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A TOPOLOGICAL DEFINITION OF HYPERTORIC MANIFOLDS AND ITS EQUIVARIANT COHOMOLOGY

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1. Introduction

In [2], Bielawski and Dancer defined the hyperKähler analogue of symplectic toric manifolds, called a *toric hyperKähler manifold* or *hypertoric manifold* (also see [1, 6, 7, 8, 9, 10]). On the other hand, there is the topological definition of toric manifolds introduced by Davis-Januszkiewicz [4], called a *quasitoric manifold* (also see [3, 5]).

In this article (extended abstract), we give a constructive definition of a *topological hypertoric manifold* (see Section 2), and compute its equivariant cohomology (see Theorem 3.2).

The details of this article will be appeared in the forthcoming paper [13].

2. Definition

Let T^n be the n-dimensional compact abelian group, i.e., n-dimensional torus, and $\mathbb{R}^n \times \mathbb{C}^n$ be the product of n-dimensional real and complex Euclidean spaces. We let $\mathcal{S} = \{S_1, \ldots, S_m\}$ denote a hypersurface arrangement in $\mathbb{R}^n$, i.e., each $S_i \in \mathcal{S}$, $i = 1, \ldots, m$, is a smooth surface in $\mathbb{R}^n$ which is diffeomorphic to $\mathbb{R}^{n-1}$. Note that $S_i \in \mathcal{S}$ might not be the affine space in $\mathbb{R}^n$. Assume that if $\cap_{i\in I} S_i \neq \emptyset$ then $\cap_{i\in I} S_i \cong \mathbb{R}^{n-|I|}$, where $|I|$ is the cardinality of $I \subset [m]$. Figure 1 is one of the examples of arrangements of hypersurfaces.

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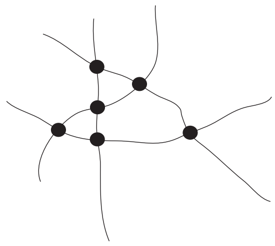

FIGURE 1. An arrangement of hypersurface $\{S_1, S_2, S_3, S_4\}$ in $\mathbb{R}^2$ whose intersections are 6 points.

We attach a label to each of S_i's by using the *characteristic function*

$$\lambda : \mathcal{S} \to \mathbb{Z}^n$$

such that

$$\det(\lambda(S_{i_1}) \cdots \lambda(S_{i_n})) = \pm 1 \quad \text{when } \cap_{j=1}^{n} S_{i_j} \neq \emptyset.$$

Now we may define the space M as follows:

$$M = T^n \times \mathbb{R}^n \times \mathbb{C}^n / \sim_\lambda .$$

Here, the equivalence relation $\sim_\lambda$ is defined by the following relations:

$$\begin{array}{ll} (t_1, r, z) \sim_\lambda (t_2, r, z) \Leftrightarrow t_1^{-1} t_2 \in T_{\lambda(S)} & \text{if } (r, z) \in S \times \lambda(S)^\perp; \\ (t_1, r, z) \sim_\lambda (t_2, r, z) \Leftrightarrow t_1 = t_2 & \text{otherwise,} \end{array}$$

where $T_{\lambda(S)}$ is the connected 1-dimensional subgroup in T^n whose Lie algebra is generated by $\lambda(S) \in \mathbb{Z}^n = \mathfrak{t}_{\mathbb{Z}}^n$ for $S \in \mathcal{S}$, and $\lambda(S)^\perp$ is the normal space of the vector $\lambda(S)$ in $\mathbb{C}^n$ with respect to the standard Hermitian inner product on $\mathbb{C}^n$, i.e., the inner product $\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i}$ for $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$. Note that $\lambda(S)^\perp$ is a complex $(n-1)$-dimensional linear subspace in $\mathbb{C}^n$.

Then we have the following theorem.

Theorem 2.1. *For the space M defined above, the following statements hold.*

1. *M is a $4n$-dimensional non-compact manifold.*
2. *M has the natural T^n-action on the T^n-factor in $T^n \times \mathbb{R}^n \times \mathbb{C}^n$.*
3. *M has the extra S^1-action on the $\mathbb{C}^n$-factor in $T^n \times \mathbb{R}^n \times \mathbb{C}^n$ by the scaler multiplication.*

We call M defined as above a *topological hypertoric manifold*.

Remark 2.2. Let X be a hypertoric manifold. Then, X can be constructed by using the above construction up to equivariant diffeomorphism, i.e., X is a topological hypertoric manifold. Moreover, in this case, all S_i's are codimension-one affine subspaces in $\mathbb{R}^n$.

Remark 2.3. For all $2n$-dimensional quasitoric manifolds V, there exists a $4n$-dimensional topological hypertoric manifold M such that M is T^n-homotopy equivalent to V.

It follows from Remark 2.2 and 2.3 that the set of all topological hypertoric manifolds is strictly wider than that of all hypertoric manifolds. Figure 2 shows one of the arrangement which does not induce any hypertoric manifold but induces topological hypertoric manifold.

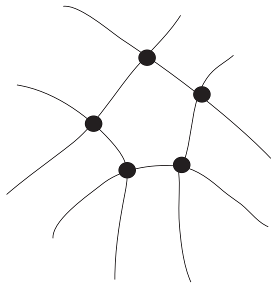

FIGURE 2. An arrangement of hypersurface in $\mathbb{R}^2$ which does not induce any hypertoric manifold because there is no affine arrangement which is combinatorially equivalent to this arrangement.

3. Equivariant cohomology

In order to compute $T^n \times S^1$-equivariant cohomologies of topological hypertoric manifolds M, we first prepare the following lemma.

Lemma 3.1. *A topological hypertoric manifold with the extra S^1-action $(M^{4n}, T^n \times S^1)$ is a GKM manifold whose GKM graph is a $2n$-valent hypertorus graph.*

Furthermore, the hypertorus graph defined by $(M^{4n}, T^n \times S^1)$ satisfies the following conditions:

(1) *For each $L \in \mathcal{L}$, there is the unique pair of a hyperfacet H and its opposite side $\overline{H}$ such that $H \cap \overline{H} = L$;*
(2) *For each subset $\mathcal{L}' \subset \mathcal{L}$, the intersection of all elements in $\mathcal{L}'$ is empty or connected,*

where $\mathcal{L} = \{L_1, \cdots, L_m\}$ is the set of all $(2n-2)$-valent hypertorus subgraphs.

Here, a *hypertorus graph* is the special type of GKM graphs defined in [12] (also see [11]), and all notations appeared in Lemma 3.1 are defined in [12].

Using Lemma 3.1 and the main theorem in [12], we have the following theorem.

Theorem 3.2. *Let $(M^{4n}, T^n \times S^1)$ be the topological hypertoric manifold defined by the labeled, hypersurface arrangement $\mathcal{S} = \{S_1, \ldots, S_m\}$ in $\mathbb{R}^n$. Then its equivariant cohomology $H_{T^n \times S^1}(M; \mathbb{Z})$ is isomorphic to the following ring:*

$$\mathbb{Z}[\tau_1, \ldots, \tau_m, \tau_{m+1}, \ldots, \tau_{2m}, \chi]/\mathcal{I},$$

where $\deg \tau_i = \deg \chi = 2$, $i = 1, \ldots, 2m$, and the ideal $\mathcal{I}$ is generated by the following homogeneous polynomials:

$$\tau_i + \tau_{m+i} - \chi \quad \text{for all } i = 1, \ldots, m;$$
$$\prod_{i \in I} \tau_i \quad \text{for } \cap_{i \in I} H_i = \emptyset \ (I \subset [2m]).$$

Here, H_j in Theorem 3.2 is defined as follows. Now $\mathbb{R}^n$ can be divided into two half (topological) spaces by S_j, $j = 1, \ldots, m$; we denote one of them and another by H_j and H_{j+m}, respectively. Namely, H_j and H_{j+m} are diffeomorphic to the upper half space $\mathbb{R}^n_+$ and the lower half space $\mathbb{R}^n_-$ (respectively) in $\mathbb{R}^n$; moreover, they satisfy that $H_j \cup H_{j+m} = \mathbb{R}^n$ and $H_j \cap H_{j+m} = S_j$.

See [13] for further studies.

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