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GKM GRAPHS INDUCED BY GKM MANIFOLDS WITH $SU(\ell+1)$-SYMMETRIES

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1. Introduction

A *GKM manifold* is a $2m$-dimensional manifold M^{2m} equipped with an effective T^n-action whose one and zero dimensional orbits have the structure of a graph, where $n \leq m$. Using the effectiveness of the action and the differentiable slice theorem around zero dimensional orbits (i.e., fixed points), we see that the induced graph of a $2m$-dimensional GKM manifold must be an *m-valent graph*, i.e., each vertex has just m outgoing edges. Moreover, each outgoing edge can be labeled by using the tangential representation (the slice representation around fixed points) of T-action, called an *axial function*. An m-valent graph labeled by an axial function is called a *GKM graph* (see Section 2 or [8, 12] for more detail).

In this article, we introduce some results in the forthcoming paper [16] which is devoted to studying the extended actions of GKM manifolds (or GKM manifolds with large symmetries). In particular, we introduce the results for GKM graphs induced by the GKM manifolds with $SU(\ell+1)$-symmetries under some assumptions, motivated by the Masuda's work [19] and the Wiemeler's work [23].

1.1. **Motivation (extended actions of torus manifolds).** One of the typical class of GKM manifolds is the case when $m = n$; this class is called a *torus manifold* (see [9]). For example, the following manifolds are examples of torus manifolds, i.e., they are also GKM manifold:

Example 1.1. Let $(t_1, \ldots, t_n) \in T^n$ be an element of the n-dimensional torus, and $(z_1, \ldots, z_n, r) \in S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}$ an element of the $2n$-dimensional sphere. We define T^n-action on S^{2n} as follows:

$$(z_1,\ \ldots,\ z_n, r) \longmapsto (t_1 z_1,\ \ldots,\ t_n z_n, r).$$

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Then its induced graph has two vertices and n edges which connect two vertices (we call this edge the *n-multiple edge*); Figure 1 shows the case when $n = 3$.

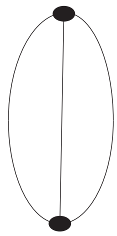

FIGURE 1. The graph of S^6 with T^3-action.

Example 1.2. We define the T^n-action on the complex projective space $\mathbb{C}P^n$ as follows:

$$[z_0 : z_1 : \cdots : z_n] \longmapsto [z_0 : t_1 z_1 : \cdots : t_n z_n].$$

where $[z_0 : \cdots : z_n] \in \mathbb{C}P^n = \mathbb{C}^{n+1} - \{o\}/\mathbb{C}^*$ and $(t_1, \ldots, t_n) \in T^n$. Then, its induced graph is the one-skeleton of the n-dimensional simplex, i.e., the complete graph with $n+1$ vertices; Figure 2 shows the case when $n = 3$.

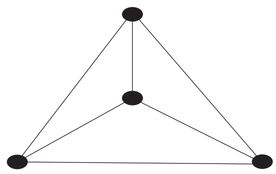

FIGURE 2. The graph of $\mathbb{C}P^3$ with T^3-action.

Note that both of the above examples are induced by the $U(n)$-action on $S^{2n} \cap \mathbb{C}^n$ (or $SO(2n)$-action on S^{2n} by regarding $\mathbb{C}^n \simeq \mathbb{R}^{2n}$) and the $PU(n+1)$-action on $\mathbb{C}P^n$, respectively. Here, $PU(n+1)$ is the projective unitary group defined by the quotient group $SU(n+1)/C$, where C is the center of $SU(n+1)$. Hence, we are naturally led to ask the following question:

Problem 1 (extended actions of torus manifolds)**.** Let (M, T) be a torus manifold and G a non-abelian group whose maximal torus is T. When does (M, T) extend to (M, G)? In other wards, characterize or classify torus manifolds with extended G-actions.

This problem has been studied by several mathematicians from several points of view; algebraic geometrical point of view [2, 3, 21] for toric varieties (we may regard torus manifolds as the generalization of non-singular toric varieties); and topological point of view [13, 14, 15, 19, 23]. In particular, one of the conclusions of Wiemeler's theorem [23] says that such a non-abelian, compact, connected group G is just a product of the three types of Lie groups; $SU(\ell+1)$, $SO(2\ell+1)$, $SO(2\ell)$, i.e., type A_ℓ, B_ℓ, D_ℓ.

1.2. **Extended actions of GKM manifolds.** As is well known, there are seven types of compact, connected, simple Lie groups denoted by A_ℓ, B_ℓ ($\ell > 1$), C_ℓ, D_ℓ ($\ell > 3$), E_ℓ ($\ell = 6, 7, 8$), F_4 and G_2 (see e.g. [20]). Therefore, the next problem is to find the class of T-manifolds with the other type of symmetries different from A_ℓ, B_ℓ, D_ℓ (the symmetries of the torus manifolds).

Fortunately, according to the results of Guillemin-Holm-Zara [4], all homogeneous spaces G/H with the natural, maximal torus T-actions are GKM manifolds, where H is a maximal rank subgroup of G, i.e., G and H have the same maximal torus. The followings are two of such examples.

Example 1.3. Because $T^{n+1} \subset Sp(n) \times Sp(1) \subset Sp(n+1)$ where $Sp(\ell)$ is the symplectic Lie group (type C_ℓ ($\ell > 1$)), it follows from the results in [4] that $Sp(n+1)/Sp(n) \times Sp(1)$ with $T^{n+1} \subset Sp(n+1)$ action is a GKM manifold. As is well known, $Sp(n+1)/Sp(n) \times Sp(1)$ is T-equivariantly diffeomorphic to the quaternionic projective space $\mathbb{H}P^n$. We can easily compute that its induced graph is the one-skeleton of the n-dimensional simplex such that every edge is the 2-multiple edge. Figure 3 shows the induced graph of the case when $n = 2$.

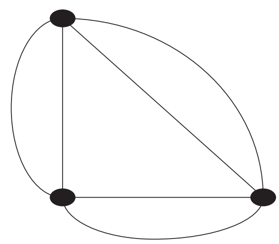

FIGURE 3. The graph of $Sp(3)/Sp(2) \times Sp(1)$ with T^3-action.

Example 1.4. Because $T^2 \subset S(U(2) \times U(1)) \subset SU(3) \subset G_2$ where G_2 is the exceptional Lie group of type G_2, it follows from the results in [4] that $G_2/SU(3)$ and $G_2/S(U(2) \times U(1))$ with $T^2 \subset G_2$ actions are GKM manifolds. We can easily compute that the induced graph of $G_2/SU(3)$ is also Figure 1 (as is well known, $G_2/SU(3) \cong S^6$), and the induced graph of $G_2/S(U(2) \times U(1))$ is the graph illustrated in Figure 4.

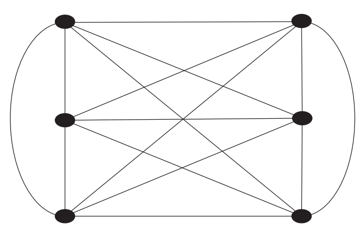

FIGURE 4. The graph of $G_2/S(U(2) \times U(1))$ with T^2-action.

From the results of Guillemin-Holm-Zara [4], one can easily expect that all types of compact, connected, simple Lie groups are appeared as extended actions of GKM manifolds. Hence, it might be interesting to ask Problem 1 for not only torus manifolds but also GKM manifolds in general, i.e.,

Problem 2 (extended actions of GKM manifolds)**.** Let (M, T) be a GKM manifold and G a non-abelian group whose maximal torus is T. When does (M, T) extend to (M, G)? In other wards, characterize or classify the GKM manifolds with extended G-actions.

In order to study this problem, as the first step, we study GKM manifolds with extended G-actions under some assumptions.

1.3. **Assumptions in this article.** Let Γ be the induced m-valent graph of the GKM manifold (M^{2m}, T^n). We may identify the fixed points of (M^{2m}, T^n) with the vertices of Γ. Moreover, we can label each outgoing edge of Γ around vertex p by its tangential representation, called an *axial function* $\mathcal{A} : E \to H^2(BT^n)$, where E is the set of oriented edges[1] of Γ. This labeled graph $(\Gamma, \mathcal{A})$ is called a *GKM graph* induced by the GKM manifold. On the other hand, a *GKM graph* can be defined abstractly by the labeled graph $(\Gamma, \mathcal{A})$ which satisfies some properties of GKM graphs induced by GKM manifolds (see Section 2 or [4, 7, 8, 12, 17] for detail).

The goal of this article is to introduce a property of GKM graphs induced by GKM manifolds with extended G-actions which satisfy the following conditions (see Theorem 4.1):

(1) a GKM manifold M^{2m} has an almost complex structure which is compatible with the T^n-action;
(2) G preserves almost complex structure $\mathcal{J}$ on M, i.e., $G \subset \mathrm{Diff}(M, \mathcal{J})$;
(3) the universal covering $\widetilde{G}$ of G has the $SU(\ell + 1)$-factor;
(4) there are codimension two characteristic submanifolds (GKM submanifolds) in (M, T); we denote the set of them by $\mathfrak{F} = \{X_1, \ldots, X_k\}$ and their orientations are induced by the T-invariant almost complex structure on M.

Example 1.5. The complex flag manifold $SU(n + 1)/T^n$ with T^n-action satisfy all conditions as above. Figure 5 shows the induced graph from $SU(3)/T^2$ with T^2-action.

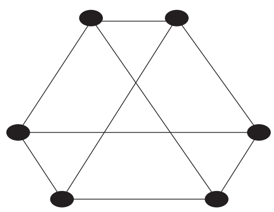

FIGURE 5. The graph of $SU(3)/T^2$ with T^2-action.

[1] In E, we distinguish the two same edges pq and qp by regarding that their orientations are different.

Remark 1.6. The manifold $Sp(n+1)/Sp(n) \times Sp(1)$ with $Sp(n+1)$-symmetry in Example 1.3 does not satisfy any condition as above. The manifold $G_2/S(U(2) \times U(1))$ with G_2-symmetry in Example 1.4 satisfies the conditions as above except (3). The manifold $G_2/SU(3)$ with G_2-symmetry in Example 1.4 satisfies the conditions as above except (3), (4).

The organization of this article is as follows. In Section 2, we recall the basic notations of GKM graphs. In Section 3, we introduce some basic facts of GKM manifolds with extended $SU(\ell+1)$-actions under some conditions. In Section 4, we introduce the main theorem (Theorem 4.1) of this article. In Section 5, we give some remarks for the main theorem.

2. Definition of GKM graphs

In this section, we recall basic notations of GKM graphs (see e.g. [4, 8, 11, 12] for detail).

We let $V(\Gamma)$ and $E(\Gamma)$ denote the set of vertices and edges of Γ, respectively (we distinct two edges pq and qp). Let $E_p(\Gamma)$ be the set of all outgoing edges from the vertex p. By the assumption (1) in Section 1, the GKM graph $(\Gamma, \mathcal{A})$ induced by the $2m$-dimensional GKM manifold has the following properties (see [8] for detail):

- Γ is an m-valent graph, i.e., $|E_p(\Gamma)| = m$ for every vertex p;
- The axial function $\mathcal{A} : E(\Gamma) \to H^2(BT)$ satisfies the following properties:
  1. $\mathcal{A}(e) = -\mathcal{A}(\bar{e})$, where e and $\bar{e}$ are the same edge but their orientations are different, e.g., if $e = pq$ then $\bar{e} = qp$;
  2. $\{\mathcal{A}(e_i) \mid e_i \in E_p(\Gamma)\}$ is pairwise linearly independent, i.e., $\mathcal{A}(e_i)$ and $\mathcal{A}(e_j)$ are linearly independent if $e_i \neq e_j$;
- if two vertices p and q are connected by an edge (called f), there is a bijective map $\triangledown_f : E_p(\Gamma) \to E_q(\Gamma)$ such that
  1. $\triangledown_{\bar{f}} = \triangledown_f^{-1}$,
  2. $\triangledown_f(f) = \bar{f}$, and
  3. $\mathcal{A}(e) - \mathcal{A}(\triangledown_f(e)) \equiv 0 \pmod{\mathcal{A}(f)}$ for $e \in E_p(\Gamma)$;

  the collection of maps $\triangledown = \{\triangledown_f \mid f \in E(\Gamma)\}$ is called a *connection* of $(\Gamma, \mathcal{A})$.

On the other hand, if the given labeled graph $(\Gamma, \mathcal{A})$, where $\mathcal{A} : E(\Gamma) \to H^2(BT^n)$, satisfies the properties above, then we call $(\Gamma, \mathcal{A})$ a *GKM graph* in this article.

Figure 6 shows an example of the GKM graph.

Here, the labels on outgoing edges in Figure 6 represent that $\mathcal{A}(pq) = \alpha$, $\mathcal{A}(qr) = -\alpha + \beta$, $\mathcal{A}(rp) = -\beta$, etc. It is easy to check that $(\Gamma, \mathcal{A})$ of Figure 6 satisfies the properties above.

We also introduce other important notions of GKM graphs. Let $H_T^*(\Gamma, \mathcal{A})$ be the following set:

$$\{f : V(\Gamma) \to H^*(BT) \mid f(p) - f(q) \equiv 0 \mod \mathcal{A}(pq)\},$$

where pq represents the edge connects two vertices p and q. Then, the set $H_T^*(\Gamma, \mathcal{A})$ has the graded ring structure induced by $H^*(BT)$. We call $H_T^*(\Gamma, \mathcal{A})$ a *graph equivariant cohomology* (also see [4, 7, 8, 11, 12]).

Let Γ' be an $(m-h)$-valent GKM subgraph of Γ. The symbol $N_p(\Gamma')$ represents the set of all normal edges of Γ' on $p \in V(\Gamma')$. Then we may define the element

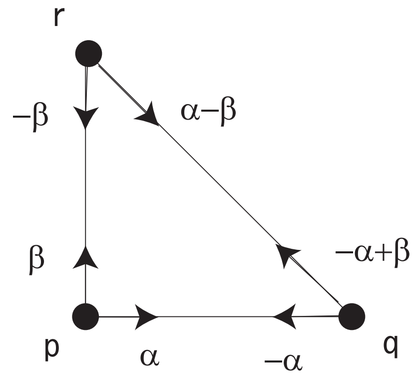

FIGURE 6. The 2-valent GKM graph. Here, α, β are the generators in $H^2(BT^2)$.

$\tau' \in H_T^{2h}(\Gamma, \mathcal{A})$ by

$$\tau'(p) = \begin{cases} \prod_{e \in N_p(\Gamma')} \mathcal{A}(e) & p \in V(\Gamma') \\ 0 & p \notin V(\Gamma') \end{cases}$$

We call τ' the *Thom class* of Γ'. Figure 7 shows examples of Thom classes of GKM subgraphs in Figure 6

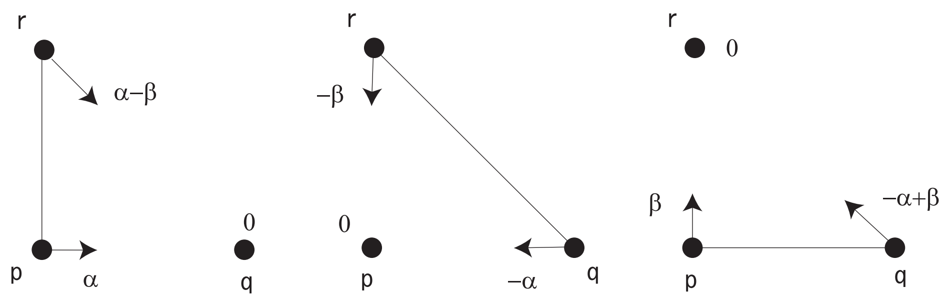

FIGURE 7. The Thom classes of 1-valent GKM subgraphs pr, qr, pq.

3. Basic properties of GKM manifolds with $SU(\ell+1)$-symmetries

Assume that the GKM manifold (M^{2m}, T^n) equipped with an extended G-action satisfying all the assumptions (1)–(4) mentioned in Section 1. Then, we may assume $\widetilde{G} = SU(\ell+1) \times G'$, where G' is a product of compact, connected, simple Lie groups and torus (see [20]). Let $\mathcal{W}_\ell$ be the Weyl group of $SU(\ell+1)$, i.e., $\mathcal{W}_\ell \simeq S_{\ell+1}$ (the symmetric group of degree $\ell+1$). Then, there is the effective $\mathcal{W}_\ell$-action on the codimension two characteristic submanifolds $\mathfrak{F} = \{X_1, \ldots, X_k\}$ induced by the $SU(\ell+1)$-action on M.

The element $\tau_i \in H_T^2(M)$, $i = 1, \ldots, k$, represents the equivariant Thom class of codimension two characteristic submanifold $X_i \in \mathfrak{F}$ (see e.g. [18]). We denote the set of such equivariant Thom classes by $\mathfrak{F}^* \subset H_T^2(M)$. Then, there is the $\mathcal{W}_\ell$-action on $\mathfrak{F}^*$ induced by the $\mathcal{W}_\ell$-action on $\mathfrak{F}$. Let $\mathfrak{X}$ be an orbit of the $\mathcal{W}_\ell$-action on $\mathfrak{F}$. Then we have

Lemma 3.1. *The cardinality of $\mathfrak{X}$ is 1 or $\ell+1$.*

On the other hand, there are simple roots $\alpha_1, \ldots, \alpha_\ell \in \mathfrak{t}^* \simeq H^2(BT^\ell;\mathbb{R})$ of $SU(\ell+1)$ (e.g. see [20, Chapter 5] for the basic facts of root systems). As is well-known, the simple root corresponds to the generators (transpositions) in $\mathcal{W}_\ell$; we let $\sigma_i \in \mathcal{W}_\ell$ denote the reflection through the hyperplane perpendicular to the simple root α_i $(i = 1, \ldots, \ell)$, i.e., $\sigma_i(\alpha_i) = -\alpha_i$ and $\sigma_i : \alpha_i^\perp \to \alpha_i^\perp$ is the identity where $\alpha_i^\perp \subset \mathfrak{t}^* \simeq H^2(BT;\mathbb{R})$ is the hyperplane perpendicular to α_i. Let $\pi : ET \times_T M \to BT$ be the projection of the Borel construction of (M,T), and $\pi^* : H^*(BT) \to H_T^*(M)$ be the induced homomorphism. The image of simple roots by π^* satisfies the following property.

Proposition 3.2. *Assume the cardinality of $\mathfrak{X}$ is $\ell+1$. Then, there is the order on $\mathfrak{X}$, say $\mathfrak{X} = \{X_1, \ldots, X_{\ell+1}\}$, such that $\sigma_i(X_i) = X_{i+1}$ and $\sigma_i(X_j) = X_j$ for $j \neq i, i+1$. Furthermore, the following equation holds:*

$$\pi^*(\alpha_i) = \tau_i - \tau_{i+1},$$

where α_i is the simple root which corresponds to σ_i and τ_i is the equivariant Thom class of X_i.

Moreover, we have the following proposition.

Proposition 3.3. *Let $\mathfrak{X}^* \subset H_T^2(M)$ be the equivariant Thom classes of $\mathfrak{X}$. Then $\mathfrak{X}^*$ is linearly independent in $H_T^2(M)$.*

4. Main theorem

Let $(\Gamma, \mathcal{A})$ be an abstract GKM graph, and $H_T^*(\Gamma, \mathcal{A})$ its graph equivariant cohomology.

Before we state the main theorem, we prepare some notations.

4.1. Preparation I, GKM-fibration.

We first recall the GKM fibration in [7].

Let Γ and B be connected graphs and $\rho : \Gamma \to B$ be a morphism of graphs. Hence ρ is a map from the vertices of Γ to the vertices of B such that if $pq \in E(\Gamma)$ then either $\rho(p) = \rho(q)$ or else $\rho(p)\rho(q) \in E(B)$. If $pq \in E(\Gamma)$ and $\rho(p) = \rho(q)$ then we will say that the edge pq is *vertical*, and if $\rho(p)\rho(q) \in E(B)$ then we will say that the edge pq is *horizontal*. For a vertex $q \in V(\Gamma)$, let $E_q^\perp(\Gamma)$ be the set of vertical edges with initial vertex q, and let $H_q(\Gamma)$ be the set of horizontal edges with initial vertex q. Then $E_q(\Gamma) = E_q^\perp(\Gamma) \cup H_q(\Gamma)$ and ρ induces canonically a map

$$(d\rho)_q : H_q(\Gamma) \to E_{\rho(q)}(B)$$

from the horizontal edges at q to the edges of B with initial vertex $\rho(q)$: if $qq' \in H_q(\Gamma)$, then $(d\rho)_q(qq') = \rho(q)\rho(q')$.

Definition. The morphism of graphs $\rho : \Gamma \to B$ is a *fibration* of graphs if for every vertex q of Γ, the map $(d\rho)_q : H_q(\Gamma) \to E_{\rho(q)}(B)$ is bijective.

Let us define the GKM fibration in [7].

Definition. A map $\rho : (\Gamma, \mathcal{A}) \to (B, \mathcal{A}_B)$ is a $(\triangledown, \triangledown_B)$-*GKM fibration*, where $\triangledown$ and $\triangledown_B$ are connections on $(\Gamma, \mathcal{A})$ and $(B, \mathcal{A}_B)$ respectively, if it satisfies the following conditions:

(1) ρ is a fibration of graphs;
(2) If e is an edge of B and $\widetilde{e}$ is any lift of e, then $\mathcal{A}(\widetilde{e}) = \mathcal{A}_B(e)$;
(3) Along every edge e of Γ, the connection $\bigtriangledown$ sends horizontal edges into horizontal edges and vertical edges into vertical edges;
(4) The restriction of $\bigtriangledown$ to horizontal edges is compatible with $\bigtriangledown_B$, in the following sense: Let $e = pq$ be an edge of B and $\widetilde{e} = p'q'$ the lift of e at p' (where $\rho(p') = p$, $\rho(q') = q$). Let $e' \in E_p(B)$ and $e'' = (\bigtriangledown_B)_e(e') \in E_q(B)$. If $\widetilde{e}'$ is the lift of e' at p' and $\widetilde{e}''$ is the lift of e'' at q', then $(\bigtriangledown)_{\widetilde{e}}(\widetilde{e}') = \widetilde{e}''$.

Figure 8 will show an example of the GKM-fibration.

4.2. **Preparation II, GKM blow-up.** We next introduce the GKM blow-up (see [8, 17]).

Let $(\Gamma', \mathcal{A}')$ be an $(m-\ell)$-valent GKM subgraph of the m-valent GKM graph $(\Gamma, \mathcal{A})$. By assumption, we have the cardinality of $N_p(\Gamma')$ is just ℓ and denote $N_p(\Gamma') = \{pp'_1, \dots, pp'_\ell\}$.

The blow-up of Γ along Γ', denoted $\widetilde{\Gamma}$, has vertex set $V(\widetilde{\Gamma}) = (V(\Gamma) - V(\Gamma')) \cup V(\Gamma)^\ell$, i.e., vertex $p \in V(\Gamma')$ is replaced by ℓ vertices $\widetilde{p}_1, \dots, \widetilde{p}_\ell$ It is convenient to regard those points as chosen close to p on edges from $N_p(\Gamma') = \{pp'_1, \dots, pp'_\ell\}$, i.e., $\widetilde{p}_i \in pp'_i$. (We also assume $\bigtriangledown_{pq}(pp'_i) = qq'_i$ if p and q are joined by an edge in Γ.) Then we have four types of edges in $\widetilde{\Gamma}$, and the corresponding values of the axial function $\widetilde{\mathcal{A}} : E(\widetilde{\Gamma}) \to H^2(BT)$:

(1) $\widetilde{p}_i\widetilde{p}_j \in E(\widetilde{\Gamma})$ for every $p \in V(\Gamma')$; $\widetilde{\mathcal{A}}(\widetilde{p}_i\widetilde{p}_j) = \mathcal{A}(pp'_j) - \mathcal{A}(pp'_i)$;
(2) $\widetilde{p}_i\widetilde{q}_i \in E(\widetilde{\Gamma})$ if $pq \in E(\Gamma')$; $\widetilde{\mathcal{A}}(\widetilde{p}_i\widetilde{q}_i) = \mathcal{A}(pq)$;
(3) $\widetilde{p}_i p'_i \in E(\widetilde{\Gamma})$ for every $p \in V(\Gamma')$; $\widetilde{\mathcal{A}}(\widetilde{p}_i p'_i) = \mathcal{A}(pp'_i)$;
(4) edges "coming from Γ", that is, $pq \in E(\Gamma)$ such that $p \notin V(\Gamma')$ and $q \notin V(\Gamma')$; $\widetilde{\mathcal{A}}(pq) = \mathcal{A}(pq)$.

Figure 9 will show an example of the GKM blow-up.

4.3. **Main theorem.** Now $\pi^* : H^*(BT^n) \to H^*_T(\Gamma, \mathcal{A})$ is defined by $\pi^*(x) = x$ $(x \in H^*(BT^n))$[2], i.e., the constant function $x : p \mapsto x$ for all vertices $p \in V(\Gamma)$.

Abstractly, we assume that $(\Gamma, \mathcal{A})$ satisfies the properties stated in Proposition 3.2 and 3.3. Namely, there exists an element $\alpha_i \in H^2(BT^n)$ $(i = 1, \dots, \ell)$ such that the following E.q. (4.1) holds.

$$\pi^* : \alpha_i \mapsto \tau_i - \tau_{i+1}, \tag{4.1}$$

where $\tau_1, \dots, \tau_{\ell+1} \in H^2_T(\Gamma, \mathcal{A})$ are linearly independent and they can be realized by Thom classes of some $(m-1)$-valent GKM subgraphs $\Gamma_1, \dots, \Gamma_{\ell+1}$, respectively ($\Gamma_i \neq \Gamma_j$ if $i \neq j$).

Now we may state the main theorem of this article.

Theorem 4.1. *Let $(\Gamma, \mathcal{A})$ be a GKM graph. Suppose that there are $\alpha_1, \dots, \alpha_\ell \in H^2(BT^n)$ which satisfy the conditions above. Then, one of the following cases occur:*

The 1^{st} case: *if $\tau_1 \cdots \tau_{\ell+1} = 0$, there is the GKM fibration $\rho : (\Gamma, \mathcal{A}) \to (K_{\ell+1}, \mathcal{A}_{\ell+1})$, where $V(K_{\ell+1}) = \{p_0, \dots, p_\ell\}$, $E(K_{\ell+1}) = \{p_ip_j \mid i \neq j\}$ and $\mathcal{A}_{\ell+1}$ is defined by $\mathcal{A}_{\ell+1}(p_0p_j) = \alpha_j$ and $\mathcal{A}_{\ell+1}(p_ip_j) = \alpha_j - \alpha_i$ for $i, j \neq 0$.*

[2]By using π^*, we may regard $H^*_T(\Gamma, \mathcal{A})$ as an $H^*(BT)$-algebra.

The 2^{nd} **case:** *otherwise, there is the GKM blow-up* $(\widetilde{\Gamma}, \widetilde{\mathcal{A}}) \to (\Gamma, \mathcal{A})$ *along* $\Gamma_1 \cap \cdots \cap \Gamma_{\ell+1}$ *such that* $(\widetilde{\Gamma}, \widetilde{\mathcal{A}})$ *satisfies the* 1^{st} *case.*

Figure 8 and 9 illustrate Theorem 4.1.

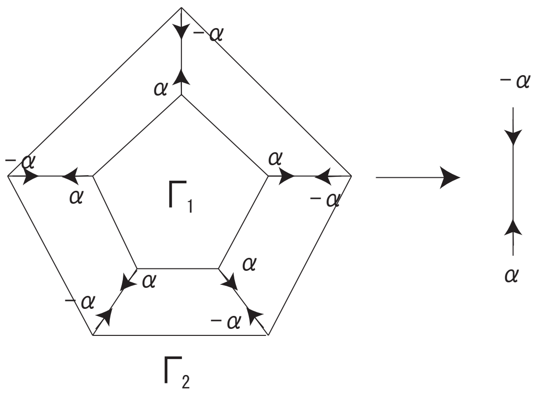

FIGURE 8. The 1^{st} case when a GKM fibration occurs. Two GKM subgraphs Γ_1 and Γ_2 (pentagons, where we omit their axial functions) satisfy $\tau_1 - \tau_2 = \alpha$ and $\tau_1\tau_2 = 0$ because $\Gamma_1 \cap \Gamma_2 = \emptyset$. Therefore, there is the GKM fibration such that Γ_1 and Γ_2 project onto bottom and top vertices of the right interval, respectively.

5. Remarks

Finally, in this section, we give some remarks.

5.1. **Geometric interpretation of Theorem 4.1.** Theorem 4.1 reminds us of the following Wiemeler's theorem (see [23], and also see [1]).

Theorem 5.1 (Wiemeler)**.** *Let* M^{2n} *be a torus manifold with* $SU(\ell+1)$*-symmetry. Then such manifold is made by the sequence of blow-downs of* $\widetilde{M} = SU(\ell + 1) \times_{S(U(\ell)\times U(1))} N$ *along the codimension-two submanifold* $\mathbb{C}P^{\ell} \times A$*, where* $A \cong M^{SU(\ell+1)}$ *and* N *is a* $(2n-2\ell)$*-dimensional torus manifold.*

Theorem 4.1 may be regarded as a generalization from combinatorial point of view of Theorem 5.1 in the following sense. The 1^{st} case may be regarded as the combinatorial generalization of that M is T^n-equivariantly diffeomorphic to the crossed product $SU(\ell + 1) \times_{S(U(\ell)\times U(1))} N$ for some $(2m - 2\ell)$-dimensional GKM manifold N. The 2^{nd} case may be regarded as the other case, i.e., M does not decompose into the crossed product; however, there is the codimension-$(2\ell + 2)$ GKM submanifold X such that there is the blow up $\widetilde{M} \to M$ along X and $\widetilde{M}$ is equivariantly diffeomorphic to the crossed product $SU(\ell + 1) \times_{S(U(\ell)\times U(1))} N$ for some N.

5.2. **Relation with root systems.** By using the identification $\mathfrak{t}^* \simeq H^2(BT^n;\mathbb{R})$, we define the inner product on $H^2(BT^n;\mathbb{R})$. Using this inner product, we can define the reflection $\sigma_\alpha : H^2(BT^n;\mathbb{R}) \to H^2(BT^n;\mathbb{R})$ through the hyperplane perpendicular to the element $\alpha \in H^2(BT^n;\mathbb{R})$. Then the following lemma holds:

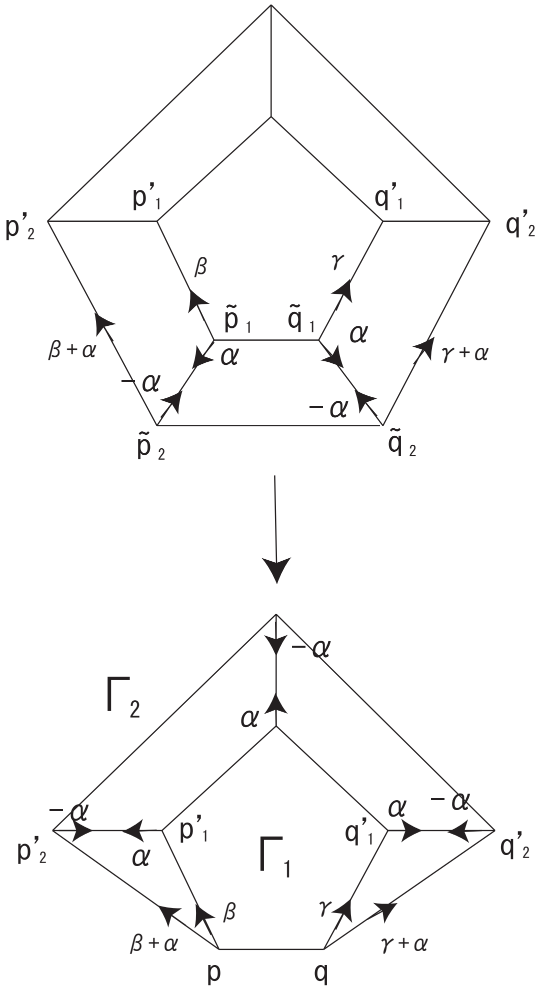

FIGURE 9. The 2^{nd} case when GKM blow-up occurs. In the bottom GKM graph, two GKM subgraphs Γ_1 and Γ_2 satisfy $\tau_1 - \tau_2 = \alpha$ but $\tau_1\tau_2 \neq 0$ because $\Gamma_1 \cap \Gamma_2 \neq \emptyset$. Therefore, there is the GKM blow-up (the top GKM graph) along $\Gamma_1 \cap \Gamma_2$.

Proposition 5.2. *Assume there are elements* $R = \{\alpha_1, \ldots, \alpha_\ell\} \subset H^2(BT^\ell) \subset H^2(BT^n)$ *which satisfy the conditions mentioned in Section 4.3. Let* $\Phi = \{\alpha \in H^2(BT^n;\mathbb{R}) \mid \pi^*(\alpha) = \tau_i - \tau_j \ for\ i \neq j, 1 \leq i,j \leq \ell\}$. *Then* Φ *is generated by the linear combination of* R *and satisfies the axiom of root systems.*

Here, the axiom of root systems is the axiom appeared in [10]. So, we may call the system Φ defined in Proposition 5.2 a *root system of type A* of a GKM graph $(\Gamma, \mathcal{A})$. This can be regarded as the generalization of Masuda's root systems of polytopes (symplectic toric manifolds) defined in [19].

The details of this article for more general cases will be appeared in the forthcoming paper [16].

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