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WEAK SOLUTIONS TO THE NAVIER-STOKES-POISSON EQUATION

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Abstract. We consider the Navier-Stokes-Poisson equations described the motion of compressible viscous isentropic gas flow under the self-gravitational force. We will prove the existence of the finite energy weak solutions in a three dimensional bounded domain

1. Introduction

The purpose of the present paper is to prove the existence of the weak solution to the Navier-Stokes-Poisson equation

$$\begin{aligned} &\rho_t + \nabla\cdot(\rho u) = 0 \\ &(\rho u)_t + \nabla\cdot(\rho u\otimes u) + \rho\nabla\Phi + a\nabla\rho^\gamma = \mu\Delta u + (\lambda+\mu)\nabla(\nabla\cdot u) \\ &\Delta\Phi = 4\pi g(\rho - \frac{1}{|\Omega|}\int_\Omega \rho) \ \text{ in } \ \Omega\times(0,T) \end{aligned} \tag{1.1}$$

with the initial-boundary condition

$$\begin{aligned} &u = 0, \frac{\partial\Phi}{\partial\nu} = 0 \ \text{ on } \ \partial\Omega\times(0,T) \\ &\rho|_{t=0} = \rho_0(x), \ \ (\rho u)|_{t=0} = q_0(x) \ \text{ in } \ \Omega, \end{aligned} \tag{1.2}$$

where $\Omega\subset R^3$ is a bounded domain with $C^{2,\theta}$ boundary $\partial\Omega$, ν the outer normal vector, $\rho=\rho(x,t)$ the density,

$$u = u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t))$$

the velocity, $\Phi=\Phi(x,t)$ the Newtonian gravitational potential, $\gamma>1$ the adiabatic constant, $\mu>0$ and λ the viscosity constants satisfying $\lambda+\frac{2}{3}\mu\geq 0$, $a=e^s$ the constant determined by the entropy S and $g>0$ the gravitational constant. Physically, this system describes the motion of compressible viscous isentropic gas flow under the self-gravitational force. Such a fluid may be formulated as the Euler-Poisson equation [16, 17, 2, 10], where the viscosity is neglected, the equation is considered in the whole space R^3, and the solution admits to have the compact support. Then, the contact angle between fluid and vaccume is not zero in the equilibrium state, and establishing the existence of the solution in an appropriate

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function space including the equilibrium state is not easy because of this, even locally in time with spherically symmetry in space.

In a series of papers [20, 18, 19], T. Makino and his co-workers studied such a problem of vaccume for spherically symmetric Navier-Stokes-Poisson equation with solid core. Here, we study the Navier-Stokes-Poisson equation on the fixed domain Ω without radial symmetry or solid core, and show the existence of the weak solution in a reasonable function space including the equilibrium state, emphasizing that the vaccume region $\{x \in \bar{\Omega} | \rho(x,t) = 0\}$ can exist inside this domain Ω.

Similarly to the above mentioned equations, equation (1) is provided with the properties of the conservation of total mass $M = \inf_\Omega \rho$ and the decrease of total energy E;

$$\begin{aligned} E &= \int_\Omega (\frac{\rho}{2}|u|^2 + \frac{P}{\gamma - 1}) + \frac{\rho}{2} \int\int_{\Omega\times\Omega} G(x,y)\rho(x)\rho(y)dxdy \\ &= \frac{a}{\gamma-1}\|\rho\|_\gamma^\gamma + \frac{1}{2}\|\sqrt{\rho}u\|_2^2 - \frac{1}{8\pi g}\|\nabla\Phi\|_2^2, \end{aligned}$$

and here $P = a\rho^\gamma$ and $G = G(x,y)$ denote the pressure and the Green's function of the Poisson part, respectively, so that $\Phi(x) = \inf_\Omega G(x,y)\rho(y)dy$ if and only if

$$\Delta\Phi = 4\pi g(\rho - \frac{1}{|\Omega|}\int_\Omega \rho) \text{ in } \Omega, \quad \frac{\partial\Phi}{\partial\nu} = 0 \text{ in } \partial\Omega, \quad \int_\Omega \Phi = 0. \tag{1.3}$$

In this Poisson equation, $\rho \in L^\gamma(\Omega)$ implies $\Phi_{x_i} \in L^2(\Omega)(i = 1,2,3)$ for $\gamma \geq \frac{6}{5}$. More precisely,

$$\|\nabla\Phi\|_2 \leq gK\|\rho\|_{\frac{6}{5}} \tag{1.4}$$

by $L^{\frac{6}{5}}$ elliptic estimate and Sobolev's inequality, where K is a constant determined by Ω. In accordance with the energy E given above, therefore, $\gamma > \frac{6}{5}$ and $\gamma = \frac{6}{5}$ are the subcritical and critical exponents of the equilibrium, respectively. In more detail, the equilibrium state is realized by $u = 0$, and hence it holds that (3) and

$$\Phi + \frac{a\gamma}{\gamma-1}\rho^{\gamma-1} = constant \text{ in } \Omega.$$

Then, this problem has a variational solution in the case of $\gamma > \frac{6}{5}$, while it does not admit a solution if $1 < \gamma < \frac{6}{5}$ and $\Omega \subset R^3$ is star-shaped [21].

Our result on the non-equilibrium stat, on the other hand, is regarded as the generalization of [12] concerning the Navier-Stokes equation without the Poisson term; more precisely,

Theorem1. Let $T > 0$ and $\gamma > \frac{3}{2}$. Then, given $\rho_0 \in L^\gamma(\Omega)$ and $|q_0^i|^2/\rho_0 \in L^1(\Omega)$ with $\rho_0 = \rho_0(x) \geq 0$ and $q_0^i(x) = 0$ for x of $\rho_0(x) = 0$, we have a finite energy weak solution ρ, u, Φ to (1) satisfying the following.

1. $\rho = \rho(x,t) \geq 0, \quad \rho \in L^\infty(0,T;L^\gamma(\Omega)), \quad u^i \in L^2(0,T;H_0^1(\Omega))$
2. $E = E(t) \in L^1_{loc}(0,T)$.
3. $\frac{dE}{dt} + \mu\|\nabla u\|_2^2 + (\lambda+\mu)\|\nabla\cdot u\|_2^2 \leq 0$ in $D'(o,T)$.
4. The first two equations of (1) hold in $D'(\Omega\times(0,T))$.
5. $\Phi(\cdot,t) = g\int_\Omega G(\cdot,y)\rho(y,t)dy$ for a.e. $t \in (0,T)$.

6. The first equation of (1) holds in $D'(R^3 \times (0,T))$ if the zero extension is taken outside Ω to ρ, u.

7. The first equation of (1) is satisfied in the sense of the renormalized solution, i.e.,

$$\frac{d}{dt}b(\rho) + \nabla \cdot (b(\rho)u) + (b'(\rho)\rho - b(\rho))\nabla \cdot u = 0 \tag{1.5}$$

in $D'(\Omega \times (0,T))$ for any $b \in C^1(R)$ such that $b'(z) = 0$ if $|z|$ is large.

Lemma1.1. If $\Phi : R \to (-\infty, +\infty]$ is a proper, lower semi-continuous, convex function, $D \subset R^m$ is a domain with bounded measure, and

$$sup_k \|v_k\|_p < +\infty \tag{1.6}$$

$$v_k \to v \ \text{ weakly in } \ L^1(D) \tag{1.7}$$

$$\Phi(v_k) \to \bar{\Phi(v)} \ \text{ weakly in } \ L^1(D) \tag{1.8}$$

$$\int_D \Phi(v) = \int_D \bar{\Phi(v)} \tag{1.9}$$

with $r > 1$, then it holds that

$$v_k \to v \ \text{ strongly in } \ L^1(D). \tag{1.10}$$

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