbb{Z}}^*$,

$$\lambda \mapsto L_\lambda := S^\infty \times_{U(1)} \mathbb{C}_\lambda \mapsto c_1(L_\lambda).$$

It is an isomorphism since L_{-x} is isomorphic to the tautological line bundle L : we have a $U(1)$ -equivariant isomorphism

$$S^\infty \times \mathbb{C}_{-x} \rightarrow \tilde{L} \quad (\vec{z}, a) \mapsto (\vec{z}, a\vec{z}).$$

Example 2.7. Let $\{x_1, \dots, x_m\}$ be the basis of $t_{\mathbb{Z}}^* \cong \text{Hom}(\mathbb{T}, U(1))$. Let \mathbb{C}_λ is the 1-dimensional representation of \mathbb{T} through the action $\lambda : \mathbb{T} \rightarrow U(1)$. The homomorphism $\text{Sym } t_{\mathbb{Z}}^* = \mathbb{Z}[x_1, \dots, x_m] \rightarrow H^*(B\mathbb{T}; \mathbb{Z})$ given by $\lambda \mapsto L_\lambda := E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}_\lambda \mapsto c_1(L_\lambda)$, is an isomorphism.

2.2. Equivariant Cohomology.

Definition 2.8. Let M be a topological space with a continuous G -action. Define the G -equivariant cohomology by

$$H_G^*(M; \mathbb{Z}) := H^*(EG \times_G M; \mathbb{Z}) \quad \text{where} \quad EG \times_G M := \frac{EG \times M}{G} \quad \text{the homotopy quotient}$$

(Borel construction).

Properties of \mathbb{T} -equivariant cohomology. Let M be a topological space with \mathbb{T} -action.

- (1) The projection $E\mathbb{T} \times_{\mathbb{T}} M \rightarrow B\mathbb{T}$ induces the action of $\mathbb{Z}[x_1, \dots, x_m]$ on $H_{\mathbb{T}}^*(M; \mathbb{Z})$ via pullback. So $H_{\mathbb{T}}^*(M; \mathbb{Z})$ is a ring over $\mathbb{Z}[x_1, \dots, x_m]$. We saw that each linear term $\lambda \in H^*(B\mathbb{T}; \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_m]$ can be regarded as a character $\lambda : \mathbb{T} \rightarrow U(1)$. Also as a cohomology class, the character λ corresponds to the first Chern class of the adjoint line bundle $L_\lambda := E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}_\lambda$ where \mathbb{T} acts on \mathbb{C}_λ by λ . By the functoriality of Chern classes, $\lambda \cdot 1 \in H_{\mathbb{T}}^*(M; \mathbb{Z})$ is the 1st Chern class of the adjoint bundle

$$L_\lambda^M := \frac{E\mathbb{T} \times M \times \mathbb{C}_\lambda}{\mathbb{T}} \rightarrow E\mathbb{T} \times_{\mathbb{T}} M,$$

since L_λ^M is the pullback of L_λ along $E\mathbb{T} \times_{\mathbb{T}} M \rightarrow B\mathbb{T}$.

- (2) If \mathbb{T} acts on M freely, then the fibers of the projection $\pi : E\mathbb{T} \times_{\mathbb{T}} M \rightarrow M/\mathbb{T}$ is the contractible space $E\mathbb{T}$, therefore the pullback induces the isomorphism

$$H_{\mathbb{T}}^*(M; \mathbb{Z}) \cong H^*(M/\mathbb{T}; \mathbb{Z}).$$

If the action of \mathbb{T} on M is *locally free*, i.e. the stabilizer of each point in M is finite, then the fiber of π is a quotient of contractible space by a finite group, and this implies that the pullback induces an isomorphism over \mathbb{Q} -coefficients

$$H_{\mathbb{T}}^*(M; \mathbb{Q}) \cong H^*(M/\mathbb{T}; \mathbb{Q}).$$

Cohomology of Orbifolds and Equivariant Cohomology. Let M be a smooth manifold with a smooth locally free action of a compact Lie group G . Then there is the induced smooth orbifold structure on the quotient topological space M/G . For each $x \in M$, let G_x be the stabilizer subgroup x in G . By the assumption of locally freeness, G_x is a finite group. The normal space $T_x M / T_x(G \cdot x)$ of the G -orbit $G \cdot x$, together with the action of G_x , defines an orbifold chart around $[x] \in M/G$. We denote this orbifold by $[M/G]$. In the language of the *groupoid*, the orbifold $[M/G]$ is also presented by the Lie groupoid $M \rtimes G \rightrightarrows M$, called the action groupoid or translation groupoid. More abstractly, the orbifold $[M/G]$ can be defined as a quotient stack, which is a category of principal G -bundles over manifolds, $P \rightarrow U$, together with a G -equivariant map $P \rightarrow M$.

Definition 2.9. The cohomology of $[M/G]$ as a stack is defined by

$$H^*([M/G]; \mathbb{Z}) := H_G^*(M; \mathbb{Z}).$$

The well-definedness of this definition, please see, for example, [15]. By Section 2.2 (2), we have

$$H^*([M/G]; \mathbb{Q}) \cong H^*(M/G; \mathbb{Q}).$$

Let $0 \rightarrow G \rightarrow \tilde{G} \rightarrow K \rightarrow 0$ be a short exact sequence of Lie groups and that the action of G is extended to the action of \tilde{G} on M . Then obviously K acts on the topological space M/G , but not only that, the quotient group K acts on the orbifold $[M/G]$ (as a stack). For details, please refer [29] or [38]. Similarly to the definition of cohomology, we can adopt the following definition of K -equivariant cohomology of $[M/G]$:

Definition 2.10. The K -equivariant cohomology of $[M/G]$ as a stack can be defined by

$$H_K^*([M/G]; \mathbb{Z}) := H_{\tilde{G}}^*(M; \mathbb{Z}).$$

Remark 2.11. Consider the following projection maps:

$$EK \times_K M/G \xleftarrow{g} EK \times_K (E\tilde{G} \times_G M) \xrightarrow{f} E\tilde{G} \times_{\tilde{G}} M.$$

The fibers of f are contractible spaces, so f^* is an isomorphism. The fibers of g are quotients of contractible spaces by finite groups, so it induces an isomorphism g^* only if we work over \mathbb{Q} . Thus $H_K^*([M/G]; \mathbb{Q}) \cong H_{\tilde{G}}^*(M; \mathbb{Q})$.

2.3. Relations between equivariant cohomology and ordinary cohomology. The followings are the well-know theorems:

- **(Poincare Duality** c.f. Theorem 4.1 [32])

Let M be a closed (i.e. compact and no boundary), orientable, connected manifold of dimension n . Then

$$H^i(M; \mathbb{Z}) \cong H_{n-i}(M; \mathbb{Z}).$$

- **(Universal Coefficient Theorem** c.f. Theorem 4.4 [32]) For any abelian group A , there is a split exact sequence:

$$0 \rightarrow \text{Ext}(H_{i-1}(M; \mathbb{Z}), A) \rightarrow H^i(M; A) \rightarrow \text{Hom}(H_i(M; \mathbb{Z}), A) \rightarrow 0.$$

- **(Leray-Hirsch Theorem** c.f. Theorem 4D.1 [23])

For a fiber-bundle $\pi : E \rightarrow B$ with the fiber $\iota : F \hookrightarrow E$, if $H^p(F; \mathbb{Z})$ is a free \mathbb{Z} -module of finite rank for each p and $i^* : H^*(E; \mathbb{Z}) \rightarrow H^*(F; \mathbb{Z})$ is surjective,

then there is the isomorphism $\Phi : H^*(B; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(F; \mathbb{Z}) \rightarrow H^*(E; \mathbb{Z})$ as $H^*(B; \mathbb{Z})$ -modules, defined by

$$\sum_{a \in I} \beta_a \otimes \gamma_a \mapsto \sum_a \pi^*(\beta_a) \cup \tilde{\gamma}_a$$

where $\{\gamma_a\}$ is the basis of $H^*(F; \mathbb{Z})$ and $\tilde{\gamma}_a$ is an element in $(\iota^*)^{-1}(\gamma_a)$. In particular, $H^*(E; \mathbb{Z})$ is a free $H^*(B; \mathbb{Z})$ -module. The theorem holds with \mathbb{Q} -coefficients.

- **(Serre Spectral Sequence** c.f. Section 5.1 [34])

For a fiber-bundle $\pi : X \rightarrow B$ with the fiber $\iota : F \hookrightarrow X$, there is a spectral sequence such that

$$E_2^{pq} \cong H^p(B, H^q(F; \mathbb{Z}))$$

and E_∞^{pq} is isomorphic to the associated graded algebra of the filtration

$$F_p^{p+q} = \ker(H^{p+q}(X; \mathbb{Z}) \rightarrow H^{p+q}(X_{p-1}; \mathbb{Z}))$$

of $H^*(X; \mathbb{Z})$ (“converges to $H^*(X; \mathbb{Z})$ ”) where B_p is the p -skelton of B and $X_p := \pi^{-1}(B_p)$.

Lemma 2.12. *Let M be a closed, connected, orientable manifold of dimension n . If $H^{odd}(M; \mathbb{Z}) = 0$, then n is even and $H^*(M; \mathbb{Z})$ has no \mathbb{Z} -torsions.*

Proof. If n is odd, by Poincare duality, $H^n(M; \mathbb{Z}) = H_0(M; \mathbb{Z}) = 0$, which is a contradiction (c.f. Proposition 2.4 [32]). Now assume that n is even. By the Poincare duality, $H_{odd}(M; \mathbb{Z}) = 0$. By setting $A = \mathbb{Z}$, the UCT implies that

$$H^{even}(M; \mathbb{Z}) \cong \text{Ext}(H_{odd}(M; \mathbb{Z}), \mathbb{Z}) \oplus \text{Hom}(H_{even}(M; \mathbb{Z}), \mathbb{Z}).$$

Since $\text{Hom}(A, \mathbb{Z})$ is a free \mathbb{Z} -module for any \mathbb{Z} -module A , $H^*(M; \mathbb{Z}) = H^{even}(M; \mathbb{Z})$ is free. \square

Lemma 2.13 (Lemma 2.1 [33]). *Let M be a closed, connected, orientable manifold with a smooth action of a torus \mathbb{T} .*

- (1) *If $H^{odd}(M; \mathbb{Z}) = 0$, then $H_\mathbb{T}^*(M; \mathbb{Z})$ is a free $H^*(B\mathbb{T}; \mathbb{Z})$ -module. In particular,*

$$H_\mathbb{T}^*(M; \mathbb{Z}) \cong H^*(B\mathbb{T}; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(M; \mathbb{Z}) \quad \text{and} \quad H^*(M; \mathbb{Z}) \cong H_\mathbb{T}^*(M; \mathbb{Z}) \otimes_{H^*(B\mathbb{T}; \mathbb{Z})} \mathbb{Z}.$$

- (2) *If the fixed points $M^\mathbb{T}$ are isolated and $H_\mathbb{T}^*(M; \mathbb{Z})$ is free over $H^*(B\mathbb{T}; \mathbb{Z})$, then $H^{odd}(M; \mathbb{Z}) = 0$.*

Proof. (1) By Lemma 2.12, $H^*(M; \mathbb{Z})$ has no \mathbb{Z} -torsion. On the other hand, Since $H^{odd}(M; \mathbb{Z}) = H^{odd}(B\mathbb{T}; \mathbb{Z}) = 0$ implies that the Serre Spectral sequence for the bundle $M \hookrightarrow E\mathbb{T} \times_\mathbb{T} M \rightarrow B\mathbb{T}$ collapses at E_2 term. This implies that the pullback to the fiber $H^*(E\mathbb{T} \times_\mathbb{T} M; \mathbb{Z}) \rightarrow H^*(M; \mathbb{Z})$ is surjective (c.f. Proof of Lemma 5.1 [18]). Together with the vanishing of \mathbb{Z} -torsions in $H^*(M; \mathbb{Z})$, Leray-Hirsch for the bundle implies that $H^*(E\mathbb{T} \times_\mathbb{T} M; \mathbb{Z})$ is a free $H^*(B\mathbb{T}; \mathbb{Z})$ -module.

- (2) By freeness, we have $\text{Tor}_{i>0}^{H^*(B\mathbb{T}; \mathbb{Z})}(H_\mathbb{T}^*(M; \mathbb{Z}), \mathbb{Z}) = 0$. This implies that the Eilenberg-Moore Spectral Sequence of $M \hookrightarrow E\mathbb{T} \times_\mathbb{T} M \rightarrow B\mathbb{T}$ collapses at $E_2 = \text{Tor}_0^{H^*(B\mathbb{T}; \mathbb{Z})}(H_\mathbb{T}^*(M; \mathbb{Z}), \mathbb{Z}) = H_\mathbb{T}^*(M; \mathbb{Z}) \otimes_{H^*(B\mathbb{T}; \mathbb{Z})} \mathbb{Z}$, and so $H_\mathbb{T}^*(M; \mathbb{Z}) \rightarrow H^*(M; \mathbb{Z})$ is surjective (c.f. Proof of Theorem 1.1 [19]). By Localization Theorem 6.7 and the freeness, we have $H_\mathbb{T}^{odd}(M; \mathbb{Z}) = 0$ so that the surjectivity implies $H^{odd}(M; \mathbb{Z}) = 0$. \square

Lemma 2.14. *Let M be a closed, connected, orientable manifold with a smooth action of a torus \mathbb{T} . Let \mathbb{G} be a subgroup of \mathbb{T} that acts on M locally freely and let $\mathbb{R} := \mathbb{T}/\mathbb{G}$.*

- (i) If $H_G^{odd}(M; \mathbb{Z}) = 0$, then $H_T^*(M; \mathbb{Z}) \rightarrow H_G^*(M; \mathbb{Z})$ is surjective. (If $H_T^{odd}(M; \mathbb{Z}) = 0$, then this is clearly if-and-only-if condition.)
- (ii) If $H_G^{odd}(M; \mathbb{Q}) = 0$, then $H_T^*(M; \mathbb{Q})$ is free and $H_T^*(M; \mathbb{Q}) \cong H_G^*(M; \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(BR; \mathbb{Q})$ as modules over $H^*(BR; \mathbb{Q})$.
- (iii) If $\text{Tor}_{i>0}^{H^*(BR; \mathbb{Z})}(H_T^*(M; \mathbb{Z})) = 0$, then $H_T^*(M; \mathbb{Z}) \rightarrow H_G^*(M; \mathbb{Z})$ is surjective.
- (iv) If $\text{Tor}_{i>0}^{H^*(BR; \mathbb{Q})}(H_T^*(M; \mathbb{Q})) = 0$, then $H_T^*(M; \mathbb{Q})$ is free and $H_T^*(M; \mathbb{Q}) \cong H_G^*(M; \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(BR; \mathbb{Q})$ as modules over $H^*(BR; \mathbb{Q})$.

Proof.

- (i) The Serre Spectral sequence for the bundle $ET \times_G M \hookrightarrow ER \times_R (ET \times_G M) \rightarrow BR$ collapses at E_2 term. Thus the pullback to the fiber $H^*(ER \times_R (ET \times_G M)) \rightarrow H^*(ET \times_G M)$ is surjective.
- (ii) Since any torsion is killed over \mathbb{Q} , we can apply Leray-Hirsch.
- (iii) The Eilenberg-Moore spectral sequence for $ET \times_G M \hookrightarrow ER \times_R (ET \times_G M) \rightarrow BR$ collapses at E_2 .
- (iv) The Eilenberg-Moore spectral sequence for $ET \times_G M \hookrightarrow ER \times_R (ET \times_G M) \rightarrow BR$ collapses at E_2 , so $H^*(ER \times_R (ET \times_G M)) \rightarrow H^*(ET \times_G M)$ is surjective with \mathbb{Q} . We can apply Leray-Hirsch. \square

2.4. Appendix: Tor and Koszul complex. Let M be a $\mathbb{Z}[\bar{u}] := \mathbb{Z}[u_1, \dots, u_n]$ -module. $\text{Tor}_*^{\mathbb{Z}[\bar{u}]}(M, \mathbb{Z})$ is defined as follows: let (F_\bullet, d) be a free resolution of \mathbb{Z} in the category of $\mathbb{Z}[\bar{u}]$ -modules. Then $\text{Tor}_*^{\mathbb{Z}[\bar{u}]}(M; \mathbb{Z})$ is defined as the homology $H_*(M \otimes_{\mathbb{Z}[\bar{u}]} F_\bullet, d)$ of the induced complex $(M \otimes_{\mathbb{Z}[\bar{u}]} F_\bullet, d)$. Here we give a popular example of the free resolution.

Definition 2.15 (Koszul complex). Let $\mathbb{Z}\langle \bar{\xi} \rangle := \mathbb{Z}\langle \xi_1, \dots, \xi_n \rangle$ be the exterior algebra with $\deg \xi_i = 1$. The degree k part is given by

$$(\mathbb{Z}\langle \bar{\xi} \rangle)_k = \bigoplus_{i_1 < \dots < i_k} \mathbb{Z} \cdot \xi_{i_1} \wedge \dots \wedge \xi_{i_k}.$$

Now consider the tensor product $\mathbb{Z}[\bar{u}] \otimes_{\mathbb{Z}} \mathbb{Z}\langle \bar{\xi} \rangle$. It is naturally a free $\mathbb{Z}[\bar{u}]$ -module. Define the differential

$$d : (\mathbb{Z}[\bar{u}] \otimes_{\mathbb{Z}} \mathbb{Z}\langle \bar{\xi} \rangle)_k \rightarrow (\mathbb{Z}[\bar{u}] \otimes_{\mathbb{Z}} \mathbb{Z}\langle \bar{\xi} \rangle)_{k-1}$$

$$d(\xi_{i_1} \wedge \dots \wedge \xi_{i_k}) := \sum_{h=1}^k (-1)^{h-1} u_{i_h} \otimes \xi_{i_1} \wedge \dots \wedge \xi_{i_{h-1}} \wedge \xi_{i_{h+1}} \wedge \dots \wedge \xi_{i_k}.$$

It is clear that $d \circ d = 0$. It is a free resolution of \mathbb{Z} in the category of $\mathbb{Z}[\bar{u}]$ -module, c.f. [17, Section 2.3]. Now

Definition 2.16 (Cohomological degree). Note that the homological grading of $\text{Tor}_*^{\mathbb{Z}[\bar{u}]}(M; \mathbb{Z})$ comes from $\deg \xi_i = 1$. We can actually equip cohomological bigrading if M is a graded $\mathbb{Z}[\bar{u}]$ -module with $\deg \bar{u} = 2$. Consider the Koszul complex $M^\bullet \langle \bar{\xi} \rangle := M^\bullet \otimes_{\mathbb{Z}[\bar{u}]} \mathbb{Z}[\bar{u}] \otimes \mathbb{Z}\langle \bar{\xi} \rangle$. Set

$$\text{bideg}(m \otimes \xi_{i_1} \wedge \dots \wedge \xi_{i_k}) := (-k, \deg m + 2k).$$

This makes $M^\bullet \langle \bar{\xi} \rangle$ into a bigraded differential complex of $\mathbb{Z}[\bar{u}]$ -modules. In particular, $\deg d = (1, 0)$. When we write $\text{Tor}_{\mathbb{Z}[\bar{u}]}^*(M; \mathbb{Z})$, the grading is the total degree of this bigrade.

3. COHOMOLOGY OF SYMPLECTIC TORIC MANIFOLDS AND ORBIFOLDS

Let (M, ω) be a symplectic manifold. An action of a torus T on M is *Hamiltonian* with a *moment map* $\mu : M \rightarrow \mathfrak{t}^*$ if

$$\omega(\xi_M, -) = d\mu^\xi, \quad \forall \xi \in \mathfrak{t}$$

where $\mu^\xi(x) := \langle \mu(x), \xi \rangle$ and ξ_M is the infinitesimal action (a.k.a. fundamental vector field) of ξ . Let $\mathfrak{a} \in \mathfrak{t}^*$ be a regular value of μ . The regularity implies that the action of G on the level set $\mu^{-1}(\mathfrak{a})$ is locally free.

Theorem 3.1 (MarsdenWeinstein). *The reduced space $[\mu^{-1}(\mathfrak{a})/G]$ is a symplectic orbifold.*

We will define toric orbifolds as reduction spaces of \mathbb{C}^m with the standard symplectic structure by a subgroup of $U(1)^m$.

3.1. From Polytopes to Toric Manifolds/Orbifolds. Let $m > n$. Let $R = U(1)^n$ be the n -dimensional torus and $\mathfrak{r} = \mathbb{R}^n$ its Lie algebra. Let $T = U(1)^m$ be the m -dimensional torus and $\mathfrak{t} = \mathbb{R}^m$ its Lie algebra. A surjective homomorphism $T \rightarrow R$ corresponds bijectively to a $n \times m$ integral matrix $B = (B_{ij}) \in \text{Mat}_{n,m}(\mathbb{Z})$ of rank n . Namely (t_1, \dots, t_m) sent to (r_1, \dots, r_n) where $r_i := \prod_{j=1}^m t_j^{B_{ij}}$. A subgroup G of T of dimension $m - n$ defines a surjective homomorphism $T \rightarrow R$ by choosing an identification $T/G \cong R$. The following claims are left for readers for exercises:

- (1) Let $B : T \rightarrow R$ be a surjective homomorphism. Let $G := \ker B$ and let G_1 be the identify component of G . Then $G/G_1 \cong \mathfrak{r}_{\mathbb{Z}}/B(\mathfrak{t}_{\mathbb{Z}})$. In particular, G is connected if and only if $B(\mathfrak{t}_{\mathbb{Z}}) = \mathfrak{r}_{\mathbb{Z}}$.
- (2) A homomorphism $A : G \rightarrow T$ of tori is injective if and only if $\mathfrak{t}_{\mathbb{Z}}/A(\mathfrak{g}_{\mathbb{Z}})$ is free.

Definition 3.2. Let Δ be a bounded, convex, simple, rational polytope in $\mathbb{R}^n = \mathfrak{r}^*$. Let H_1, \dots, H_m be the facets of Δ . A polytope is simple if and only if each vertex is the intersection of exactly n facets. A polytope Δ is given by the inequalities: $\vec{u} \in \Delta$ if and only if

$$\langle \vec{u}, \beta_i \rangle + \eta_i \geq 0, \quad i = 1, \dots, m,$$

where $\beta_i \in \mathbb{R}^n = \mathfrak{r}$ is an inward normal vector to H_i and $\eta_i \in \mathbb{R}$. A polytope is rational if and only if $\beta_i \in \mathfrak{r}_{\mathbb{Z}}$. Let ρ_i be the inward primitive normal vector to each facet H_i . In this section, a polytope is always bounded convex simple rational.

Construction 3.3 ([30]). A labeled polytope (Δ, \mathfrak{b}) is a polytope Δ with facets labeled by positive integers $\mathfrak{b}_1, \dots, \mathfrak{b}_m$. Let $\vec{\eta} = (\eta_i) \in \mathfrak{t}^*$ be the unique vector such that

$$\Delta = \{\vec{u} \in \mathfrak{r}^* \mid \langle \vec{u}, \mathfrak{b}_i \rho_i \rangle + \eta_i \geq 0, \forall i = 1, \dots, m\}$$

Consider the matrix $B := [\mathfrak{b}_1 \rho_1, \dots, \mathfrak{b}_m \rho_m]$ and regard it as the surjective homomorphism $B : T \rightarrow R$. Let $G := \ker B$. Let $\mu_T : \mathbb{C}^m \rightarrow \mathfrak{t}^*$ be the standard moment map for the canonical Hamiltonian T -action on \mathbb{C}^m . Let $A : \mathfrak{g} \rightarrow \mathfrak{t}$ be the inclusion. The induced moment map for the G -action on \mathbb{C}^m is denoted by $\mu_G := A^* \circ \mu_T$. The toric orbifold $\mathcal{X}_{\Delta, \mathfrak{b}}$ associated to (Δ, \mathfrak{b}) is given as the reduction of \mathbb{C}^m by G at the regular value $A^*(\vec{\eta})$, i.e. the quotient orbifold

$$\mathcal{X}_{\Delta, \mathfrak{b}} := \mathbb{C}^m //_{A^*(\vec{\eta})} G := [\mu_G^{-1}(A^*(\vec{\eta}))]/G.$$

Consider an affine embedding $\widetilde{B}^* : \mathfrak{r}^* \rightarrow \mathfrak{t}^*$ defined by $\widetilde{B}^*(\vec{u}) := B^*(\vec{u}) + \vec{\eta}$. Then $\widetilde{\Delta} := \widetilde{B}^*(\Delta) = \widetilde{B}^*(\mathfrak{r}^*) \cap \mathfrak{t}_{\geq 0}^*$ where $\mathfrak{t}_{\geq 0}^*$ is the first quadrant of \mathfrak{t}^* . It is not too difficult to see that

$\mu_{\mathbb{G}}^{-1}(A^*(\vec{\eta})) = \mu_{\mathbb{T}}^{-1}(\widetilde{\Delta})$. The following map defines the moment map $\mu_{\mathbb{R}} : \mathcal{X}_{\Delta, \mathbf{b}} \rightarrow \mathfrak{r}^*$ for the toric orbifold/manifold with the Hamiltonian \mathbb{R} -action:

$$\mu_{\mathbb{G}}^{-1}(A^*(\vec{\eta})) \xrightarrow{\mu_{\mathbb{T}}} \widetilde{B}^*(\mathfrak{r}^*) \xrightarrow{\widetilde{B}^{-1}} \mathfrak{r}^*$$

Note that the image of $\mu_{\mathbb{R}}$ is exactly Δ .

Definition 3.4. The labeled polytope (Δ, \mathbf{b}) is called *Delzant* if for each vertex $v = H_{i_1} \cap \cdots \cap H_{i_n}$, the collection $\{\mathbf{b}_{i_1} \rho_{i_1}, \cdots, \mathbf{b}_{i_n} \rho_{i_n}\}$ is a \mathbb{Z} -basis of $\mathfrak{r}_{\mathbb{Z}}^*$ (in particular, $\mathbf{b}_i = 1, \forall i$).

Definition 3.5 (Simplicial complex associated to a simple polytope). A simplicial complex K on $[m] := \{1, \cdots, m\}$ is a subset of the power set $2^{[m]}$ such that if $\sigma \subset K$, then every subset of σ is an element of K . The simplicial complex K_{Δ} associated to a simple polytope Δ with facets H_1, \cdots, H_m , is a simplicial complex on $[m]$ defined as follows

$$\sigma \in K_{\Delta} \text{ if and only if } \bigcap_{i \in \sigma} H_i \neq \emptyset.$$

Definition 3.6. For a simplicial complex K on $[m]$, associate the following topological space called the *moment angle complex* (c.f. Chapter 6 [10]):

$$(3.1) \quad \mathcal{Z}_K := \bigcup_{\sigma \text{ facet of } K} D^{\sigma} \times (\partial D)^{\sigma \setminus [m]} \text{ in } \mathbb{C}^m$$

where $D = \{|z| \leq 1\}$, $\partial D = \{|z| = 1\} \subset \mathbb{C}$ are the unit disk and unit circle in \mathbb{C} .

Proposition 3.7. $\mathcal{X}_{\Delta, \mathbf{b}}$ is a toric manifold if and only if (Δ, \mathbf{b}) is Delzant. In this case, we say Δ is a Delzant polytope and denote $\mathcal{X}_{\Delta} := \mathcal{X}_{\Delta, \mathbf{b}}$.

Proof. It is known that $\mathcal{Z}_{K_{\Delta}}$ is \mathbb{T} -equivariantly homeomorphic to $\mu_{\mathbb{R}}^{-1}(A^*(\vec{\eta}))$ (c.f. Section 6.1 [10]). Therefore, to prove the proposition, it suffices to show that \mathbb{G} acts on $\mathcal{Z}_{K_{\Delta}}$ freely if and only if (Δ, \mathbf{b}) is Delzant. Let σ be a facet of K_{Δ} and $\mathbb{T}_{\sigma} := U(1)^{\sigma} \times \{1\}^{[m] \setminus \sigma}$. Let $i_{\sigma} : U(1)^n \rightarrow \mathbb{T}$ be the obvious embedding such that $\text{Im } i_{\sigma} = \mathbb{T}_{\sigma}$. If we let $\{\mathbf{e}_i, i \in [m]\}$ be the standard basis of \mathbb{Z}^m , then $i_{\sigma} : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$ is given by the matrix $[\mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_n}]$ where $\sigma = \{i_1, \cdots, i_n\}$. The following are equivalent

- \mathbb{G} acts on $D^{\sigma} \times (\partial D)^{\sigma \setminus [m]}$ freely.
- $\mathbb{G} \cap \mathbb{T}_{\sigma} = \{1\}$
- The map $\Phi_{\sigma} : \mathbb{G} \times U(1)^n \rightarrow \mathbb{T}, (g, t) \mapsto g \cdot i_{\sigma}(t)$ is injective.

Thus \mathbb{G} acts on $\mathcal{Z}_{K_{\Delta}}$ freely if and only if Φ_{σ} is injective for every facet σ of K_{Δ} . If Φ_{σ} is injective, it is actually an isomorphism by the dimensional reason and so \mathbb{G} must be connected. Therefore $B : \mathbb{Z}^m \rightarrow \mathbb{Z}^n$ is surjective. If we let $\{v_j\}$ be a \mathbb{Z} -basis of $\mathfrak{g}_{\mathbb{Z}}$, then $A(v_1), \cdots, A(v_{m-n}), \mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_n}$ is a \mathbb{Z} -basis of \mathbb{Z}^m . Since $B : \mathbb{Z}^n \rightarrow \mathbb{Z}^m$ is surjective and $B \circ A = 0$, $\{B(\mathbf{e}_i) = \mathbf{b}_i \mathbf{e}_i, i \in \sigma\}$ must be a \mathbb{Z} -basis of \mathbb{Z}^n . Thus the Delzant condition follows. On the other hand, if $\{B(\mathbf{e}_i) = \mathbf{b}_i \mathbf{e}_i, i \in \sigma\}$ is a \mathbb{Z} -basis of \mathbb{Z}^n , then $A(v_1), \cdots, A(v_{m-n}), \mathbf{e}_{i_1}, \cdots, \mathbf{e}_{i_n}$ must be a \mathbb{Z} -basis of \mathbb{Z}^m and so \mathbb{G} is connected. Thus Φ_{σ} is an isomorphism. \square

3.2. Cell Decomposition of Toric Manifolds. In this section, we will describe the even dimensional cell decomposition of toric manifolds, following [13, p.431]. The consequence is that the ordinary cohomology of a toric manifold is concentrated in even degree and is a free \mathbb{Z} -module. This implies that the ordinary cohomology is the quotient of the equivariant cohomology.

Construction 3.8. Let Δ be a Delzant polytope in $\mathfrak{r}^* \cong \mathbb{R}^n$ and choose a generic linear form $\xi \in \mathfrak{r}$ so that the values of vertices are distinct and so we have a total order on the vertices

(from “bottom” to “top” as ξ defines a height function). This induces the direction on every edge of Δ . For a face F , let a_F and b_F be the top and the bottom vertices ($a_F \leq b_F$). Let d_v be the number of edges coming in to v ($n - d_v$ is the number of edges coming out of v). Let F_v be the smallest face containing all incoming edges to v . Then $\dim F_v = d_v$. Also if F' is a face of Δ with $v := a_{F'}$, then F' is a face of \widehat{F}_v . Let \widehat{F}_v be the union of relative interiors of faces contained in F_v and containing v . Then \widehat{F}_v is diffeomorphic to $\mathbb{R}_{\geq 0}^{d_v}$ (positive quadrant in \mathbb{R}^{d_v}) and

$$\Delta = \coprod_{v \text{ vertex}} \widehat{F}_v.$$

Let $\mu_{\mathbb{R}} : \mathcal{X}_{\Delta} \rightarrow \mathfrak{r}^*$ be the moment map defined in Construction 3.3. Then $\mu_{\mathbb{R}}^{-1}(\widehat{F}_v) \cong \mathbb{R}^{2d_v}$. This gives a cell decomposition in even dimension.

Corollary 3.9 (See Lemma 2.13). *Let \mathcal{X}_{Δ} be a toric manifold of dimension $2n$. $H^*(\mathcal{X}_{\Delta}; \mathbb{Z})$ is a free \mathbb{Z} -module only in even degree. Moreover $H_{\mathbb{R}}^*(\mathcal{X}_{\Delta}; \mathbb{Z})$ is a free $H^*(\mathbb{B}\mathbb{R}; \mathbb{Z})$ -module.*

3.3. Equivariant Cohomology of Toric Manifolds/Orbifolds.

Definition 3.10 (Stanley-Reisner rings). Let K be a simplicial complex on $[m]$. The *Stanley-Reisner ring* (a.k.a. face ring) $\mathbb{Z}[K]$ of K is defined as a quotient of the polynomial ring by the ideal generated by square-free monomials of non-faces:

$$\mathbb{Z}[K] := \frac{\mathbb{Z}[x_1, \dots, x_m]}{\langle x_{\sigma}, \sigma \notin K \rangle}$$

where $x_{\sigma} := \prod_{i \in \sigma} x_i$.

In this section, we prove

Theorem 3.11. *Let K be a simplicial complex on $[m]$ and \mathcal{Z}_K the associated moment angle complex defined at (3.1).*

$$H_{\mathbb{T}}^*(\mathcal{Z}_K; \mathbb{Z}) \cong \mathbb{Z}[K]$$

Corollary 3.12. *Let (Δ, \mathbf{b}) be a labeled polytope. Then $H_{\mathbb{R}}^*(\mathcal{X}_{\Delta, \mathbf{b}}; \mathbb{Z}) \cong \mathbb{Z}[K_{\Delta}]$.*

Proof. $H_{\mathbb{R}}^*(\mathcal{X}_{\Delta, \mathbf{b}}; \mathbb{Z}) = H_{\mathbb{R}}^*(\mu_{\mathbb{G}}^{-1}(A^*(\vec{\eta}))/\mathbb{G}; \mathbb{Z}) = H_{\mathbb{T}}^*(\mu_{\mathbb{G}}^{-1}(A^*(\vec{\eta})); \mathbb{Z}) \cong H_{\mathbb{T}}^*(\mathcal{Z}_{K_{\Delta}}; \mathbb{Z}) \cong \mathbb{Z}[K_{\Delta}]$. \square

Proof of Theorem 3.11.

Definition 3.13. The cellular complex of $\mathbb{C}\mathbb{P}^{\infty}$ is given by

$$o := [1 : 0 : 0 : \dots] \subset [z_1 : 1 : 0 : \dots] \subset [z_1 : z_2 : 1 : 0 : \dots] \subset [z_1 : z_2 : z_3 : 1 : 0 : \dots]$$

Let $\tilde{o} = (1, 0, \dots) \in S^{\infty}$ be a point in the fiber of o of $S^{\infty} \rightarrow \mathbb{C}\mathbb{P}^{\infty}$.

For $\sigma \subset [m]$, let $\mathbb{T}_{\sigma} := \mathbb{U}(1)^{\sigma} \times \{1\}^{[m] \setminus \sigma}$. Then we can regard $\mathbb{B}\mathbb{T}_{\sigma} \subset \mathbb{B}\mathbb{T} = (\mathbb{C}\mathbb{P}^{\infty})^m$ as the subcellular complex

$$\mathbb{B}\mathbb{T}_{\sigma} = (\mathbb{C}\mathbb{P}^{\infty})^{\sigma} \times \{\tilde{o}\}^{[m] \setminus \sigma} \subset \mathbb{B}\mathbb{T}.$$

For a simplicial complex K on m , define the cellular subcomplex $\text{DJ}(K)$ called “*Daivis-Januszkiewicz space*” by

$$\text{DJ}(K) := \bigcup_{\sigma \subset K} \mathbb{B}\mathbb{T}_{\sigma} \subset \mathbb{B}\mathbb{T}.$$

The following proposition is an immediate corollary from the definition of Stanley-Reisner ring and CW cohomology.

Proposition 3.14. *The cellular chain complex $C^*(\text{DJ}(K), \mathbb{Z})$ and its cohomology $H^*(\text{DJ}(K); \mathbb{Z})$ is isomorphic as an algebra to $\mathbb{Z}[K]$.*

Theorem 3.11 follows from

Theorem 3.15 (c.f. Theorem 6.29 [10]). *There is an embedding $i : \text{DJ}(K) \hookrightarrow \text{ET} \times_{\mathbb{T}} \mathcal{Z}_K$ such that*

$$\begin{array}{ccc} & \text{ET} \times_{\mathbb{T}} \mathcal{Z}_K & \\ & \uparrow i & \searrow \\ \text{DJ}(K) & \longrightarrow & \text{BT}. \end{array}$$

is commutative and there is a deformation retraction $\text{ET} \times_{\mathbb{T}} \mathcal{Z}_K \rightarrow \text{DJ}(K)$ preserving the commutative diagram.

Proof. Recall that $\mathcal{Z}_K := \bigcup_{\sigma \in K} \text{D}^{\sigma} \times (\partial \text{D})^{[m] \setminus \sigma}$. Then each component in the union is \mathbb{T} -invariant, so

$$\text{ET} \times_{\mathbb{T}} \mathcal{Z}_K = \bigcup_{\sigma \in K} \text{ET} \times_{\mathbb{T}} (\text{D}^{\sigma} \times (\partial \text{D})^{[m] \setminus \sigma}).$$

Then

$$\text{ET} \times_{\mathbb{T}} (\text{D}^{\sigma} \times (\partial \text{D})^{[m] \setminus \sigma}) = (\text{ET}_{\sigma} \times_{\mathbb{T}_{\sigma}} \text{D}^{\sigma}) \times \text{ET}_{[m] \setminus \sigma}$$

We can define the inclusion

$$i_{\sigma} : \text{BT}_{\sigma} \cong \text{ET}_{\sigma} \times_{\mathbb{T}_{\sigma}} \{0\}^{\sigma} \times \{\partial\}^{[m] \setminus \sigma} \subset (\text{ET}_{\sigma} \times_{\mathbb{T}_{\sigma}} \text{D}^{\sigma}) \times \text{ET}_{[m] \setminus \sigma}.$$

and the projection

$$\rho_{\sigma} : (\text{ET}_{\sigma} \times_{\mathbb{T}_{\sigma}} \text{D}^{\sigma}) \times \text{ET}_{[m] \setminus \sigma} \rightarrow \text{BT}_{\sigma}.$$

The map of ρ_{σ} is the projection to the first factor and its fiber is contractible, so there is a deformation retract with respect to i_{σ} . Actually we can choose the deformation retract so that it preserves the projection map $\text{ET} \times_{\mathbb{T}} (\text{D}^{\sigma} \times (\partial \text{D})^{[m] \setminus \sigma}) \rightarrow \text{BT}$. The maps i_{σ} and ρ_{σ} are patched together to define the desired maps in the claim. \square

3.4. Cohomology of Toric Orbifolds. By the Kirwan surjectivity Theorem 4.8, the cohomology of a toric orbifold $\mathcal{X}_{\Delta, \mathbf{b}}$ with \mathbb{Q} -coefficients doesn't have odd degree. Thus by Lemma 2.14, $H_{\mathbb{R}}(\mathcal{X}_{\Delta, \mathbf{b}}; \mathbb{Q})$ is free over $H^*(\text{BR}; \mathbb{Q})$ and $H^*(\mathcal{X}_{\Delta, \mathbf{b}}; \mathbb{Q}) \cong \mathbb{Q}[K_{\Delta}] / \langle u_1, \dots, u_n \rangle$. From the algebraic geometry side, we can find this fact in [12]. In this section, we survey on the integral cohomology of toric orbifolds.

Example 3.16 (Weighted Projective Space, [27]). Let $[\mathbb{C}\mathbb{P}_{\mathbf{a}}^{m-1}]$ be a weighted projective space as a quotient stack orbifold.

$$H_{\mathbb{R}}^*([\mathbb{C}\mathbb{P}_{\mathbf{a}}^{m-1}]; \mathbb{Z}) = \mathbb{Z}[y] / \langle \mathbf{a}_1 \cdots \mathbf{a}_m y^m \rangle.$$

Theorem 3.17 ([31]). *Let K be a simplicial complex and Let $\mathbb{G} \subset \mathbb{T}$ be a subgroup ($\mathbb{R} := \mathbb{T}/\mathbb{G}$). Then there is an isomorphism*

$$\Phi_{K, \mathbb{G}} : H_{\mathbb{G}}^*(\mathcal{Z}_K; \mathbb{Z}) \cong \text{Tor}_{H^*(\text{BR}; \mathbb{Z})}^*(H_{\mathbb{T}}^*(\mathcal{Z}_K; \mathbb{Z}), \mathbb{Z}) = \text{Tor}_{\mathbb{Z}[u_1, \dots, u_n]}^*(\mathbb{Z}[K], \mathbb{Z}),$$

where $\{u_1, \dots, u_n\}$ is a \mathbb{Z} -basis of $\mathfrak{r}_{\mathbb{Z}}^*$. Furthermore there is a commutative diagram of obvious maps:

$$\begin{array}{ccc} H_{\mathbb{T}}^*(\mathcal{Z}_K; \mathbb{Z}) & \longrightarrow & \text{Tor}_{\mathbb{Z}}^*(\mathbb{Z}[K]; \mathbb{Z}) \cong \mathbb{Z}[K] \\ \downarrow & & \downarrow \\ H_{\mathbb{G}}^*(\mathcal{Z}_K; \mathbb{Z}) & \longrightarrow & \text{Tor}_{\mathbb{Z}[u_1, \dots, u_n]}^*(\mathbb{Z}[K], \mathbb{Z}) \end{array}$$

The proof of this theorem is based on the following fact and the application of [17, Theorem 5.5].

Theorem 3.18 ([16]). *The singular cochain complex $C^*(ET \times_{\mathbb{T}} \mathcal{Z}_K; \mathbb{Z})$ is formal in the category of $H_{\mathbb{T}}^*(BT; \mathbb{Z})$ -modules up to homotopy, i.e. there is a sequence of quasi-isomorphism between $C^*(ET \times_{\mathbb{T}} \mathcal{Z}_K; \mathbb{Z})$ and $H^*(ET \times_{\mathbb{T}} \mathcal{Z}_K; \mathbb{Z})$ as modules over cobar construction of $H^*(BT; \mathbb{Z})$.*

Thus Theorem 3.17 actually holds in general for any \mathbb{T} -space with the same formality. Now we can describe a stronger version of Lemma 2.14

Theorem 3.19 ([31]). *The following are equivalent:*

- (1) $H_{\mathbb{G}}^{\text{odd}}(\mathcal{Z}_K; \mathbb{Z}) = 0$
- (2) $H_{\mathbb{T}}^*(\mathcal{Z}_K; \mathbb{Z}) \rightarrow H_{\mathbb{G}}^*(\mathcal{Z}_K; \mathbb{Z})$ is surjective, i.e. $H_{\mathbb{G}}^*(\mathcal{Z}_K; \mathbb{Z}) = \mathbb{Z}[K]/\langle u_1, \dots, u_n \rangle$.
- (3) $\text{Tor}_{i>0}^{\mathbb{Z}[u_1, \dots, u_n]}(\mathbb{Z}[K], \mathbb{Z}) = 0$

(3) can be shown to be equivalent to the condition: $\{u_1, \dots, u_n\}$ is a regular sequence for $\mathbb{Z}[K]$ as a module over $\mathbb{Z}[u_1, \dots, u_n]$. This is referred to Cohen-Macaulayness if the module is finitely generated. In our case, it may not be finitely generated; it could have infinitely many torsions as in the case of weighted projective spaces. Thus, if (3) is satisfied, we call the module a *big Cohen-Macaulay* module over $\mathbb{Z}[u_1, \dots, u_n]$.

Corollary 3.20. *Let (Δ, \mathbf{b}) be a labeled polytope. Recall that the toric orbifold \mathcal{X}_{Δ} is given as the symplectic reduction of \mathbb{C}^m by the action of \mathbb{G} . The Kirwan map $\iota^* : H_{\mathbb{G}}^*(\mathbb{C}^m; \mathbb{Z}) \rightarrow H^*(\mathcal{X}_{\Delta}; \mathbb{Z})$ is given by pulling back along the inclusion of the level set $\iota : \mu_{\mathbb{G}}^{-1}(A^*(\vec{\eta})) \hookrightarrow \mathbb{C}^m$. The Kirwan surjectivity over \mathbb{Z} holds if and only if one of the condition in Theorem 3.19 holds.*

Summary. Let $\mathcal{X}_{\Delta, \mathbf{b}}$ be a toric orbifold with a labeled polytope (Δ, \mathbf{b}) .

- $H_{\mathbb{T}}^*(\mathcal{X}_{\Delta, \mathbf{b}}; \mathbb{Z})$ as a cohomology of stack is isomorphic to $\mathbb{Z}[K_{\Delta}]$.
- $H^*(\mathcal{X}_{\Delta, \mathbf{b}}; \mathbb{Q})$ is isomorphic to $\mathbb{Q}[K_{\Delta}]/\langle u_1, \dots, u_n \rangle$.
- $H^*(\mathcal{X}_{\Delta, \mathbf{b}}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}[K_{\Delta}]/\langle u_1, \dots, u_n \rangle$ if one of the conditions in Theorem 3.19 holds.
- If Δ is a Delzant polytope, then $H^*(\mathcal{X}_{\Delta}; \mathbb{Z})$ is isomorphic to $\mathbb{Z}[K_{\Delta}]/\langle u_1, \dots, u_n \rangle$.

4. GENERAL METHODS TO DESCRIBE (EQUIV) COHOMOLOGY OF HAMILTONIAN \mathbb{T} -SPACE

4.1. Injectivity Theorem and Chang-Skjelbred Lemma for Hamiltonian Torus Actions. In this section, we explain the injectivity theorem and the Chang-Skjelbred Lemma. For the Hamiltonian torus actions, we follow [39] and [26]. Let M be a connected compact symplectic manifold with a Hamiltonian torus \mathbb{T} -action. Let $M^{\mathbb{T}}$ be the submanifold of \mathbb{T} -fixed points. The following condition is needed for the injectivity theorem and Chang-Skjelbred lemma to hold:

- (Z1) For each connected component F of $M^{\mathbb{T}}$, the weights of the \mathbb{T} -action on the normal bundle of F in M are primitive.

Note that the weights are the element of $\mathfrak{t}_{\mathbb{Z}}^* \cong \mathbb{Z}^{\dim \mathbb{T}}$. An element $\lambda \in \mathfrak{t}_{\mathbb{Z}}^*$ is primitive if $a = \pm 1$ for all $\lambda' \in \mathfrak{t}_{\mathbb{Z}}^*$ and $a \in \mathbb{Z}$ such that $a\lambda' = \lambda$.

Theorem 4.1 (Injectivity, c.f. Prop 7.2. [39]). *Let $\iota : M^{\mathbb{T}} \hookrightarrow M$ be the inclusion. Then $\iota^* : H_{\mathbb{T}}^*(M; \mathbb{Q}) \rightarrow H_{\mathbb{T}}^*(M^{\mathbb{T}}; \mathbb{Q})$ is injective. If (Z1) is satisfied, then ι^* is injective with \mathbb{Z} -coefficients.*

Theorem 4.2 (Chang-Skjelbred Lemma, c.f. Prop 7.3. [39]). *Let M_1° be the set of all 1-dimensional orbits of \mathbb{T} and let $M_1 := M_1^{\circ} \cup M^{\mathbb{T}}$. Let $j : M_1 \rightarrow M$ be the inclusion. Then the image of $j^* : H_{\mathbb{T}}^*(M_1; \mathbb{Q}) \rightarrow H_{\mathbb{T}}^*(M^{\mathbb{T}}; \mathbb{Q})$ and the image of ι^* coincide. If (Z1) is satisfied, the claim holds with \mathbb{Z} -coefficients.*

Orbifold case. Let M be a smooth manifold with a torus T -action. Let G be a subgroup of T that acts on M locally freely so that $[M/G]$ is an orbifold. Let $R := T/G$.

Definition 4.3. Let $\omega \in \Omega^2(M)^T$ be a T -invariant 2-form. Then ω induces a G_x -invariant 2-form $\bar{\omega}$ on $T_x M/T_x(G \cdot x)$. The pair $([M/G], \omega)$ is a symplectic orbifold with a Hamiltonian R -action if

- (1) $\bar{\omega}$ is closed and non-degenerate
- (2) $\omega(\xi_M, -) = d\mu^\xi$ for all $\xi \in \mathfrak{t}$.

Let M_0 be the submanifold of all $(\dim G)$ -dimensional T -orbits and let M_1° be the submanifold of all $(\dim G + 1)$ -dimensional T -orbits. Then $[M_0/G]$ can be viewed as the suborbifold of R -fixed orbitoid points and $[M_1^\circ/G]$ as the suborbifold of 1-dimensional R -orbits in the orbifold $[M/G]$. Now let $M_1 := M_1^\circ \cup M_0$. We have the following generalization of the Injectivity Theorem and Chang-Skjelbred Lemma:

Theorem 4.4 (Theorem 4.10, 5.5 [26]). *Let $\iota : M_0 \rightarrow M$ and $j : M_1 \rightarrow M$ be the inclusion. Then $\iota^* : H_T^*(M; \mathbb{Q}) \rightarrow H_T^*(M_0; \mathbb{Q})$ is injective and the image of ι^* coincides with the image of $j^* : H_T^*(M_1; \mathbb{Q}) \rightarrow H_T^*(M_0; \mathbb{Q})$. If the condition*

- (Z2) *The stabilizer T_x of each point of $x \in M_0$ is connected and the weights of the action of T_x on the normal bundle of M_0 in M is primitive.*

is satisfied, then the claim holds with \mathbb{Z} -coefficients.

Remark 4.5. For each connected component F of M_0 , the identity components of the stabilizer groups T_x of any points x in F are the same.

4.2. Kirwan Surjectivity for Hamiltonian Torus Actions.

Definition 4.6 (Introduction [22]). An action of a n -dimensional torus T on a compact manifold M is *quasi-free* if there is a subset $S \subset M$ where T acts freely and if, moreover, for every point x in $M \setminus S$, the stabilizer group of x is of dimension greater or equal to 1.

Remark 4.7. The stabilizer of every point is connected. If such an S exists, it is unique and open and dense. If a Hamiltonian T -action on M is quasi-free, then the action of T on a regular level set of the moment map is free. Since the regularity implies that the stabilizer of a point in the level set is finite, so it has to be trivial.

Theorem 4.8 (Kirwan Surjectivity over \mathbb{Q}). *Let M be a symplectic manifold with a Hamiltonian G -action with a proper moment map. Assume that the set of connected components of the fixed point set is finite. If $\mathfrak{a} \in \mathfrak{t}^*$ is a regular value of the moment map μ , then $\iota_* : H_G^*(M; \mathbb{Q}) \rightarrow H_G^*(\mu^{-1}(\mathfrak{a}); \mathbb{Q})$ is a surjection where $\iota : \mu^{-1}(\mathfrak{a}) \hookrightarrow M$ is the obvious inclusion.*

Theorem 4.9 (Kirwan Surjectivity over \mathbb{Z} , c.f. Prop 7.6 [40]). *Let M be a symplectic manifold with a Hamiltonian G -action with a proper moment map. Assume that the T -action is quasi-free and the set of connected components of the fixed point set is finite. If $\mathfrak{a} \in \mathfrak{t}^*$ is a regular value of the moment map μ , then $\iota^* : H_G^*(M; \mathbb{Z}) \rightarrow H_G^*(\mu^{-1}(\mathfrak{a}); \mathbb{Z})$ is a surjection.*

Remark 4.10. If $H_G^{odd}(M; \mathbf{k}) = 0$, then the Kirwan surjectivity implies that $H_G^{odd}(\mu^{-1}(\mathfrak{a}); \mathbf{k}) = 0$. In the quasi-free case, G acts on $\mu^{-1}(\mathfrak{a})$ freely and $H_G^*(\mu^{-1}(\mathfrak{a}); \mathbb{Z}) \cong H^*(\mu^{-1}(\mathfrak{a})/G; \mathbb{Z})$. Thus by Lemma 2.12, $H^*(\mu^{-1}(\mathfrak{a})/G; \mathbb{Z})$ has no \mathbb{Z} -torsion.

5. GKM DESCRIPTION

5.1. Preliminary.

Local Normal Form and Euler class of normal bundles to a fixed point.

Lemma 5.1. *Let M be a $2n$ -dimensional symplectic manifold with a Hamiltonian torus \mathbb{T} -action. Let $p_0 \in M^\mathbb{T}$ be an isolated fixed point. Let $T_{p_0}M = \bigoplus_\lambda W_\lambda$ be the weight decomposition of the \mathbb{T} -action. Suppose that the weights are primitive. Then each λ is the outward primitive integral vector spanning an edge coming out of the vertex corresponding to p in the moment polytope.*

Proof. The local normal form of the Hamiltonian \mathbb{T} -action (c.f. [11, Theorem 5.22]) implies this². There is a neighborhood U with a local chart $\phi : U \cong V \subset \mathbb{C}^n$ centered at p_0 such that the moment map is given by

$$(5.1) \quad \mu_{\mathbb{T}}(p) = \phi(p_0) + \sum_{i=1}^n |z_i|^2 \cdot \lambda_i \in \mathfrak{t}^*, \quad \text{for } \phi(p) = \vec{z} \in V.$$

□

Remark 5.2. Without loss of generality, we can assume $\mu(p) = 0$. Then the moment map μ locally around p is the moment map for \mathbb{C}^n with respect to the action $\lambda : \mathbb{T} \rightarrow \text{U}(1)^n$. Namely,

$$\mathbb{C}^n \xrightarrow{\mu} (\mathbb{R}^*)^n \xrightarrow{\lambda} \mathfrak{t}^*$$

where $\lambda(\mathbf{e}_i) = \lambda_i$. So the edges are spanned by λ_i 's.

Equivariant cohomology of $\mathbb{C}\mathbb{P}^1$. Let $\mathbb{R} = \text{U}(1)$ and consider the usual action on $\mathbb{C}\mathbb{P}^1$: $t \cdot [x : y] = [x : ty]$. Let $p_1 := \{[1 : 0]\}$ and $p_2 := \{[0 : 1]\}$. The tautological line bundle on $\mathbb{C}\mathbb{P}^1$ is given by $L := \{([x : y], (x, y))\} \subset \mathbb{C}\mathbb{P}^1 \times \mathbb{C}^2$. Let $u \in \mathfrak{t}_{\mathbb{Z}}^*$ be the generator corresponding to the identity character $\text{id} : \text{U}(1) = \mathbb{R} \rightarrow \text{U}(1)$.

Lemma 5.3. *The image of $H_{\mathbb{T}}^*(\mathbb{C}\mathbb{P}^1; \mathbb{Z}) \hookrightarrow \mathbb{Z}[u] \oplus \mathbb{Z}[u]$ is given by*

$$\{(f(u), g(u)) \mid f(u) - g(u) \text{ is divisible by } u\}.$$

Proof. The ordinary cohomology of projective space is generated by the 1st Chern class $c_1(L)$ of the tautological line bundle L and $H^*(\mathbb{C}\mathbb{P}^1; \mathbb{Z}) \cong \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot c_1(L)$. (We have $H^*(\mathbb{C}\mathbb{P}^1; \mathbb{Z}) \cong \mathbb{Z}[y]/\langle y^2 \rangle$ c.f. Theorem 14.4 [37]). Lift the $\text{U}(1)$ -action on $\mathbb{C}\mathbb{P}^1$ to L by

$$t \cdot ([x : y], (x, y)) := ([x : ty], (x, ty)).$$

and consider the line bundle $\widetilde{L} := E\mathbb{T} \times_{\mathbb{T}} L$. Then L is the pullback of \widetilde{L} along the fiber inclusion $i : \mathbb{C}\mathbb{P}^1 \hookrightarrow E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}\mathbb{P}^1$. By the Leray-Hirsch,

$$\Phi : H^*(\mathbb{C}\mathbb{P}^1; \mathbb{Z}) \otimes \mathbb{Z}[u] \rightarrow H_{\mathbb{T}}^*(\mathbb{C}\mathbb{P}^1; \mathbb{Z}), \quad c \otimes f \mapsto \tilde{c} \cup \pi^*(f), \quad \text{for } i^*(\tilde{c}) = c.$$

defines an isomorphism of $\mathbb{Z}[u]$ -module where $\pi : E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}\mathbb{P}^1 \rightarrow B\mathbb{T}$. We have $\Phi(c_1(L) \otimes 1) = c_1(\widetilde{L})$. By the definition of the $\text{U}(1)$ -action,

$$c_1(\widetilde{L}|_{p_1}) = 0 \in \mathbb{Z}[u], \quad c_1(\widetilde{L}|_{p_2}) = u \in \mathbb{Z}[u].$$

Therefore the image of $H_{\mathbb{T}}^*(\mathbb{C}\mathbb{P}^1; \mathbb{Z})$ is generated by $(1, 1)$ and $(0, u)$ as a free module over $\mathbb{Z}[u]$. For $g_1, g_2 \in \mathbb{Z}[u]$,

$$(f_1, f_2) := g_1(1, 1) + g_2(0, u) = (g_1(u), g_1(u) + ug_2(u))$$

We have $u \mid f_1 - f_2$. If (f_1, f_2) satisfies $u \mid f_1 - f_2$, so let $f_2 - f_1 := uh$. Then

$$(f_1, f_2) = f_1(1, 1) + h(0, u).$$

□

²More general local normal form can be found in, for example, [25, Lemma 2.1]

5.2. GKM description of the equivariant cohomology. Let M be a closed $2m$ -dimensional symplectic manifold with a Hamiltonian torus T -action. Suppose that

- (a) The fixed points M^T are finite
- (b) The weights of the T action on $T_x M, x \in M^T$ are primitive and pairwise-linearly-independent.

Then by Injectivity Theorem 4.1 and Chang-Skjelbred Lemma 4.2, $H_T^*(M; \mathbb{Z})$ is isomorphic to the image of

$$H_T^*(M; \mathbb{Z}) \rightarrow \bigoplus_{v \in M^T} \mathbb{Z}[x_1, \dots, x_n]$$

where $\{x_1, \dots, x_n\}$ is a \mathbb{Z} -basis of t_v^* . By those assumptions above, M_1 is a chain of $\mathbb{C}P^1$'s. More precisely, let v be a fixed point and let λ_v be a weight of the T -action on the tangent space. The closure of a connected component of the one dimensional orbits M_1° in this direction will add another fixed point w and this closure is isomorphic to $\mathbb{C}P^1$ where T acts by $\lambda : T \rightarrow U(1)$. By the computation in the previous section, we arrive at the following combinatorial description of the T -equivariant cohomology of M .

Lemma 5.4 (GKM description). *The pullback of the inclusion $M^T \hookrightarrow M$ induces the isomorphism*

$H_T^*(M; \mathbb{Z}) \cong \{(f_v)_{v \in M^T} \mid \lambda_{(v,w)} \text{ divides } f_v - f_w \text{ for each edge } (v,w) \text{ of the moment polytope}\}$, where $\lambda_{(v,w)}$ is the weight of the T -action in the normal direction at v toward w .

Remark 5.5. In the case of toric manifolds, this enable us to present the Stanley-Reisner ring as a subring of a direct sum of the polynomial ring $H^*(BR; \mathbb{Z})$. This presentation is known as the ring of “continuous piecewise polynomials” on the associated fan. See [5, 6, 7]. It should be stressed here that this presentation is a presentation of the Stanley-Reisner ring *not as a ring over $H^*(BT; \mathbb{Z})$ but as a ring over $H^*(BR; \mathbb{Z})$* . In [26], the authors deal with toric orbifolds and present $\mathbb{Z}[K_\Delta]$ as a ring over $H^*(BT; \mathbb{Z})$.

GKM Description of Toric Manifold. Let Δ be a Delzant polytope in $\mathfrak{r}^* \cong \mathbb{R}^n$. Let \mathcal{X}_Δ be the corresponding toric manifold with the torus \mathbb{R} -action.

Theorem 5.6. *Under the map*

$$\mathbb{Z}[K_\Delta] = H_{\mathbb{R}}^*(\mathcal{X}_\Delta; \mathbb{Z}) \rightarrow H_{\mathbb{R}}^*(\mathcal{X}_\Delta^{\mathbb{R}}; \mathbb{Z}) = \bigoplus_{v \in \mathcal{X}_\Delta^{\mathbb{R}}} \mathbb{Z}[u_1, \dots, u_n],$$

x_i goes to $(f_i^v)_{v \in \mathcal{X}_\Delta^{\mathbb{R}}}$ where

$$f_i^v = \begin{cases} 0 & \text{if } v \notin H_i \\ \lambda_i^v & \text{if } v \in H_i \end{cases}$$

where λ_i^v is the primitive integral outward generator of the unique edge not contained in H_i .

Proof. Let $p_v \in \mathcal{X}_\Delta^{\mathbb{R}}$ be the fixed point corresponding to a vertex $v = \bigcap_{i \in \sigma} H_i = H_{i_1} \cap \dots \cap H_{i_n}$. Under the \mathbb{R} -equivariant homeomorphism $\mathcal{X}_\Delta \cong \mathcal{Z}_{K_\Delta}/\mathbb{G}$, $p_v = \mathcal{Z}_\sigma/\mathbb{G}$ where $\mathcal{Z}_\sigma = \{0\}^\sigma \times (\partial D)^{[m] \setminus \sigma}$.

$$\begin{array}{ccc} H_{\mathbb{R}}^*(\mathcal{X}_\Delta) & \longrightarrow & H_{\mathbb{R}}^*(p_v) = H^*(BR) = \mathbb{Z}[u_1, \dots, u_n] \\ \cong \downarrow & & \cong \downarrow \\ H_T^*(\mathcal{Z}_{K_\Delta}) & \longrightarrow & H_T^*(\mathcal{Z}_\sigma) = H^*(BT_\sigma) = \mathbb{Z}[x_i \in \sigma]. \end{array}$$

The right vertical map is given by $\mathbb{T}_\sigma \rightarrow \mathbb{T} \rightarrow \mathbb{R}$ which is an isomorphism defined by $B_\sigma = [\rho_{i_1}, \dots, \rho_{i_n}]$ (Delzant property). Namely

$$\mathbb{Z}[u_1, \dots, u_n] \rightarrow \mathbb{Z}[x_1, \dots, x_m] \rightarrow \frac{\mathbb{Z}[x_1, \dots, x_m]}{\langle x_j = 0, j \notin \sigma \rangle} = \mathbb{Z}[x_i \in \sigma], \quad u_i \mapsto \sum_{j=1}^m B_{ij}x_j \mapsto \sum_{j \in \sigma} B_{ij}x_j.$$

Let $C := B_\sigma^{-1}$. By $C \cdot B_\sigma = I_n$, the k -th row denoted by λ_{i_k} is perpendicular to ρ_{i_h} , $h \neq k$ and $\langle \lambda_{i_k}, \rho_{i_k} \rangle = 1$. This means that λ_{i_k} is the primitive integral vector positively generating the edge defined by $\cap_{h \neq k} H_{i_h}$. Also the image of x_{i_k} is given by λ_{i_k} as an element of $\text{span}_{\mathbb{Z}}\{u_1, \dots, u_n\}$. \square

GKM Description for toric orbifolds. The diagram in the proof of Theorem works for toric orbifolds. Let (Δ, \mathbf{b}) be a labeled polytope and $\mathcal{X}_{\Delta, \mathbf{b}} = [M_{\Delta, \mathbf{b}}/\mathbb{G}]$ the corresponding toric orbifold where $M_{\Delta, \mathbf{b}} := \mu_{\mathbb{G}}^{-1}(A^*(\vec{\eta}))$. Let $X_{\Delta, \mathbf{b}}$ be the underlying topological space of $\mathcal{X}_{\Delta, \mathbf{b}}$. Under the identification $M_{\Delta, \mathbf{b}} \cong \mathcal{Z}_{K_\Delta}$, let $F_{\sigma_v} := \mathcal{Z}_{\sigma_v}$ where $v := \cap_{i \in \sigma_v} H_i$ is a vertex of Δ . Let $p_v := F_{\sigma_v}/\mathbb{G}$ be the fixed point in $X_{\Delta, \mathbf{b}}^{\mathbb{R}}$ corresponding to the vertex v . Consider

$$ER \times_{\mathbb{R}} X_{\Delta, \mathbf{b}} \xleftarrow{f} ER \times_{\mathbb{R}} (ET \times_{\mathbb{G}} M_{\Delta, \mathbf{b}}) \xrightarrow{g} ET \times_{\mathbb{T}} M_{\Delta, \mathbf{b}}$$

where g is the projection to 2nd and 3rd factors and f is the projection to 1st factor. The fiber of g is contractible so its pullback is an isomorphism and the fiber of f is a quotient of contractible space by a finite group, so its pullback is an isomorphism only if we work over \mathbb{Q} . Consider the following diagram

$$\begin{array}{ccc} H_{\mathbb{R}}^*(X_{\Delta, \mathbf{b}}; \mathbb{Z}) & \longrightarrow & \bigoplus_{v: \text{vertex}} H_{\mathbb{R}}^*(p_v; \mathbb{Z}) \\ \downarrow f^* \circ (g^*)^{-1} & & \downarrow f^* \circ (g^*)^{-1} \\ H_{\mathbb{T}}^*(\mathcal{Z}_{K_\Delta}; \mathbb{Z}) & \xrightarrow{\iota^*} & \bigoplus_{v: \text{vertex}} H_{\mathbb{T}}^*(\mathcal{Z}_{\sigma_v}; \mathbb{Z}) \\ \parallel \text{def} & & \parallel \text{def} \\ H_{\mathbb{R}}^*(X_{\Delta, \mathbf{b}}; \mathbb{Z}) & \longrightarrow & H_{\mathbb{R}}^*([F_{\sigma_v}/\mathbb{G}]; \mathbb{Z}) \end{array}$$

where $\iota : \mathcal{Z}_{\sigma_v} \hookrightarrow \mathcal{Z}_K$ is the obvious inclusion. Consider the following map

$$\Phi : \mathbb{Z}[x_1, \dots, x_m] \rightarrow \bigoplus_{\sigma: \text{facet of } K} \frac{\mathbb{Z}[x_1, \dots, x_m]}{\langle x_i = 0, i \notin \sigma \rangle}, \quad x_i \mapsto (x_i)_\sigma.$$

We can show that Φ factor through ι^* . We can also show that the kernel of Φ is exactly the Stanley-Reisner ideal. It follows that ι^* is injective. By the same argument as in the proof of Theorem 5.2, we can show that x_i can be mapped to λ_k , but in this case only with \mathbb{Q} -coefficients.

5.3. Class of \mathbb{T} -invariant submanifolds. Let M be a manifold with a \mathbb{T} -action. Let N be a \mathbb{T} -invariant submanifold with codimension $2r$ and let $f : N \hookrightarrow M$ be the inclusion. Suppose that its normal bundle ν_N is oriented. The class $[N]_{\mathbb{T}}$ of N in $H_{\mathbb{T}}^{2r}(M; \mathbb{Z})$ is defined by the image of 1_N by

$$H_{\mathbb{T}}^*(N; \mathbb{Z}) \xrightarrow{\text{Thom}} H_{\mathbb{T}}^{*+2r}(\nu_N, \nu_N - N; \mathbb{Z}) \xleftarrow{\cong, \text{excision}} H_{\mathbb{T}}^{*+2r}(M, M - N; \mathbb{Z}) \xrightarrow{\text{pullback}} H_{\mathbb{T}}^{*+2r}(M; \mathbb{Z}).$$

This composition $f_* : H_{\mathbb{T}}^*(N; \mathbb{Z}) \rightarrow H_{\mathbb{T}}^*(M; \mathbb{Z})$ is the cohomology push-forward map and is a homomorphism of modules over $H_{\mathbb{T}}^*(M; \mathbb{Z})$ whose action on $H_{\mathbb{T}}^*(N; \mathbb{Z})$ is given by the

pullback map $f^* : H_{\mathbb{T}}^*(M; \mathbb{Z})$:

$$f_*(\alpha_N \cup f^* \beta_M) = f_* \alpha_N \cup \beta_M \quad (\text{projection formula}).$$

Furthermore, $f^* f_* 1_N = f^*[N]_{\mathbb{T}} = \text{Euler}_{\mathbb{T}}(\nu_N)$.

Theorem 5.7. *The class $[X_i]_{\mathbb{T}}$ of the toric submanifold $X_i := \mu_{\mathbb{R}}^{-1}(H_i) \subset X_{\Delta}$ is the class represented by $x_i \in \mathbb{Z}[K]$. In particular, the (equivariant) cohomology is generated by the class of those submanifolds.*

Proof. Let $v \in H_i$ be a vertex and p_v the corresponding fixed point in X_i . The local normal form (5.1) and Lemma 5.1 implies that the weight of \mathbb{R} -action on $\nu_{X_i}|_{p_v}$ is λ_i which is the outward primitive integral vector spanning the unique edge coming out of v and not in H_i . Therefore the pullback of $\text{Euler}_{\mathbb{T}}(\nu_N)$ to p_v is λ_i . Since the pullback of $[X_i]_{\mathbb{T}}$ to X_i is $\text{Euler}_{\mathbb{T}}(\nu_N)$ and by the naturality of the pullback, the pullback of $[X_i]_{\mathbb{T}}$ to p_v is λ_i . The claim now follows from Theorem 5.2. \square

5.4. Cartan Model and the class of symplectic form. Let M be a manifold with a compact Lie group \mathbb{G} -action. The de Rham model for the equivariant cohomology of the \mathbb{G} -action on M is called the Cartan complex and defined as follows: let $\text{Sym}^{\bullet}[\mathfrak{g}^*]$ be the symmetric algebra on \mathfrak{g}^* (polynomial functions on \mathfrak{g}) and let $(\Omega^{\bullet}(M), d)$ be the deRham complex on M and let $\Omega(M)^{\mathbb{G}}$ be the \mathbb{G} -invariant forms on M . The tensor product $\mathbb{R}[\mathfrak{g}^*] \otimes \Omega(M)$ is a \mathbb{G} -module via the coadjoint action on \mathfrak{g}^* and the induced action on the differential form. The Cartan model is the cochain complex

$$\Omega_{\mathbb{G}}^n(M) := \sum_{i \in \mathbb{Z}_{\geq 0}} (\text{Sym}^i(\mathfrak{g}^*) \otimes \Omega^{n-2i}(M))^{\mathbb{G}}$$

with the differential $d_{\mathbb{G}}(\alpha \otimes \omega) := \alpha \otimes d\omega + \sum_k u_k \alpha \otimes i_{\xi_k} \omega$ where $\{\xi_k\}$ is the basis of \mathfrak{g} and $\{u_k\}$ dual basis of \mathfrak{g}^s , ξ_k^M is the infinitesimal action (fundamental vector field) for each $\xi_k \in \mathfrak{g}$. As an \mathbb{G} -equivariant polynomial function $d_{\mathbb{G}}(\alpha \otimes \omega) : \mathfrak{g}^* \rightarrow \Omega(M)$, we have

$$d_{\mathbb{G}}(\alpha \otimes \omega)(\xi) := \alpha(\xi)(d\omega + i_{\xi} \omega).$$

There is a natural projection $\Omega_{\mathbb{G}}^{\bullet}(M) \rightarrow \Omega(M)$ which commutes with the differentials. There is the natural isomorphism $H_{\mathbb{G}}^*(M; \mathbb{R}) \cong H^*(\Omega_{\mathbb{G}}^{\bullet}(M); d_{\mathbb{G}})$. Please refer, for example, Section 1.2 [21] for the more detailed properties of Cartan models (functoriality, etc).

Class of Symplectic Form. Let (M, ω) be a Hamiltonian \mathbb{T} -manifold with a moment map $\mu : M \rightarrow \mathfrak{t}^*$. We can think of $\mu \in (\mathfrak{t}^* \otimes \Omega^0(M))^{\mathbb{T}}$ where we have used the \mathbb{T} -invariance of the moment map (see [21, Proposition 2.9]).

Lemma 5.8. $d_{\mathbb{G}}(\omega - \mu) = 0$. In particular, it defines a class $[\omega - \mu]_{\mathbb{T}} \in H_{\mathbb{G}}^*(M; \mathbb{R})$.

Proof. For $\xi \in \mathfrak{t}$, we have $d_{\mathbb{G}}(\omega - \mu)(\xi) = d\omega + i_{\xi} \omega - d(\mu^{\xi})$ where $\mu^{\xi}(p) = \langle \mu(p), \xi \rangle$ for $p \in M$. Thus $d_{\mathbb{G}}(\omega - \mu) = 0$ iff $d\omega = 0$ and $i_{\xi} \omega = d(\mu^{\xi})$. \square

Lemma 5.9. *The equivariant class $[\omega - \mu]_{\mathbb{T}}$ pulls back to $-\mu(p) \in \text{Sym}[\mathfrak{t}^*]$ at a \mathbb{T} -fixed point $p \in M$.*

Proof. The natural pullback of the Cartan model maps $\omega - \mu$ to $-\mu(p)$. Note that pulling back a positive degree differential form to a point is always zero. \square

Corollary 5.10. *Let X_{Δ} be a toric manifold with the torus \mathbb{R} -action. Then $[\omega - \mu_{\mathbb{R}}]_{\mathbb{R}}$ is given by*

$$(-\mu_{\mathbb{R}}(p_v))_{v: \text{vertex}} \in \bigoplus_{v: \text{vertex}} H^*(\mathbb{B}\mathbb{R}; \mathbb{R}).$$

in the GKM description.

Lemma 5.11. *Let Δ be a Delzant polytope with facets H_1, \dots, H_m , which is given by the inequalities*

$$\langle \vec{u}, \rho_i \rangle + \eta_i \geq 0, \quad i = 1, \dots, m,$$

where ρ_i is the primitive inward normal vector to H_i . For a vertex $v = H_{i_1} \cap \dots \cap H_{i_n}$ of Δ , let $\lambda_{i_1}^v, \dots, \lambda_{i_n}^v$ be the primitive outward generator of the edge coming out of v where λ_{i_k} corresponds to the unique edge not in H_{i_k} . Then

$$(5.2) \quad v = - \sum_{k=1}^n \eta_{i_k} \lambda_{i_k}^v.$$

Proof. A vertex v is the unique solution to $\langle v, \rho_{i_k} \rangle + \eta_{i_k} = 0$ for $k = 1, \dots, n$. Recall that $\lambda_{i_k}^v$ is given by

$$(5.3) \quad \begin{bmatrix} - & \lambda_{i_1}^v & - \\ & \vdots & \\ - & \lambda_{i_n}^v & - \end{bmatrix} \begin{bmatrix} | & & | \\ \rho_{i_1} & \cdots & \rho_{i_n} \\ | & & | \end{bmatrix} = I_n$$

Thus (5.2) satisfies the first set of equations. \square

Remark 5.12. This lemma immediately generalizes to a simple rational polytope. In that case, $\lambda_{i_k}^v$ is also given by (5.3) and is the unique rational vector such that lies on the edge not contained in H_{i_k} and $\langle \lambda_{i_k}^v, \rho_{i_k} \rangle = 1$.

Corollary 5.13. *The class $[\omega - \mu_R]_{\mathbb{R}} \in H_{\mathbb{R}}^*(\mathcal{X}_{\Delta}; \mathbb{R})$ is represented as $-\sum_{i=1}^m \mu_i x_i$ in $\mathbb{R}[K_{\Delta}]$. Also the class $[\omega] \in H^*(\mathcal{X}_{\Delta}; \mathbb{R})$ is represented as $-\sum_{i=1}^m \mu_i x_i$ in $\mathbb{R}[K_{\Delta}]/\langle u_1, \dots, u_n \rangle$.*

6. LOCALIZATION THEOREM AND INTEGRATION FORMULA

In this section, we prove the localization theorem for a torus action on an orbifold. The main references are [3] and [28]. Let M be a smooth manifold with a smooth action of an m -dimensional torus T . Let G be an n -dimensional subgroup of T that acts on M locally freely, so that $R := T/G$ acts on the orbifold stack $[M/G]$. Let $B : T \rightarrow R$ be the quotient map. Let M_0 be the $(m - n)$ -dimensional orbits of T so that $[M_0/G]$ can be viewed as the sub-orbifold $[M/G]^R$ of R -fixed orbifold-points of $[M/G]$. Let $\iota : M_0 \hookrightarrow M$ be the inclusion which induces the inclusion of orbifolds $[M_0/G] \hookrightarrow [M/G]$. The goal of this section is to prove the following localization theorems for the orbifold $[M/G]$:

Theorem 6.1. *Assume that $[M/G]$ is compact. Then the kernel and the co-kernel of*

$$\iota^* : H_{\mathbb{R}}^*([M/G]; \mathbb{Z}) \rightarrow H_{\mathbb{R}}^*([M_0/G]; \mathbb{Z}), \quad \text{and} \quad \iota_* : H_{\mathbb{R}}^*([M_0/G]; \mathbb{Z}) \rightarrow H_{\mathbb{R}}^*([M/G]; \mathbb{Z})$$

are torsion-modules over $H^*(BR; \mathbb{Z})$.

As the \mathbb{R} -equivariant cohomology of $[M/G]$ is defined as $H_{\mathbb{R}}^*(M)$ (Definition 2.10), we will work in the world of T -equivariant geometry of manifolds, so Theorem 6.1 becomes Theorem 6.7 and 6.8.

6.1. Preparation. If $x \in M_0$, then $\dim T_x = n$ where T_x is the stabilizer of x in T . On the other hand, if $x \notin M_0$, then $\dim T_x < n$. Thus $B(T_x)$ is a proper subgroup of R .

Remark 6.2 (tubular neighborhood). For each $x \in M$, there is a neighborhood U of $T \cdot x \subset M$ such that there is a T -equivariant diffeomorphism

$$\phi_x : U \rightarrow T \times_{T_x} (T_x M / T_x(T \cdot x)).$$

where T_x is the stabilizer group of x in T .

If $x \in M_0$, then $R_x = R$ but if $x \notin M_0$, then R_x is a proper subgroup of R .

Lemma 6.3. Let K be a subgroup of T such that $B(K)$ is a proper subgroup of R . Suppose that there is a T -equivariant map $\psi : M \rightarrow T/K$. Then $H_T^*(M, \mathbb{Z})$ is a torsion-module over $H^*(BR; \mathbb{Z})$.

Proof. The action of $H^*(BK)$ is given by

$$H^*(BR) \longrightarrow H^*(BT) \longrightarrow H_T^*(T/K) = H^*(BK) \longrightarrow H_T^*(M)$$

The third term is given by $ET \times_T T/K = ET/K$. Since the image of K in R under the map B is a proper subgroup, there is non-trivial kernel of $H^*(BR) \rightarrow H^*(BK)$. Thus $H_T^*(M, \mathbb{Z})$ is a torsion-module over $H^*(BR)$. \square

Corollary 6.4. If $x \notin M_0$, then $H_T^*(T \times_{T_x} (T_x M / T_x(T \cdot x)); \mathbb{Z})$ is a torsion-module over $H^*(BR; \mathbb{Z})$.

Proof. There is a T -equivariant map $\psi : T \times_{T_x} (T_x M / T_x(T \cdot x)) \rightarrow T/T_x$ and since $x \notin M_0$, $B(T_x)$ is a proper subgroup of R . \square

Lemma 6.5. If $A \rightarrow B \rightarrow C$ is an exact sequence of $H^*(BR; \mathbb{Z})$ -modules and A and C are torsion-modules over $H^*(BR; \mathbb{Z})$, then B is a torsion-module over $\mathbb{Z}[R^*]$.

Proof. Let $b \in B$. Let $f \neq 0 \in \mathcal{R}$ such that $f \cdot \beta(b) = 0$. Then let $a \in A$ such that $\alpha(a) := f \cdot b$ and let $f' \neq 0 \in \mathcal{R}$ such that $f' \cdot a = 0$. Then $f' \cdot f \cdot b = 0$: $f' \cdot f \cdot b = f' \cdot \alpha(a) = \alpha(f' \cdot a) = 0$. \square

Proposition 6.6. Assume that M is compact. Then $H_T^*(M - M_0; \mathbb{Z})$ is a torsion-module over $H^*(BR; \mathbb{Z})$. Furthermore $H_T^*(M - U, \partial(M - U); \mathbb{Z})$ is a torsion-module over $H^*(BR; \mathbb{Z})$.

Proof. Let U be a T -stable tubular neighborhood of M_0 , which is T -equivariantly diffeomorphic to the normal bundle of M_0 . We can take a finite open covering of M by U and $U_i, i = 1, \dots, r$ where U_i is a tubular neighborhood of an orbit. We have that $\cup U_i \subset M - M_0$. Let $U_{1 \dots k} := \cup_{i=1}^k U_i$. Since we have the T -equivariant map $U_{1 \dots k} \cap U_{k+1} \hookrightarrow U_{k+1}$, $H_T^*(U_{1 \dots k} \cap U_{k+1})$ is also a torsion-module over $H^*(BR; \mathbb{Z})$. The Mayer-Vietoris gives an exact sequence

$$H_T^*(U_{1 \dots k} \cap U_{k+1}) \rightarrow H_T^*(U_{1 \dots k+1}) \rightarrow H_T^*(U_{1 \dots k}) \oplus H_T^*(U_{k+1})$$

By Lemma 6.5 and the induction on k , $H_T^*(U_{1 \dots r})$ is a torsion-module. Take U small enough so that there is a T -equivariant deformation retract of $X - U \subset X - X_0$. Then since $X - U \subset U_1 \cup \dots \cup U_r \subset X - X_0$, we can conclude that $H_T^*(M - M_0; \mathbb{Z})$ is also a torsion-module over $H^*(BR; \mathbb{Z})$. Cover $\partial(M - U)$ by $V_i := \partial(M - U) \cap U_i$ (open in the relative sense). Then $H_T^*(V_i)$ is a torsion-module since $V_i \rightarrow U_i$. By the same argument above, $H_T^*(\partial(M - U))$ is a torsion-module. By the relative cohomology long exact sequence, we can conclude that $H_T^*(M - U, \partial(M - U); \mathbb{Z})$ is a torsion-module over $H^*(BR; \mathbb{Z})$. \square

6.2. Localization Theorems.

Theorem 6.7 (Localization for pullback). *Let M be a connected compact oriented manifold with a smooth torus \mathbb{T} -action. Then the kernel and the co-kernel of $\iota^* : H_{\mathbb{T}}^*(M; \mathbb{Z}) \rightarrow H_{\mathbb{T}}^*(M_0; \mathbb{Z})$ are torsion-modules over $H^*(B\mathbb{R}; \mathbb{Z})$.*

Proof. Since $M_0 \subset U$ is a \mathbb{T} -equivariant deformation retract, so $H_{\mathbb{T}}^*(M, M_0) \cong H_{\mathbb{T}}^*(M - U, \partial(M - U))$. Now consider the relative cohomology sequence

$$\rightarrow H^*(M, M_0) \rightarrow H_{\mathbb{T}}^*(M) \rightarrow H_{\mathbb{T}}^*(M_0) \rightarrow H^{*+1}(M, M_0) \rightarrow$$

The kernel is the image of a torsion module and the co-kernel is a submodule of a torsion-module. Therefore the theorem holds. \square

Let $r := \text{codim } M_0 \subset M$. Define the pushforward $\iota_* : H_{\mathbb{T}}^*(M_0; \mathbb{Z}) \rightarrow H_{\mathbb{T}}^*(M; \mathbb{Z})$ by the following maps³

$$H_{\mathbb{T}}^*(M_0; \mathbb{Z}) \xrightarrow[\text{thom}]{\cong} H_{\mathbb{T}}^*(\nu_{M_0}, \nu_{M_0} \setminus M_0; \mathbb{Z}) \xleftarrow[\text{exc}]{\cong} H_{\mathbb{T}}^*(M, M \setminus M_0; \mathbb{Z}) \xrightarrow[\text{pullback}]{\cong} H_{\mathbb{T}}^*(M; \mathbb{Z}).$$

The image of 1 is called the Thom class and its pullback to M_0 is the Euler class of the bundle. By the functoriality of pullbacks, we have $\iota^* \iota_* 1 = \text{Euler}_{\mathbb{T}}(\nu_{M_0})$. The projection formula $\iota^*(\beta \cup \iota_* \alpha) = (\iota^* \beta) \cup \alpha$ (c.f. [20, B.1]) means that ι_* is a $H_{\mathbb{T}}^*(M; \mathbb{Z})$ -module homomorphism and it implies

$$(6.1) \quad \iota^* \iota_* \alpha = \alpha \cup \text{Euler}_{\mathbb{T}}(\nu_{M_0})$$

Theorem 6.8 (Localization for pushforward). *Let M be a connected compact oriented manifold with a smooth torus \mathbb{T} -action. Then the kernel and the co-kernel of $\iota_* : H_{\mathbb{T}}^*(M_0; \mathbb{Z}) \rightarrow H_{\mathbb{T}}^*(M; \mathbb{Z})$ are torsion-modules over $H^*(B\mathbb{R}; \mathbb{Z})$.*

Proof. Consider the long exact sequence of relative cohomology

$$\rightarrow H_{\mathbb{T}}^{*-1}(M - M_0) \rightarrow H^*(M, M - M_0) \xrightarrow{\star} H_{\mathbb{T}}^*(M) \rightarrow H_{\mathbb{T}}^*(M - M_0) \rightarrow$$

Replacing $H^*(M, M - M_0)$ by $H_{\mathbb{T}}^*(M_0; \mathbb{Z})$ so that the map \star becomes ι_* . The claim follows from the same argument in the previous proof. \square

6.3. Integration formula. The consequence of the localization theorem is that if we localize $H_{\mathbb{T}}^*(M; \mathbb{Z})$ and $H_{\mathbb{T}}^*(M_0; \mathbb{Z})$ as $H^*(B\mathbb{R}; \mathbb{Z})$ -modules, then ι_* and ι^* are actually isomorphisms. Note that if we localize those modules, we obtain $H_{\mathbb{T}}^*(M; \mathbb{Q}) \otimes_{\mathbb{Q}} H_{(\mathbb{R})}$ and $H_{\mathbb{T}}^*(M_0; \mathbb{Q}) \otimes_{\mathbb{Q}} H_{(\mathbb{R})}$ where $H_{(\mathbb{R})}$ is the fraction field of the polynomial ring $H^*(B\mathbb{R}; \mathbb{Q})$. It follows from $\iota^* \iota_* 1 = \text{Euler}_{\mathbb{T}} \nu_{M_0}$ (6.1) that $\text{Euler}_{\mathbb{T}} \nu_{M_0}$ is invertible in $H_{\mathbb{T}}^*(M_0; \mathbb{Q}) \otimes_{\mathbb{Q}} H_{(\mathbb{R})}$.

Now we have an equality $\iota_* \left(\frac{\iota^* \gamma}{\text{Euler}_{\mathbb{T}}(\nu_{M_0})} \right) = \gamma$ for all $\gamma \in H_{\mathbb{T}}^*(M; \mathbb{Q}) \otimes_{\mathbb{Q}} H_{(\mathbb{R})}$. More precisely

$$(6.2) \quad \gamma = \sum_F \iota_{F*} \left(\frac{\iota_F^* \gamma}{\text{Euler}_{\mathbb{T}}(\nu_F)} \right)$$

where F runs over all connected component of M_0 and $\iota_F : F \hookrightarrow M$ and ν_F is the normal bundle of $F \subset M$.

³Note that M_0 is again a compact oriented manifold (c.f. [2, Sec 5.2]). Since the normal bundle of a connected component F of M_0 has the action of $H_F \subset \mathbb{T}$ where H_F is the unique connected stabilizer of points in M_0 , it decomposes into complex line bundles of irreducible representations of H_F (recall that M_0 is the set of fixed dimensional orbits, so the complement stabilizer acts on the normal direction non-trivially so that the induced representations can be made into complex representation). This implies that the normal bundle has even rank and oriented. Now the Thom isomorphism can be found at [8, Theorem 11.3, p.368]. For equivariant version, we approximate ET and BT by S^{2n+1} and $\mathbb{C}\mathbb{P}^n$ (c.f. [9, Sec 1]).

((*Manifold Case*)). Suppose that $\mathbf{G} = \{1\}$ so that $\mathbf{T} = \mathbf{R}$. In this case, $M_0 = M^\top$. The integration \int_M of cohomology classes over M is just the pushforward $\pi_* : H_{\mathbf{T}}(M) \rightarrow H_{\mathbf{T}}(\text{pt})$ to a point (using approximation and poincare duality). By the functoriality of the pushforward maps, Equation (6.2) gives

Theorem 6.9 (Integration Formula).

$$\int_M \gamma = \sum_F \int_F \frac{\iota_F^* \gamma}{\text{Euler}_{\mathbf{T}}(V_F)} \quad \text{in } H_{(\mathbf{T})}.$$

((*Orbifold Case*)). In the case of orbifolds, it is unclear how to define the pushforward along $EG \times_{\mathbf{G}} M \rightarrow \text{pt}$. Never the less, using the de Rham model of cohomology of orbifolds, the integration can be defined (c.f. [1, Section 2.1] or [4]). In terms of this integration over an orbifold, we actually have the integration formula for more general orbifolds (See [35, Thm 2.1]).

Theorem 6.10.

$$\frac{1}{d_{[M/\mathbf{G}]}} \int_{M/\mathbf{G}} \gamma = \sum_F \frac{1}{d_{[F/\mathbf{G}]}} \int_{F/\mathbf{G}} \frac{\iota_F^* \gamma}{\text{Euler}_{\mathbf{T}}(V_F)} \quad \text{in } H_{(\mathbf{R})}$$

where $d_{[M/\mathbf{G}]}$ and $d_{[F/\mathbf{G}]}$ are the cardinalities of the global stabilizers.

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