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LYAPUNOV STABLE CLASSES AND RESIDUAL ATTRACTORS

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1. Main results

Let M be a compact smooth manifold without boundary. Denote by $\mathrm{Diff}^1(M)$ the set of C^1 diffeomorphisms on M, endowed with the C^1 topology.

Given a C^1 diffeomorphism $f : M \to M$, $\{x_i\}_{i=0}^n$ is called an ϵ-chain from x_0 to x_n if $d(f(x_i), x_{i+1}) < \epsilon, i = 0, 1, \cdots, n-1$. x is *chain-recurrent* if for any $\epsilon > 0$, there exists an ϵ-chain $\{x_i\}_{i=0}^n$ from x to x. Denote by $\mathrm{C}(f)$ the set of all chain-recurrent points of f. Given $x, y \in \mathrm{C}(f)$, x and y are *chain-equivalent* (denoted as $x \leftrightsquigarrow y$) if for any $\epsilon > 0$, there exists an ϵ-chain from x to y and an ϵ-chain from y to x. It is easy to see that $\leftrightsquigarrow$ is a closed equivalence relation over $\mathrm{C}(f)$, and $\mathrm{C}(f)$ is decomposed into the equivalent classes:

$$\mathrm{C}(f) = \bigcup_{\alpha} C_\alpha,$$

where $C_\alpha(f)$'s are *chain classes* of f.

Let P be a hyperbolic periodic orbit of f. Denote by $H(P, f) = H(P)$ the closure of the set of transverse homoclinic points of P, i.e.,

$$H(P) = \overline{W^s(P) \pitchfork W^u(P)}.$$

$H(P)$ is the *homoclinic class* of P. C^1 generically, $H(P)$ coincides with the chain class $C(P)$ containing P ([BC, GW]). If $C_\alpha(f)$ does not contain any periodic orbit of f, it is called an *aperiodic class*.

Let Λ be a compact invariant set of f. Λ is *Lyapunov stable* if for any neighborhood U of Λ, there exists a neighborhood V of Λ s.t. $\forall n \geq 0$, $f^n V \subset U$. Λ is a *residual attractor* of f if Λ is chain-transitive (for any $\epsilon > 0$, and any $x, y \in \Lambda$, there exists an ϵ-chain in Λ from x to y) and there exists a neighborhood U of Λ and a residual set $R \subset U$ s.t. $f(\overline{U}) \subset \mathrm{Int}U$ and $\forall x \in R$, $\omega(x) \subset \Lambda$.

Lyapunov stable chain class is called Lyapunov Stable Class (LSC). By using the result in [MP], one can prove that C^1 generically, a LSC C is a residual attractor if there exists a neighborhood U of C s.t. C is the unique LSC contained in U.

Now, we can state the main results of the paper.

Theorem A. *C^1 generically, every LSC C admitting a partially hyperbolic splitting*

$$T_C M = E^s \oplus E^c \oplus E^u, \quad \dim E^c = 1,$$

is a residual attractor.

Theorem B. *For C^1 generic $f \notin \overline{\mathrm{HT}(M^3)}$, every LSC of f is a residual attractor. Here $\mathrm{HT}(M)$ denote the set of diffeomorphisms on M exhibiting homoclinic tangency.*

We will use many (but finite) C^1 generic properties, one of which is:

Generic Property: *For any hyperbolic periodic orbit P such that*

$$T_P M = E^{ss} \oplus E^c \oplus E^{uu}, \quad \dim E^c \geq 1,$$

we have

$$W^{ss}(P) \cap W^{uu}(P) = \{P\}.$$

This is a Kupka-Smale type genericity.

2. Motivation and background

Recently, Bonatti, Li and Yang ([BLY]) constructed a local generic diffeomorphisms without attractors. Precisely,

Theorem (Bonatti, Li & Yang) *There exists a non-empty open set $\mathcal{U} \subset \mathrm{Diff}^1(M)(\dim M \geq 3)$ and a residual set $\mathcal{R} \subset \mathcal{U}$ s.t. $\forall f \in \mathcal{R}$, f has* no attractor.

Their example suggested that the notation "attractor" may be too strong for general diffeomorphisms and then they introduced the concept "residual attractor" and asked if generic diffeomorphism contains at least one residual attractor. We prove the main results while trying to give an answer to their question.

Another background is the study of diffeomorphisms far away from homoclinic tangencies. Bonatti formed the following important and ambitious conjecture:

Conjecture (Bonatti) *C^1 generic $f \notin \overline{\mathrm{HT}(M)}$ is* tame, *i.e., f has only finite many homoclinic classes.*

Because of the importance of this conjecture, we would like to mention some important progress in studying Bonatti's conjecture.

Pujals and Sambarino ([PS]) proved that for surface diffeomorphisms, the conjecture is true. For higher dimensional diffeomorphisms, Wen ([W02]) proved that $\overline{P_i}$ has an i-dominated splitting. D. Yang, Gan and Wen ([YGW]) proved that every minimally nonhyperbolic set Λ is index-complete, and there exists a partially hyperbolic splitting:

$$T_\Lambda M = E^s \oplus E_1^c \oplus \cdots \oplus E_k^c \oplus E^u,\ \dim E_i^c = 1.$$

This generalized a previous result of Wen ([W04]) about diffeomorphisms far away from homoclinic tangencies and heterodimensional cycles. Bonatti, Gan and Wen ([BGW]) proved that for C^1 generic 3-dimensional diffeomorphism f far away from homoclinic tangencies, every LSC is a homoclinic class. J. Yang ([Y07-a]) generalized this result to general diffeomorphisms, and he ([Y07-b]) proved that for generic $f \notin \overline{\mathrm{HT}(M)}$, every aperiodic class admits a splitting $T_C M = E^s \oplus E^c \oplus E^u$, with $\dim E^c = 1$.

Both Theorem A and Theorem B are proved by analyzing the dynamics of 1-dimensional central manifolds. The difference is that, in proving Theorem A, we use the technique developed in [BGW], while in proving Theorem B, besides the result of Theorem A, we use the technique of central model developing by Crovisier in [Cr06-p].

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