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LECTURES ON EQUIVARIANT SCHUBERT CALCULUS, GKM THEORY AND POSET PINBALL

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Contents

Overview

These are notes from a series of 7 lectures on equivariant Schubert calculus, Goresky-Kottwitz-MacPherson theory, and Hessenberg varieties, delivered at the Korea Advanced Institute of Science and Technology in November 2011. In preparing this document I have preserved the informal, concrete, and example-oriented style of the actual lectures. In particular, this is not meant to be in any way a comprehensive introduction to the topics mentioned. Rather, these notes are intended to provide a very brief overview of the underlying perspective(s) motivating the manuscripts [15, 14, 1, 2, 7].

I take this opportunity to thank Professor Dong Youp Suh for the invitation to KAIST, his kind and generous hospitality during my stay, and his support and encouragement during the preparation of this document.

1. A quick introduction to Schubert calculus

We begin with some remarks on history. On the one hand, one can think of *Schubert calculus* as being a subfield within *Enumerative Geometry*, which is itself a subfield of the large and active research area of *Algebraic Geometry*. Here by *enumerative geometry* we mean roughly the counting of geometric objects of a given type satisfying a given collection of geometric conditions (concrete examples to follow!). On the other hand, modern Schubert calculus also has many links with research areas that are not strictly within the realm of traditional algebraic geometry, such as combinatorics, equivariant topology, and symplectic geometry. Thus, Schubert calculus is also a beautiful subject lying at the intersection of *combinatorics, geometry*, and *algebra*. In this context one can roughly describe Schubert calculus as the study of the *(equivariant) intersection theory on Kac-Moody flag varieties G/P (and various subvarieties thereof).*

1.1. **What is enumerative geometry?** Our first example of an enumerative-geometric problem is given by the famous "Appolonius' Problem" (circa roughly 200 B.C.):

(Q0) How many circles are tangent to 3 given circles in a plane?

The answer is: generically 8.[1]

We will take our second example from Schubert. (We do not mean Franz, the composer, but Herbert, the mathematician after whom 'Schubert calculus' is named. Herbert Schubert was born in 1848 and died in 1911.)

(Q1) How many lines intersect 2 given lines and a point in $\mathbb{R}^3$?

The answer is: generically 1.

What do we mean by *generic*? While we won't give a full answer here, this question gives an excuse and motivation to consider the following:

(Q2) What is the number of points of intersection of 2 lines in $\mathbb{R}^2$?

The answer to (Q2) is also "generically 1", and this can be seen intuitively. That is, if the two lines in (Q2) are not parallel – and this is true almost all of the time – then they certainly do intersect exactly once. So in this case, "generically" precisely means: "if the lines are not parallel". On the other hand, if the lines *are*, in fact, parallel, then we want the answer to still be 1; i.e., they intersect at "the point at infinity."

To get this answer to indeed be 1 even for parallel lines, we will define a space where the "points at infinity" are actually a part of the space. This motivates the following definitions:

Definition 1.1. Affine n-space, denoted by $\mathbb{A}^n$, is the following space:

$$\mathbb{A}^n := \{(a_1, \dots, a_n) \mid a_i \in \mathbb{A}\}.$$

(Here $\mathbb{A}$ can be either $\mathbb{R}$ or $\mathbb{C}$. We will mainly work with $\mathbb{C}$ for simplicity.)

Definition 1.2. Projective n-space, denoted by $\mathbb{P}^n$, is the following space:

$$\mathbb{P}^n := \{[a_0 : a_1 : \dots : a_n] \mid \text{not all } a_i = 0 \text{ and } (a_0, \dots, a_n) \sim (\lambda a_0, \dots, \lambda a_n) \text{ for } \lambda \in \mathbb{A}^*\}.$$

We can equivalently think of projective n-space as

$$\mathbb{P}^n = \left\{ \text{1-dimensional subspaces of } \mathbb{A}^{n+1} \right\}.$$

Sometimes we write $\mathbb{RP}^n$ or $\mathbb{CP}^n$ to emphasize the field we work over.

Example 1.3. *For the purposes of visualization in this example we work over $\mathbb{R}$ (not $\mathbb{C}$). The projective 2-space $\mathbb{RP}^1$ is the space of 1-dimensional (real) subspaces of $\mathbb{R}^2$. There is a concrete way to identify $\mathbb{RP}^1$ with $\mathbb{R} \cup \{\infty\}$, i.e. "the real line plus the point at infinity", by taking the intersection point with the affine line $\{y = 1\} \cong \mathbb{R}$ (and the intersection point of the subspace $\{x = 0\}$ with $\{y = 1\}$ is taken to be "∞"). See Figure 1.*

A similar picture shows that $\mathbb{RP}^2$ – the set of lines in $\mathbb{R}^3$ – can be thought of as the set of points in $\mathbb{R}^2$ with some points at infinity added. Generalizing this principle

[1]See `http://en.wikipedia.org/wiki/Appolonius%27_problem` for some nice figures!

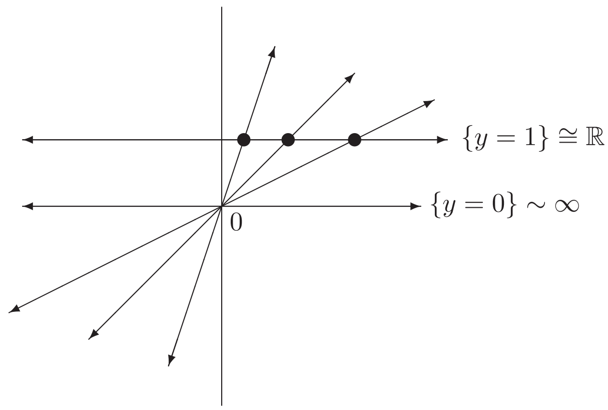


FIGURE 1. A depiction of $\mathbb{RP}^1 \cong \mathbb{R} \cup \{\infty\}$. A line in $\mathbb{R}^2$ is a point in $\mathbb{RP}^1$, and most lines intersect $\{y=1\}$ at a unique point, indicated by the highlighted dots in the figure. The one exception is the line $\{y=0\}$, which is the 'point at infinity'.

still further, by intersecting a 2-dimensional subspace of $\mathbb{A}^4 = \{(x_1, x_2, x_3, x_4) : x_i \in \mathbb{A}\}$ *with the affine subspace* $\{x_4 = 1\}$*, we can think of*

$$\{2\text{-}plane\ in\ \mathbb{A}^4\ going\ through\ the\ origin\} =: Gr(2,4)$$

(i.e. the ***Grassmannian of*** **2*-planes in*** $\mathbb{A}^4$*) as*

$$\{line\ in\ \mathbb{A}^3\ (not\ necessarily\ going\ through\ the\ origin)\}$$

plus 'some points at infinity'.

With these new spaces in hand, we now ask the following question:

(Q3) How many lines can intersect 4 given lines in $\mathbb{A}^3$?

We interpret this question as actually taking place in $Gr(2,4)$, using the correspondence informally described above. Schubert's answer to (Q3) is: generically 2. We will spend the rest of this section discussing (Q3), by relating Schubert's answer to the geometry of Grassmannians and to combinatorics (i.e., Young diagrams). In particular, to solve our problem using "modern methods," we will think about $Gr(2,4)$ in different ways. This is where algebra (and linear algebra) will start to appear. Much of this discussion is motivated from the beautiful (though dated) paper [16], which is highly recommended reading to anyone interested in Schubert calculus.

1.2. **The Grassmannian of 2-planes in $\mathbb{C}^4$.** Let $\mathbb{A} = \mathbb{C}$. Recall that a 2-plane in $\mathbb{C}^4$ is determined by a choice of basis $\{P_1, P_2\}$. Thinking of P_1 and P_2 as row vectors

$$P_1 = \begin{bmatrix} p_1(0) & p_1(1) & p_1(2) & p_1(3) \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} p_2(0) & p_2(1) & p_2(2) & p_2(3) \end{bmatrix},$$

in $\mathbb{C}^4$ written with respect to the standard basis, we get a 2×4 matrix

$$\begin{bmatrix} p_1(0) & p_1(1) & p_1(2) & p_1(3) \\ p_2(0) & p_2(1) & p_2(2) & p_2(3) \end{bmatrix} \tag{1}$$

specifying the 2-plane. Changing the basis $\{P_1, P_2\}$ of the plane to a different basis $\{P_1', P_2'\}$ does not change the 2-plane, since P_1' and P_2' are necessarily linear combinations of P_1 and P_2.

By performing Gaussian elimination (using only elementary row operations) on the rows of (1) we can always put the matrix into a standard **row-reduced echelon form** (abbreviated RREF) which is unique. In our setting of 2×4 matrices, there are 6 possibilities for where the pivots of the RREF are located, and we immediately see a connection with combinatorics (i.e., Young diagrams)! See Table 1.2.

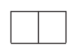
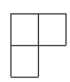
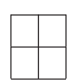
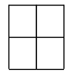
Subset of $Gr(2,4)$	**Corresponding Young Diagram**	**Cell**
$\left\{ \begin{bmatrix} * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix} \right\}$	$\emptyset$	$C_\emptyset$
$\left\{ \begin{bmatrix} * & 1 & \boxed{0} & 0 \\ * & 0 & * & 1 \end{bmatrix} \right\}$		$C_{(1)}$
$\left\{ \begin{bmatrix} 1 & \boxed{0} & \boxed{0} & 0 \\ 0 & * & * & 1 \end{bmatrix} \right\}$		$C_{(2)}$
$\left\{ \begin{bmatrix} * & 1 & 0 & \boxed{0} \\ * & 0 & 1 & \boxed{0} \end{bmatrix} \right\}$		$C_{(1,1)}$
$\left\{ \begin{bmatrix} 1 & \boxed{0} & 0 & \boxed{0} \\ 0 & * & 1 & \boxed{0} \end{bmatrix} \right\}$		$C_{(2,1)}$
$\left\{ \begin{bmatrix} 1 & 0 & \boxed{0} & \boxed{0} \\ 0 & 1 & \boxed{0} & \boxed{0} \end{bmatrix} \right\}$		$C_{(2,2)}$

Table 1. In the leftmost column, we are describing a partition of $Gr(2,4)$ into six disjoint sets of matrices in RREF (which uniquely specify a 2-plane) and where the $*$s represent free variables. (Exercise: convince yourself that $Gr(2,4)$ really is the union of these sets.) The middle column represents the corresponding Young diagrams, and the rightmost column is our notation for the corresponding set of matrices (i.e., 2-planes) in the leftmost column. These are called **cells**, and we shall return to them later.

Note that in Table 1.2, the middle column is all the possible Young diagrams that "fit" into a $2 \times (4-2)$ box

 .

Young diagrams appear in Table 1.2, so we review them now for the sake of the reader who has never seen them before. Young diagrams are a concrete and pictorial way to represent partitions of an integer.

Definition 1.4. A **partition** of a positive integer n is a representation of n as a sum of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. If

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$$

gives a partition of n, then we write $\lambda \vdash n$.

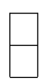
Here are some examples of partitions.

Example 1.5.

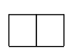
Positive integer n	*Partitions of n*
$n = 2$	$(1,1)$, (2)
$n = 3$	$(1,1,1)$, $(2,1)$, (3)
$n = 4$	$(1,1,1,1)$, $(2,1,1)$, $(2,2)$, $(3,1)$, (4)

The Young diagram corresponding to a partition $\lambda \vdash n$ is a picture containing n boxes, with λ_i boxes in the i-th row. We take the convention that the rows are left-aligned. See the following example.

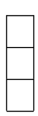
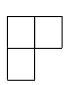
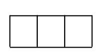
Example 1.6. *Let us write down the Young diagrams corresponding to the partitions given in Example 1.5. If $n = 2$, then the Young diagrams corresponding to the partitions $(1,1)$ and (2) are, respectively,*



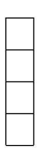
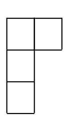
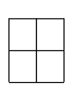
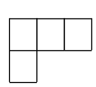
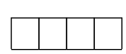
and

 .

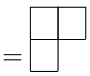
Similarly, if $n = 3$, then the corresponding Young diagrams to the partitions of 3 given in Example 1.5 are

 ,  , *and*  respectively.

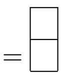
Similarly, if $n = 4$, then the corresponding Young diagrams for the partitions in Example 1.5 are

 ,  ,  ,  , *and*  .

By slight abuse of notation we often use the same letter λ for the partition of n and its corresponding Young diagram. Given Young diagrams $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$, we say that λ **contains** μ if $m \leq k$ and $\mu_i \leq \lambda_i$ for all $i \in \{1, \ldots, m\}$. Intuitively, this corresponds to the Young diagram μ "fitting inside" the Young diagram λ. For instance,

$$\lambda = (2,1,1) =$$ 

contains

$$\mu = (1,1) =$$ 

since $\mu_1 = 1 \leq 2 = \lambda_1$ and $\mu_2 = 1 \leq 1 = \lambda_2$.

Now we return to the Grassmannian. Another way to think about $Gr(2,4)$) is to embed it into $\mathbb{P}^5$, as the set of solutions to an algebraic equation (i.e. as a (projective) algebraic variety). This is analogous to thinking of the circle as being the set of solutions in $\mathbb{R}^2$ to $x^2 + y^2 = 1$. For $Gr(2,4)$ the map to $\mathbb{P}^5$ can be described concretely as follows:

$$\phi : Gr(2,4) \to \mathbb{P}^5 = \{[z_0 : z_1 : z_2 : z_3 : z_4 : z_5] \mid z_i \in \mathbb{C}\},$$

$$\begin{bmatrix} p_1(0) & p_1(1) & p_1(2) & p_1(3) \\ p_2(0) & p_2(1) & p_2(2) & p_2(3) \end{bmatrix} \mapsto \underbrace{\left[\det \begin{bmatrix} p_1(0) & p_1(1) \\ p_2(0) & p_2(1) \end{bmatrix} : \det \begin{bmatrix} p_1(0) & p_1(2) \\ p_2(0) & p_2(2) \end{bmatrix} : \text{ etc.}\right]}_{\text{determinants of all } 2\times 2 \text{ submatrices}}.$$

The reader should check that ϕ is well-defined. Moreover we have the following.

Fact 1.7. *The map ϕ is an embedding of $Gr(2,4)$ into $\mathbb{P}^5$ and the image of ϕ is precisely the set of solutions to*

$$z_0 z_5 - z_1 z_4 + z_2 z_3 = 0 \tag{2}$$

where we think of $\mathbb{P}^5$ as the space of lines in $\mathbb{C}^6 = \{(z_0, z_1, z_2, z_3, z_4, z_5) : z_i \in \mathbb{C}\}$. Equation (2) is called the **Plücker relation***, and ϕ is called the* **Plücker embedding**.

1.3. **Schubert varieties in $Gr(2,4)$.** Let L_1, L_2, L_3, L_4 be 4 fixed 2-planes in $\mathbb{C}^4$. Under our correspondence between lines in $\mathbb{C}^3$ and 2-planes in $\mathbb{C}^4$, it is clear that two lines intersect (in a point) exactly when the corresponding two planes intersect (in a line). Therefore, in order to answer (Q3) we must understand the set

(3)

$$\{L \in Gr(2,4) \mid \dim(L \cap L_i) \geq 1 \ \forall\ i = 1,2,3,4\} = \bigcap_{i=1}^{4} \{L \in Gr(2,4) \mid \dim(L \cap L_i) \geq 1\}.$$

In the right hand side of (3), each set in the intersection is a subset (actually a subvariety) of $Gr(2,4)$, and a geometric object in its own right. We will analyze these carefully next.

Let $\{e_1, \ldots, e_4\}$ denote the standard basis in $\mathbb{C}^4$. Suppose

$$L_1 = \operatorname{span}\langle e_1, e_2\rangle = \{\begin{bmatrix} * & * & 0 & 0 \end{bmatrix} \in \mathbb{C}^4\}.$$

Going back to our subsets in Table 1.2, we see that

$$\{L \in Gr(2,4) \mid \dim(L \cap L_1) \geq 1\} = C_{(1)} \cup C_{(2)} \cup C_{(1,1)} \cup C_{(2,1)} \cup C_{(2,2)} \tag{4}$$

so (4) consists of everything except $C_\emptyset$! Note also that (4) is a union over all cells with corresponding Young diagrams which contain the "smallest" Young diagram $\square$. Similarly, for the Young diagram $\square\square$, the corresponding union of cells would be

$$C_{(2)} \cup C_{(2,1)} \cup C_{(2,2)}.$$

We define

$$X_{(1)} := C_{(1)} \cup C_{(2)} \cup C_{(1,1)} \cup C_{(2,1)} \cup C_{(2,2)},$$

and

$$X_{(2)} := C_{(2)} \cup C_{(2,1)} \cup C_{(2,2)},$$

and in general, X_λ is the union of all cells C_μ such that μ contains the Young diagram λ. It turns out that X_λ is precisely the closure of the cell C_λ in $Gr(2,4)$ (technically we mean the closure in the Zariski topology, but in this case this is the same as the closure in the usual analytic topology).

Fact 1.8. *All X's are subvarieties of $Gr(2,4)$, i.e., they are cut out by algebraic equations.*

It is not hard to prove Fact 1.8, but we will not do it here, referring instead to [16].

1.4. **Computations in cohomology rings.** So far, we have seen algebra showing up as (polynomial) equations specifying subsets of $\mathbb{P}^n$ or $\mathbb{C}^n$, or in the linear algebra of matrices representing subspaces. Now we will see algebra of a different flavor. The Grassmannian $Gr(2,4)$ is an example of a manifold, and from algebraic topology we know that we can associate to it a cohomology ring

$$H^*(Gr(2,4);\mathbb{Z}) = \bigoplus_i H^i(Gr(2,4);\mathbb{Z})$$

which is a graded ring with respect to a product structure called the **cup product**. Moreover, $Gr(2,4)$ happens to be *oriented*, which means there is a natural isomorphism of the top-dimensional cohomology group with the integers. Given an element u in this top-dimensional group, we say $\deg(u)$ (the **degree** of u) is the image of u in $\mathbb{Z}$ under this natural isomorphism. Cohomology has the following very useful properties.

(I) Subvarieties of $Gr(2,4)$ are associated to elements in the ring $H^*(Gr(2,4);\mathbb{Z})$, in a way to be described more precisely below (cf. Section 3.2). We will denote this by $Y \mapsto \sigma(Y) \in H^*(Gr(2,4).\mathbb{Z})$. For the Schubert varieties X_λ introduced above we also write $\sigma_\lambda := \sigma(X_\lambda)$; these classes are called **Schubert classes**. The degree of σ_λ is $2\dim_{\mathbb{C}}(X_\lambda)$.

(II) If a subvariety Y_1 can be "moved continuously" to a subvariety Y_2, then

$$\sigma(Y_1) = \sigma(Y_2) \in H^*(Gr(2,4);\mathbb{Z}).$$

(Intuitively, one can think of Y_1 and Y_2 as being 'homotopic'.)

(III) When several subvarieties $Y_1, \ldots, Y_k$ intersect 'nicely' in a finite set of points, then the number of points (counted with multiplicity) is equal to the degree of the product of the corresponding cohomology classes. So in the case of $Gr(2,4)$, we have

$$\underbrace{\sigma(Y_1)\cdot\sigma(Y_2)\cdot\cdots\cdot\sigma(Y_k)}_{\text{product in cohomology ring!}} = (\#(Y_1\cap\cdots\cap Y_k))\cdot\sigma_{(2,2)},$$

where $\sigma_{(2,2)}$ is a special Schubert class, called the **top class**. (In general, the top class is always the class corresponding to the maximal Young diagram.)

These properties are what allow us to transform geometric computations and questions into algebraic ones about the product structure of a cohomology ring. In particular, it will allow us to give an answer to (Q3). Indeed, observe that for *any* plane L' in $\mathbb{C}^4$,

$$\{L \in Gr(2,4) \ | \ \dim(L \cap L') \geq 1\}$$

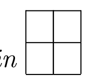
can be moved continuously to

$$X_{(1)} = \{L \in Gr(2,4) \ | \ \dim(L \cap \operatorname{span}\langle e_1, e_2\rangle) \geq 1\}.$$

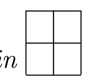
Thus, the answer to (Q3) is given by computing the constant c in the formula

$$(\star) \qquad \boxed{\sigma_{(1)} \cdot \sigma_{(1)} \cdot \sigma_{(1)} \cdot \sigma_{(1)} = c \cdot \sigma_{(2,2)}} \ .$$

Finally, we quote a beautiful combinatorial formula for computing in the cohomology ring of the Grassmannian. It reduces the cohomology computation (and hence also the geometric computation) to a very simple visual process involving Young diagrams. This result is usually called the **Pieri formula**. An exposition and further references can be found in [9].

Theorem 1.9. *Let λ be a Young diagram in*  *. Then*

$$\sigma_\lambda \cdot \sigma_{(1)} = \sum_\mu \sigma_\mu$$

where the summation is over all μ in  *obtained from λ by adding 1 box.*

Using Theorem 1.9, finding the constant c becomes quite straightforward. The reader is encouraged to verify that the following computation can be easily derived from the Pieri formula.

$$\begin{aligned} \sigma_{(1)}^2 &= \sigma_{(1)} \cdot \sigma_{(1)} = \sigma_{(2)} + \sigma_{(1,1)}, \\ \sigma_{(1)}^3 &= (\sigma_{(2)} + \sigma_{(1,1)}) \cdot \sigma_{(1)} = \sigma_{(2,1)} + \sigma_{(2,1)} = 2\sigma_{(2,1)}, \\ \sigma_{(1)}^4 &= (2\sigma_{(2,1)}) \cdot \sigma_{(1)} = 2\sigma_{(2,2)} \end{aligned} \qquad (5)$$

and hence the answer to (Q3) is indeed '2', as already indicated by Schubert.

2. An introduction to equivariant cohomology and GKM theory

We saw in the previous section that (enumerative) geometric questions can sometimes be translated into algebraic questions about cohomology rings, to great benefit. We now introduce the concept of an *equivariant* cohomology ring. While the definition is somewhat involved and perhaps initially uninuitive, there is an advantage to working equivariantly: it turns out that in nice enough situations (e.g. in Schubert calculus), there are many powerful techniques for computing in *equivariant* cohomology, and we can also re-derive the ordinary cohomology ring from the equivariant cohomology ring. Below we give a very brief sketch of these ideas. We refer the reader to e.g. [24] for a more detailed introduction.

Let G be a compact Lie group. Suppose M is a G-space (for our purposes, M will be a manifold or an algebraic variety). Roughly, the idea of equivariant cohomology is that it "ought" to be the ordinary cohomology of the quotient (orbit) space M/G.

However, if the G-action is not free, then M/G can be quite bad (e.g., there might be singularities). The agreed-upon solution to this problem is to alter the space M in such a way that the G-action is 'forced' to be free, but without changing the homotopy type of M.

Carrying out this goal depends on the following fact. For any compact Lie group G, there exists a principal G-bundle $\begin{matrix} EG \\ \downarrow \\ BG \end{matrix}$ with EG contractible. In particular, G acts freely on EG. The usual first example to consider is $G = S^1$. Then $ES^1 = S^\infty$, $BS^1 = \mathbb{C}P^\infty$, and $\begin{matrix} S^\infty \\ \downarrow \\ \mathbb{C}P^\infty \end{matrix}$. (This is the infinite-dimensional version of the well-known **Hopf fibration** $S^3 \to \mathbb{CP}^1$.)

Using this G-bundle, we now consider the space $M \times EG$ in place of the original space M. Notice that since EG is contractible, the new space has the same homotopy type of M, as desired. Equip $M \times EG$ with the diagonal G-action. We now define

$$H^*_G(M) := H^*(M \times EG/G)$$

i.e. the G-equivariant cohomology of M is the ordinary cohomology of the quotient space $M \times EG/G$. (The space $M \times EG/G$ is often denoted $M \times_G EG$, and is frequently called **the Borel construction** associated to the G-action on M.) Note that we could use different coefficients for this cohomology, such as $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$, etc., and in general it is an interesting, well-studied and subtle question as to which of the techniques described below apply to $H^*_T(-;k)$ for different choices of k. For what follows, we will take $\mathbb{C}$ coefficients for simplicity.

Remark 2.1. *If the G-action on M is free, then $H^*_G(M) \cong H^*(M/G)$, since EG is contractible. At the opposite extreme, if G acts on $\{\text{pt}\}$ trivially, then*

$$H^*_G(\text{pt}) \cong H^*(BG).$$

A special case is if we take $G = T^n = (S^1)^n$. Then,

$$\begin{aligned} H^*_{T^n}(\text{pt}) = H^*(BT^n) &= H^*((\mathbb{C}P^\infty)^n) \\ &\cong \mathbb{C}[u_1, \ldots, u_n]. \end{aligned}$$

The above computation shows that $H^*_{T^n}(M)$ is, in general, an $H^*_{T^n}(\text{pt}) \cong \mathbb{C}[u_1, \ldots, u_n]$-module.

2.1. **Computations in equivariant cohomology.** The main idea underlying many of the known techniques for actual computations in equivariant cohomology is that of "localization". Suppose now that the group G that acts on M is in fact a compact torus T.

Let

$$M^T := T\text{-fixed points in } M,$$

and let

$$i : M^T \hookrightarrow M$$

denote the inclusion map, and

$$i^* : H^*_T(M) \to H^*_T(M^T)$$

the induced map in (equivariant) cohomology. The following is a crucial result. An exposition and further references can be found in [13].

Theorem 2.2 (Borel)**.** *With the assumptions and notation above, the kernel and cokernel of i^* are torsion $H_T^* = H_T^*(\mathrm{pt})$-modules.*

Just to check our understanding, let us consider what this theorem says when the T-action on M is free. In this case, there is an isomorphism $H_T^*(M) \cong H^*(M/T)$ as graded rings, so in particular the degrees of the elements in $H_T^*(M)$ are bounded above. Since $H_T^* \cong H_T^*(\mathrm{pt}) \cong \mathbb{C}[u_1, \ldots, u_n]$ is a polynomial ring and the degrees are *not* bounded above, this means that for any $\alpha \in H_T^*(M)$, there exists some positive integer k such that $u^k \cdot \alpha = 0$ in $H_T^*(M)$. Thus $H_T^*(M)$ is a H_T^*-torsion module, and hence i^* is the zero map. This makes sense since $M^T = \emptyset$ if the T-action is free. On the other extreme, it turns out that in many of the cases of interest in Schubert calculus, the equivariant cohomology $H_T^*(M)$ is in fact a *free* $H_T^*(\mathrm{pt})$-module. Motivated by this, we explicitly record the following immediate corollary of Borel's theorem above.

Corollary 2.3. *If $H_T^*(M)$ is a free H_T^*-module, then*

$$i^* : H_T^*(M) \hookrightarrow H_T^*(M^T)$$

is injective.

There are two essential points here: firstly, there are many concrete criteria that guarantee that $H_T^*(M)$ is a free H_T^*-module (such as in the 'GKM theory' to be discussed below), and secondly, the T-action on M^T is trivial, so $H_T^*(M^T)$ is easily computed to be

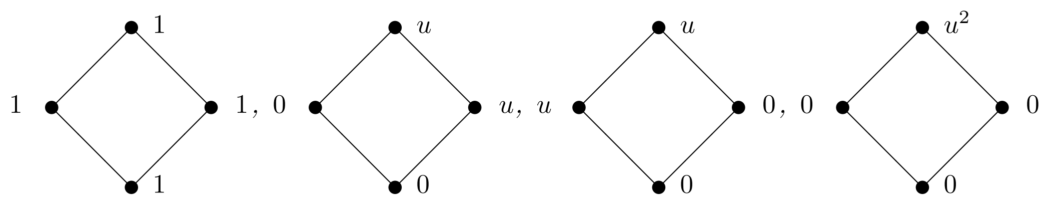
$$H_T^*(M^T) \cong H^*(M^T) \otimes H_T^*(\mathrm{pt}).$$

Also note that we cannot expect Corollary 2.3 to hold in ordinary cohomology, since in many cases (e.g. if M^T consists of isolated points) M^T is too small to hold all the information coming from the larger space M. This illustrates the power of working with equivariant cohomology. To take full advantage of Corollary 2.3, however, we need to concretely describe the image of i^*. This is the main goal of the so-called Goresky-Kottwitz-MacPherson ('GKM') theory which we sketch below, but first, we illustrate with some concrete examples.

Example 2.4. *Let S^1 act in the standard fashion (by rotation around a fixed axis) on the unit sphere S^2 in $\mathbb{R}^3$. Let N and S denote the north and south poles of S^2, respectively; these are the two (isolated) fixed points of the S^1 action on S^2. Now consider the space $M = S^2 \times S^2$ equipped with the diagonal action of $T = S^1$. Then it is straightforward that*

$$M^T = \{(N,N), (N,S), (S,N), (S,S)\}.$$

Here are some examples of elements of $im(i^*)$:



where we have pictorially encoded each of the 4 fixed points in M^T as the vertices *of the square, and we have labelled each vertex with an element in* $H_T^*(\mathrm{pt}) \cong \mathbb{C}[u]$*. Here we are thinking of* $H_T^*(M^T)$ *as isomorphic to* $\oplus_{i=1}^4 H_T^*(\mathrm{pt})$*. It is not yet obvious that these are in the image of* i^**, but we will explain this in Example 2.9.*

2.2. **Goresky-Kottwitz-MacPherson (GKM) theory.** As mentioned above, our goal now is to describe $\mathrm{im}(i^*)$ in Corollary 2.3 in concrete combinatorial terms. In order to do so, we need to make further simplifying assumptions. (We also warn the reader that there exists many versions of the so-called GKM theory in the literature, and the terminology and notation are not entirely consistent. In this note we will sketch one of the simplest versions.) Our first assumption is:

(Assumption 1) $\qquad M^T$ is nonempty and consists of isolated points.

As already noted above, this implies that

$$H_T^*(M^T) \cong \bigoplus_{p\in M^T} H_T^* \cong \bigoplus_{p \in M^T} \mathbb{C}[u_1, \ldots, u_{\mathrm{rank}(T)}].$$

Now, observe that at every $p \in M^T$, we have the isotropy representation given by T acting on T_pM. Moreover, since T is abelian, we get a decomposition into 1-dimensional representations

$$T_pM \cong \mathbb{C}_{\alpha_{1,p}} \oplus \cdots \oplus \mathbb{C}_{\alpha_{d},p},$$

where $d = \frac{1}{2}\dim_{\mathbb{R}} M$, the $\alpha_{i,p} \in \mathfrak{t}_{\mathbb{Z}}^*$ are isotropy weights, $\mathbb{C}_\alpha$ is the 1-dimensional representation of the weight α, and the isomorphism is as T-representations.

Our second assumption is

(Assumption 2) $\qquad$ At every $p \in M^T$, the isotropy weights $\{\alpha_{i,p}\}_{i=1}^d$ are pairwise linearly independent in $\mathfrak{t}_{\mathbb{Z}}^*$.

Definition 2.5. Let T be a compact torus and M a T-space. If both Assumptions 1 and 2 are satisfied, then we say the T-action on M is **GKM**.

There is also a useful characterization of Assumption 2 (given Assumption 1): namely, Assumption 2 says that the "equivariant 1-skeleton"

$$M^{(1)} := \{x \in M \ \mid \ \dim_{\mathbb{R}}(\underbrace{T\cdot x}_{T\text{-orbit}}) \leq 1\}$$

is 2-dimensional. That is, $M^{(1)}$ is a union of S^2's, equipped with a T-action given by weight $\alpha_{i,p}$ intersecting at fixed points. (In fact, there is a sign ambiguity here, but it does not affect the later computations so we will ignore this ambiguity.)

Using this information, we can now define the following combinatorial object.

Definition 2.6. The **GKM graph** of T acting on M is the labelled graph $\Gamma = (V, E, \alpha)$, where

$V = \text{vertices} = M^T$,

$E = \text{edges} = \left\{(p,q) \in M^T \times M^T \ \mid \ \exists \text{ an embedded } S^2 \subset M^{(1)} \text{ with } (S^2)^T = \{p,q\}\right\}.$

We also label each edge with the weight $\alpha = \alpha_{(p,q)} \in \mathfrak{t}_{\mathbb{Z}}^*$ of the T-action corresponding to that S^2.

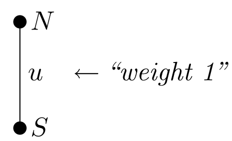
Next, note that a T-weight $\alpha \in \mathfrak{t}_{\mathbb{Z}}^* \subseteq H_T^*(\mathrm{pt})$ can be interpreted as an element in T-equivariant cohomology. Indeed, α is a linear polynomial in the variables $u_1, \ldots, u_n$. We can now state the following theorem, due to Goresky, Kottwitz, and MacPherson [11].

Theorem 2.7 (Goresky, Kottwitz, MacPherson). *Suppose that the T-action on M satisfies Assumptions 1 and 2. Then,*

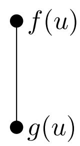
$$im(i^*) = \left\{ (f_p) \in \bigoplus_{p \in M^T} H_T^*(\mathrm{pt}) : \underbrace{\overbrace{\alpha_{(p,q)} | f_p - f_q}^{\text{divisible in } H_T^*} \text{ for all edges } (p,q) \in E}_{\text{often called ``GKM conditions''}} \right\} \cong H_T^*(M).$$

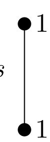
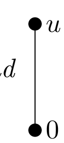
Here are some example computations using this theorem.

Example 2.8. *Let S^1 act on $S^2 = M$ in the standard way, as in Example 2.4. Then $M^{S^1} = \{N, S\}$. The GKM graph is*

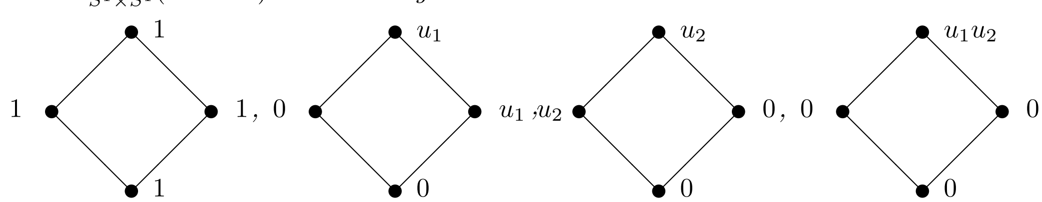


where we label the edge connecting N and S with the weight $u \in \mathfrak{t}^* \subseteq H_T^*(\mathrm{pt})$. Thus,

$im(i^*) = \{$  such that u divides $f(u) - g(u)$ in $\mathbb{C}[u]$ $\} \cong H_{S^1}^*(S^2)$.

This implies that, as an $H_{S^1}^*$-module, $H_{S^1}^*(S^2)$ has generators  *and*  .

Example 2.9. *Now consider $S^1 \times S^1$ acting on $S^2 \times S^2$. Similar reasoning shows that $H_{S^1 \times S^1}^*(S^2 \times S^2)$ has module generators*



Note that the elements displayed in Example 2.4 were obtained via the projection $H_{T^2}^*(\mathrm{pt}) \to H_{S^1}^*(\mathrm{pt})$ given by $u_1 \mapsto u$ and $u_2 \mapsto u$, thus justifying why the elements displayed in Example 2.4 are in fact in the image of i^*. Also, this example illustrates a general phenomenon: module generators for equivariant cohomology can often be arranged to have an "increasing number of zeroes," starting from the "bottom" (with respect to some choice of ordering). This is computationally very convenient, just as upper-triangular matrices are comptuationally easy to work with in linear algebra.

In fact, and most importantly for us, the (equivariant) Schubert classes will have this property.

3. GKM theory in Schubert calculus

Our next goal is to concretely illustrate GKM theory 'in action' in the setting of Schubert calculus, where even more is known and is combinatorially quite beautiful. For concreteness, we only discuss the so-called 'Lie type A' (i.e., $GL(n,\mathbb{C})$) case, although the theory exists for all types.

In the first section we worked with $Gr(2,4)$. Now let's generalize this to the **Grassmannian of k-planes in $\mathbb{C}^n$**, i.e.,

$$Gr(k,n) := \{V \subset \mathbb{C}^n \mid \dim_{\mathbb{C}} V = k\}.$$

Note that $GL(n,\mathbb{C})$ acts transitively on $Gr(k,n)$ by matrix multiplication. Moreover, the stabilizer of the standard k-plane $\mathrm{span}\{e_1,\ldots,e_k\}$ (where $\{e_1,\ldots,e_n\}$ is the standard basis of $\mathbb{C}^n$) is the parabolic subgroup P of the form

$$P = \left\{ \begin{bmatrix} \bigstar_1 & \bigstar_2 \\ \mathbf{0} & \bigstar_3 \end{bmatrix} \right\} \subset GL(n,\mathbb{C})/P,$$

where $[\bigstar_1]$ is some $k\times k$ submatrix, $[\bigstar_2]$ is some $(n-k)\times(n-k)$ submatrix, $[\bigstar_3]$ is some $(n-k)\times(n-k)$ submatrix, and $[\mathbf{0}]$ is an $(n-k)\times k$ zero (sub)matrix. Thus $Gr(k,n)$ is $GL(n,\mathbb{C})$-equivariantly identified with the coset space $GL(n,\mathbb{C})/P$. In fact, in modern Schubert calculus, the main geometric object of study is in fact a somewhat more expanded version of the Grassmannian, namely, the **flag variety** which is defined as

$$\mathcal{F}\ell ags(\mathbb{C}^n) := \{0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n \mid \dim_{\mathbb{C}} V_i = i\}.$$

Notice that there is a natural projection $\mathcal{F}\ell ags(\mathbb{C}^n) \to Gr(k,n)$ for any k between 1 and $n-1$. In this sense $\mathcal{F}\ell ags(\mathbb{C}^n)$ contains *more* information than $Gr(k,n)$.

We encourage the reader to work out the following .

Exercise 3.1. *In analogy to the fact that $Gr(k,n)$ is isomorphic to $GL(n,\mathbb{C})/P$, prove that $\mathcal{F}\ell ags(\mathbb{C}^n) \cong GL(n,\mathbb{C})/B$, where $B \subseteq GL(n,\mathbb{C})$ is the Borel subgroup of upper-triangular matrices. (*Hint: *Which flag is stabilized by B? Here, we are thinking of a flag as being represented by an invertible matrix $g = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$, where the v_i's are column vectors, and $V_i := span\{v_1,\ldots,v_i\}$.)*

We will focus on $\mathcal{F}\ell ags(\mathbb{C}^n) \cong GL(n,\mathbb{C})/B$ from now on. The group in question is the maximal torus T^n of diagonal (unitary) matrices in $U(n,\mathbb{C})$ (which is the compact form of $GL(n,\mathbb{C})$). This T^n acts on $\mathcal{F}\ell ags(\mathbb{C}^n)$ by multiplication on cosets in $GL(n,\mathbb{C})/B$. The rest of this section is devoted to analyzing this particular $T = T^n$ action on $\mathcal{F}\ell ags(\mathbb{C}^n)$ using the GKM techniques sketched above. For this we need the following.

Fact 3.2. *a) The T^n-fixed points are precisely the flags which are represented by permutation matrices (in particular, they are isolated). (By abuse of notation we often use w to denote both an element of S_n = Weyl group of $GL(n,\mathbb{C})$ and the permutation matrix representing it, e.g., $w = 132$, so $w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. We also denote by $[w] \in GL(n,\mathbb{C})/B$ (and sometimes just w) the corresponding coset.)*

b) *The T^n-isotropy weights at any $[w]$ are pairwise linearly independent (more on this below).*

Fact 3.2 implies that the T^n-action on $\mathcal{F}\ell ags(\mathbb{C}^n)$ is GKM.

3.1. **GKM graph for $\mathcal{F}\ell ags(\mathbb{C}^n)$.** We now describe the GKM graph for the $T = T^n$-action on $\mathcal{F}\ell ags(\mathbb{C}^n)$ very concretely. We refer the reader to [12] for more details and general statements. First we set some notation.

- Given $w \in S_n$, we will often use the one-line notation of w. Specifically, recall that $w \in S_n$ is a bijection
$$\begin{pmatrix} 1 & 2 & 3 & \cdots & \cdots & n \\ \downarrow & \downarrow & \downarrow & \cdots & \cdots & \downarrow \\ w(1) & w(2) & w(3) & \cdots & \cdots & w(n) \end{pmatrix}$$
on the set $\{1, 2, \ldots, n\}$. The one-line notation is the ordered list $w = \begin{pmatrix} w(1) & w(2) & \cdots & w(n) \end{pmatrix}$. For example, if $w \in S_3$ is given by $\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix}$ then its one-line notation is (213). In particular, one-line notation is *different* from the cycle notation which is also frequently used in group theory.
- Given $i, j \in \{1, 2, \ldots, n\}$, let s_{ij} denote the **simple transposition** which interchanges i and j only. For example, $s_{37} \in S_8$ has one-line notation (12745638). (In cycle notation, $s_{37} = (37)$.)
- Given $i \in \{1, 2, \ldots, n-1\}$, we denote by s_i the simple transposition $s_{i,i+1}$.

Remark 3.3. *It is a well-known fact, which we will use without proof, that the elements $\{s_i\}_{i=1}^{n-1}$ generate S_n.*

Based on the notation and terminology above we can now state the following. See [12] for more details and more generalities.

Theorem 3.4. *Let $n > 0$. The GKM graph of $\mathcal{F}\ell ags(\mathbb{C}^n)$ with respect to the maximal torus T^n-action is given as follows:*

- *Vertices $= \mathcal{F}\ell ags(\mathbb{C}^n)^{T^n} \cong$ Weyl group $= S_n$.*
- *For $v, w \in S_n$, there exists an edge between $[v]$ and $[w]$ precisely when $v = ws_{ij}$ for the simple transposition s_{ij}.*
- *For v, w as above, the label on the edge connecting v and w is $u_{w(i)} - u_{w(j)}$.*

Remark 3.5. *The labelling $u_{w(i)} - u_{w(j)}$ in the last item above does not look symmetric in w and v. If we instead think of w as vs_{ij}, we get*
$$u_{v(i)} = u_{w(j)} = -(u_{w(i)} - u_{w(j)}).$$
However, since the ambiguity of sign does not affect the GKM conditions, we ignore this.

Figure 2 contains an illustration of the GKM graph of $\mathcal{F}\ell ags(\mathbb{C}^3)$.

We close this section with an exercise to familiarize the reader with some of the computations that arise in this context.

Exercise 3.6. *We can compute directly the set of T-isotropy weights at the T-fixed points in $GL(n, \mathbb{C})/B$. For example, consider the identity coset $[id] \in GL(n, \mathbb{C})/B$. The point $[id]$ corresponds to the* **standard flag**
$$\{0 \subset V_1 = span\{e_1\} \subseteq V_2 = span\{e_1, e_2\} \subseteq \cdots \subseteq V_n = span\{e_1, \ldots, e_n\} = \mathbb{C}^n\},$$

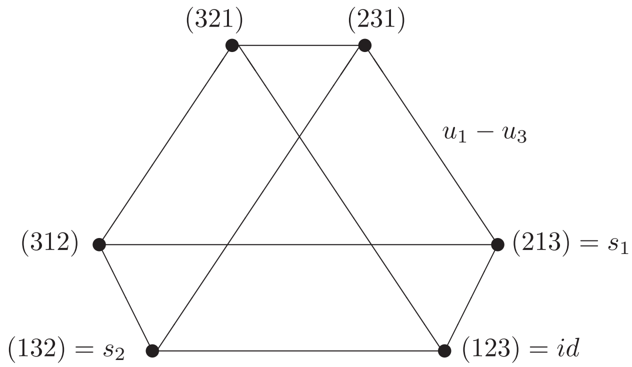


FIGURE 2. Note that parallel edges in this GKM graph have the same label.

where $\{e_1, \ldots, e_n\}$ *is the standard basis of* $\mathbb{C}^n$. *The corresponding matrix with respect to this basis is the* $n \times n$ *identity matrix. Since* $GL(n, \mathbb{C})/B$ *is a homogeneous space, the tangent space* $T_{[id]}GL(n, \mathbb{C})/B$ *is isomorphic to* $\mathfrak{gl}(n, \mathbb{C})/\mathfrak{b} \cong \mathfrak{n}^-$*, where* $\mathfrak{n}^-$ *denotes the Lie subalgebra of* strictly *lower-triangular matrices. Hence, the tangent space* $T_{[id]}GL(n, \mathbb{C})/B$ *has a basis* $\{E_{i,j}\}_{i>j, 1 \leq i,j \leq n}$*, where* $E_{i,j}$ *is the* $n \times n$ *matrix with a* 1 *at the* (i, j)*-th entry, and zeroes elsewhere.*

(1) *Prove that the* T*-action on the* $E_{i,j}$ *above is the usual conjugation action:*

$$t \cdot E_{i,j} = tE_{i,j}t^{-1} = t_i t_j^{-1} E_{i,j},$$

so the corresponding weight is $u_i - u_j$. *Show that each* $E_{i,j}$ *spans a* T*-invariant 1-dimensional subspace of* $T_{[id]}GL(n, \mathbb{C})/B$.

(2) *At the other fixed points* $[w] \in GL(n, \mathbb{C})/B$*, the tangent space* $T_{[w]}GL(n, \mathbb{C})/B$ *is isomorphic to* $w\mathfrak{n}^- w^{-1}$ *instead, and a basis is given by* $\{wE_{i,j}w^{-1}\}_{i>j,\ 1 \leq i,j \leq n}$*, with corresponding weight* $u_{w(i)} - u_{w(j)}$. *Thus, the* T*-weights at each* $[w]$ *are* pairwise *linearly independent in* $\mathfrak{T}^*_{\mathbb{Z}} = span\{u_i\}_{i=1}^n$. *(Note that it is important to say "pairwise." The dimension of* $\mathfrak{t}^*$ *is* n *while the number of* T*-weights we have for* $T_{[w]}GL(n, \mathbb{C})/B$ *is* $\dim_{\mathbb{C}} GL(n, \mathbb{C})/B = \frac{n(n-1)}{2}$*, which is usually much larger than* n.*)*

3.2. **(Equivariant) Schubert classes and their images in $H_T^*\left((GL(n, \mathbb{C})/B)^T\right)$.** Our next topic is the famous **Billey formula** which gives a concrete combinatorial computation of the images the equivariant Schubert classes under the restriction map $H_T^*(\mathcal{F}\ell ags(\mathbb{C}^n)) \to H_T^*((\mathcal{F}\ell ags(\mathbb{C}^n))^T)$. This formula is one of the most important tools in equivariant Schubert calculus and it sets the stage for all of the work in [15, 14, 1, 2, 7], so we take the time to describe it in some detail.

First recall that in $Gr(2, 4)$, the Schubert classes σ_λ described in Section 1 were associated to (closures of) Bruhat cells C_λ, arising from RREFs of a "given type," e.g.,

$$\left\{ \begin{bmatrix} * & 1 & 0 & 0 \\ * & 0 & * & 1 \end{bmatrix} \right\}.$$

Exercise 3.7. *Prove that these sets* C_λ *are in fact the* B*-orbits of the 6 special flags in* $Gr(2, 4)$ *of the form* $span\{e_i, e_j\}$*, where* e_i *is the* i*-th standard basis vector;*

e.g.,

$$span\{e_2, e_4\} \leftrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

*That is, under the identification $Gr(2,4) \cong GL(4,\mathbb{C})/P$, C_λ is a double coset $Bw_\lambda P$, where w_λ is a permutation matrix uniquely specified by λ. This is a general phenomenon that occurs for any $Gr(k,n)$. (*Hint: *Take the transpose of the 2×4 matrix (or the $k \times n$ matrix for the general case), representing for instance $span\{e_i, e_j\}$ for, say, some $i < j$. Act on this by B, and then use P to put the result in RREF.)*

For $\mathcal{F}\ell ags(\mathbb{C}^n)$, we can do something similar to Exercise 3.7. Namely, given $w \in S_n$, the **Bruhat cell** C_w is by definition the double coset $BwB \subseteq GL(n,\mathbb{C})/B$. For instance, in $\mathcal{F}\ell ags(\mathbb{C}^3)$,

$$\left\{ \begin{bmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

is $C_{(312)}$.

Exercise 3.8. *(1) Write out all 6 Bruhat cells in $\mathcal{F}\ell ags(\mathbb{C}^3)$.*

(2) Prove that $C_w \cong \mathbb{C}^{\ell(w)}$, where $\ell(w)$ is the **Bruhat length of w**. *(This is generally true, not just for $n = 3$). We define Bruhat length precisely below; for now, you can use the fact that*

$$\begin{aligned} \ell(w) &= \#\{\textit{inversions in (the one-line notation of) } w\} \\ &= \#\{\textit{pairs } i < j \mid w(i) > w(j)\}. \end{aligned}$$

For example, $w = (231)$ has inversions $(1,2)$ and $(1,3)$, so $\ell(w) = 2$.

Now we define the Schubert variety X_w as the closure in $\mathcal{F}\ell ags(\mathbb{C}^n)$ (in either the Zariski topology or the usual analytic topology - the result is the same) of C_w: in symbols, $X_w := \overline{C_w}$. Notice that these Schubert varieties are T-invariant by definition of the C_w. We can use these varieties X_w to specify (equivariant) cohomology classes, as we now sketch. For details on what follows, see [9, Appendix B]. Let X be a nonsingular projective n-complex-dimensional variety. Suppose V is an irreducible closed k-complex-dimensional subvariety of X. Then V determines a "fundamental class" $[V]$ in $H_{2k}(X)$. Recall that capping with the fundamental class $[X]$ in $H_{2n}(X)$ gives an isomorphism

$$\begin{aligned} H^i(X) &\to H_{2n-i}(X) \\ \alpha &\mapsto \alpha \cap [X] \end{aligned}$$

for all i, so under this isomorphism there exists a unique cohomology class in $H^{2n-2k}(X)$ associated to $[V]$. This is the cohomology class associated to V, and by abuse of notation we also denote it by $[V]$. Using this general procedure, we define the equivariant Schubert class $\sigma_w \in H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n);\mathbb{C})$ to be

$$\sigma_w := [B^- wB] = X_{w_0 w},$$

where B^- is the set of lower-triangular invertible matrices, and w_0 is the so-called 'full inversion' in S_n, i.e.

$$w_0 = \begin{pmatrix} n & n-1 & n-2 & \cdots & 2 & 1 \end{pmatrix}$$

in one-line notation. Note that

$$\dim_{\mathbb{C}}[B^- wB] = \operatorname{codim}_{\mathbb{C}}[BwB],$$

so

$$\deg(\sigma_w) = 2\dim_{\mathbb{C}}(X_w) = 2\ell(w).$$

The next fact is crucial.

Fact 3.9. *The* $\{\sigma_w\}_{w\in S_n}$ *form a* $H^*_T(\mathrm{pt})$*-module basis for* $H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n))$.

We saw in Section 1 that Schubert calculus problems can be translated into computations in the associated cohomology ring $H^*(Gr(2,4))$, with respect to the basis of Schubert classes. In line with this philosophy, 'equivariant Schubert calculus for $\mathcal{F}\ell ags(\mathbb{C}^n)$' is concerned with deriving "nice" formulas (where the Pieri formula in Theorem 1.9 counts as "nice"!) for the structure constants $c^u_{wv} \in H^*_T(\mathrm{pt}) = \mathbb{C}[u_1,\ldots,u_n]$ in the equation

$$\sigma_w \cdot \sigma_v = \sum_{u\in S_n} c^u_{wv}\sigma_u, \tag{6}$$

where the left-hand side of this equation is a product in $H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n);\mathbb{C})$. A great deal of current research concerns this question and very closely related analogues, e.g., for more general coset spaces G/P, for equivariant K-theory instead of equivariant cohomology, for equivariant quantum QH^*_T, and so on. However, my goal in this and the next two lectures is not to give a survey of this list, but rather to motivate the addition of one more possible 'generalized Schubert calculus' to this list: namely, to also consider *interesting subvarieties* of $GL(n,\mathbb{C})/P$. However, before starting that discussion, we need still to describe the **Billey formula**.

We need some notation. From now on we denote also by σ_v the image of σ_v in

$$H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n)^T) \cong \bigoplus_{w\in S_n} H^*_T(\mathrm{pt}).$$

As an element of $\bigoplus_{w\in S_n} H^*_T(\mathrm{pt})$, the class σ_v is nothing else than a list of $|S_n| = n!$ polynomials in $H^*_T(\mathrm{pt})$. In order to compute σ_v for any v, it therefore suffices to explicitly compute each $\sigma_v(w)$ for all $v, w \in S_n$, where here $\sigma_v(w)$ denotes the value of σ_v (really $i^*(\sigma_v)$)) at the fixed point w (really $[w]$).

Definition 3.10. For any $w \in S_n$, we say that an expression of w in terms of the simple transpositions s_i,

$$w = s_{i_1}s_{i_2}\cdots s_{i_k} \tag{7}$$

with $i_j \in \{1,\ldots,n-1\}$, is a **reduced word decomposition** of w if the length k of the word is minimal (i.e., there does not exist an expression of the above form with fewer factors). This minimal length is called the **Bruhat length** $\ell(w)$ of w. The ordered list $\mathcal{I} = \{i_1, i_2, \ldots, i_k\}$ corresponding to the indices in (7) is also called a reduced word decomposition of w.

Example 3.11. *If* $w = (321) \in S_3$*, then* w *has a reduced word decomposition* $w = s_1s_2s_1$*. Note that reduced word decompositions are* not *necessarily unique. In this example, we could have also written* $w = s_2s_1s_2$*.*

Given a word $\mathcal{I} = \{i_1, i_2, \ldots, i_k\}$, we say that $\mathcal{J}$ is a **subword** if it is an ordered list obtained by deleting some of the elements of $\mathcal{I}$, i.e., $\mathcal{J} = \{i_{n_1}, i_{n_2}, \ldots, i_{n_s}\}$ for some subset $\{n_1 < n_2 < \cdots < n_s\}$ of $\{1, 2, \ldots, k\}$. We will say that a subword **has product** $v \in S_n$ if

$$v = s_{i_{n_1}} s_{i_{n_2}} \cdots s_{i_{n_s}}$$

and we will say $\mathcal{J}$ is a **reduced subword with product** v if the above is in fact a reduced word decomposition of v. (Informally, it is also often said that v **appears** as a subword of w.) Note that there may be several subwords of a given word $\mathcal{I}$ with product v. For example, if $w = s_1 s_2 s_1$ and $v = s_1$, then v appears as a subword in 2 different ways:

$$\underbrace{\boxed{s_1}}_{=v} s_2 s_1 \tag{8}$$

and

$$s_1 s_2 \underbrace{\boxed{s_1}}_{=v}. \tag{9}$$

Finally, note that there is a natural S_n-action on $\mathbb{C}[u_1, \ldots, u_n]$ which permutes indices, e.g., $s_i(u_i) = u_{i+1}$. We can now state the Billey formula [3]. Although the actual statement is more general, we restrict to the $GL(n, \mathbb{C})$ case. We use the formulation given in [17].

Theorem 3.12 (Billey [3, 17])**.** *Let $\mathcal{I}$ be a reduced word decomposition of $w \in S_n$ whose k-th entry is the transposition $s_{i_k} =: r_k$. Then, for any $v \in S_n$,*

$$\sigma_v(w) = \sum_{\mathcal{J} \subseteq \mathcal{I}} \prod_{\mathcal{I}} (\hat{\alpha_k}^{[r_k \in \mathcal{J}]} r_k) \cdot 1, \tag{10}$$

where

- *α_k denotes the simple root $u_{i_k} - u_{i_k+1}$,*
- *the sum is over all reduced subwords $\mathcal{J}$ with product v, and*
- *$\hat{\alpha_k}$ denotes the multiplication-by-α_k operator on $\mathbb{C}[u_1, \ldots, u_n]$, included in the ordered product only if $r_k \in \mathcal{J}$, i.e., if the transposition r_k is part of the subword $\mathcal{J}$ under consideration.*

We work out several concrete examples to illustrate how to compute the right hand side of the Billey formula.

Example 3.13. *Suppose $\mathcal{I} = (1, 2, 1)$, so that $i_1 = 1$, $i_2 = 2$, and $i_3 = 1$, and the corresponding permutation is $w = (321) = s_1 s_2 s_1$. Let $v = s_1 = (213)$. The right hand side of the Billey formula for $\sigma_{(213)}((321))$ will be a sum over 2 subwords: $\mathcal{J} = i_1$ and $\mathcal{J} = i_3$, as in (8) and (9). Therefore,*

$$\begin{aligned}
\sigma_{(213)}((321)) &= (\widehat{u_1 - u_2}) \cdot s_1 s_2 s_1 \cdot 1 + s_1 s_2 (\widehat{u_1 - u_2}) s_1 \cdot 1 \\
&= (u_1 - u_2) + s_1 s_2 (u_1 - u_2) \\
&= (u_1 - u_2) + s_1 (u_1 - u_3) \\
&= (u_1 - u_2) + (u_2 - u_3) = u_1 - u_3.
\end{aligned}$$

The next example illustrates that the summation and the summands in the right hand side of the Billey formula can depend on the choice of the reduced word decomposition of w.

Example 3.14. *Let $w = (321)$ as above, but now suppose that we chose the reduced word decomposition $\mathcal{I} = (2, 1, 2)$ for $w = (321)$ instead. Then, with $v = s_1$, since*

$$s_2 \underbrace{\boxed{s_1}}_{=v} s_2,$$

then the sum in the right hand side of the Billey formula is over just 1 subword:

$$\begin{aligned}\sigma_{(213)}((321)) &= s_2\widehat{(u_1 - u_2)}s_1 s_2 \cdot 1 \\ &= s_2(u_1 - u_2) \\ &= u_1 - u_3.\end{aligned}$$

Of course, implicit in Theorem 3.12 is the statement that – as Examples 3.13 and 3.14 illustrate in a specific example – the value of the sum on the right-hand side of (10) is independent of the choice of reduced word decomposition.

Example 3.15. *This example is to warn the reader that one has to be careful, in general, to find* all *possible reduced subwords in a given word $\mathcal{I}$ which product to v. This is because – as mentioned above – reduced word decompositions for v are also not necessarily unique. For example, suppose $\mathcal{I} = (1, 2, 3, 2, 1)$ for $w = s_1 s_2 s_3 s_2 s_1$ in S_4 and suppose $v = s_1 s_3 = s_3 s_1$. Then the Billey formula for $\sigma_v(w)$ would have 2 summands, one for the subword $\boxed{s_1} s_2 \boxed{s_3} s_2 s_1$, and one for $s_1 s_2 \boxed{s_3} s_2 \boxed{s_1}$.*

Given $v, w \in S_n$, we say $v < w$ in **Bruhat order** if and only if v appears as a subword of a (and hence any) reduced word decomposition of w. Note that Bruhat order is just a partial order. Since the Billey formula for $\sigma_v(w)$ has a right hand side which is non-zero precisely when v appears as a subword of w, the following is immediate.

Corollary 3.16. *Let $v, w \in S_n$. If $v \not\leq w$ in Bruhat order, then $\sigma_v(w) = 0$.*

The following exercise is highly recommended to gain practice with the Billey formula.

Exercise 3.17. *Compute explicitly the polynomials $\sigma_v(w)$ for $v, w \in S_3$. In other words, compute explicitly the images under i^* of the* 6 *different equivariant Schubert classes in $H_T^*(\mathcal{F}\ell ags(\mathbb{C}^3))$.*

4. Hessenberg varieties and GKM compatibility

We are now in a position to discuss the more recent research presented in the manuscripts [15, 14, 1, 2, 7]. Again, for simplicity, we will restrict our discussion to Lie type A (i.e., $GL(n, \mathbb{C})$).

We begin by introducing the main geometric objects of study, for which we need the following data: a linear operator $X : \mathbb{C}^n \to \mathbb{C}^n$ and $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, a function satisfying

- $h(i + 1) \geq h(i)$ for all $1 \leq i \leq n - 1$
- $h(i) \geq i$ for all $1 \leq i \leq n$.

Such an h is called a **Hessenberg function**. Hessenberg functions have historically arisen in many contexts, including geometric representation theory [23, 21, 10], numerical analysis [6], mathematical physics [18, 20], combinatorics [8], and algebraic geometry [4, 5, 25]. The **Hessenberg variety** $\mathrm{Hess}(X,h)$ associated to the data of X and h above is defined to be

$$\mathrm{Hess}(X,h) := \{V_\bullet \in \mathcal{F}\ell ags(\mathbb{C}^n) \mid XV_i \subseteq V_{h(i)} \text{ for all } i\}.$$

Since $\mathrm{Hess}(X,h)$ and $\mathrm{Hess}(gXg^{-1},h)$ for $g \in GL(n,\mathbb{C})$ are isomorphic varieties, we assume without loss of generality that X is in Jordan form with respect to the standard basis of $\mathbb{C}^n$. One can then obtain a description of $\mathrm{Hess}(X,h)$ by intersecting the Bruhat cells C_w with $\mathrm{Hess}(X,h)$ and taking their union, as in the next example.

Example 4.1. *Suppose*

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$h(i) = i$$

for $i = 1,2,3$. *Then, by explicit computation, one can check that*

$$\mathrm{Hess}(X,h) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} * & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & * & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\} \subseteq \mathcal{F}\ell ags(\mathbb{C}^n).$$

Certain subclasses of Hessenberg varieties have also been studied in other contexts, as the following examples illustrate.

Example 4.2. *If* $h(i) = i$ *is the identity function, the corresponding* $\mathrm{Hess}(X,h)$ *is called a* **Springer variety**. *These appear in geometric representation theory. The well-known Springer theory constructs geometrically the irreducible* S_n*-representations on the cohomology rings of these Springer varieties.*

Example 4.3. *On the other hand, in the special case when* X *is the linear operator with a single Jordan block of eigenvalue 0, i.e.,* X *has the form*

$$X = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix},$$

then $\mathrm{Hess}(X,h)$ *is called a* **regular nilpotent Hessenberg variety**. *The special case of* $h(i) = i+1$ *is called a* **Peterson variety**, *which has been studied by Peterson, Kostant, and Rietsch (e.g.* [18, 20]*). Kostant in particular showed that there exists a dense subvariety of the Peterson variety whose coordinate ring is isomorphic to* $QH^*(\mathcal{F}\ell ags(\mathbb{C}^n))$. *Much of this should be better understood.*

The first goal of the recent research in [15, 14, 1, 7] is to build explicit $H^*_{S^1}(\mathrm{pt})$-module bases for the S^1-equivariant cohomology rings of Hessenberg varieties which are

- concretely computable at each S^1-fixed point (as an element in $H^*_{S^1}(\mathrm{pt}) \cong \mathbb{C}[u]$) in the spirit of the Billey formula of Theorem 3.12, and

- computationally convenient in the sense of being as 'upper-triangular' as possible (with respect to some natural partial order).

Of course, the question arises: why build such module bases? To answer this we take a moment to recall the broader context. In the classical Schubert calculus on $Gr(k,n)$, the Schubert classes σ_λ have corresponding structure constants

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c^\nu_{\lambda\mu} \sigma_\nu, \tag{11}$$

where $c^\nu_{\lambda\mu}$ are the famous **Littlewood-Richardson numbers/coefficients**, which appear in many fields: they appear in the combinatorics of skew tableau, and in representation theory they are tensor product multiplicities. In particular, they are all **positive** and **integral**. Of course, one cannot expect such 'beautiful' structure constants to arise for some arbitrary choice of module basis; it is only because (11) is written with respect to the geometrically-defined Schubert classes that the Littlewood-Richardson coefficients appear there (so one can also interpret the $c^\nu_{\lambda,\mu}$ also as geometric intersection numbers).

Motivated by this, the following is a sort of 'guiding principle' for researchers in Schubert calculus in a broad context:

(12)

> Find a 'good' module basis for equivariant cohomology, and also give a general formula for structure constants with respect to this basis which is
>
> - combinatorial, i.e., given by a counting argument and/or manifestly positive (so each summand in a sum is positive),
> - elegant (e.g., exhibits natural symmetry in the structure constants which reflect the underlying geometry).

Of course (12) is much too vague to be considered in a precise mathematical sense, but it is nevertheless a philosophy ('mantra') which motivates much of the recent work in Schubert calculus and related areas. Thus the goal of the work in [15, 14, 1, 2, 7] can be interpreted as first steps towards achieving (12) in the context of Hessenberg varieties.

As a final motivational point, we also note that in the related field of geometric representation theory, the existence of a good basis frequently allows one to explicitly build an action of (for example) a Weyl group on an (ordinary or equivariant) cohomology ring. This can be viewed as a related benefit to achieving the first part of (12). Indeed, Tymoczko and I construct in [14] an equivariant lift of the classical Springer action on ordinary cohomology to one on S^1-equivariant cohomology in a special case of Springer varieties.

Next we introduce, using the Hessenberg varieties as our working example, the concept of **GKM compatibility** introduced in [14]. To use our methods we must now restrict our attention to a special case of Hessenberg varieties. Suppose $X : \mathbb{C}^n \to \mathbb{C}^n$ is a *nilpotent* linear operator, i.e., its Jordan canonical form has only zeroes on its diagonal. The corresponding $\mathrm{Hess}(X,h)$ are called **nilpotent Hessenberg varieties**. The following is an easy fact left to the reader.

Exercise 4.4. *Let $X : \mathbb{C}^n \to \mathbb{C}^n$ be nilpotent and in Jordan canonical form with respect to the standard basis of $\mathbb{C}^n$. The circle subgroup S^1 of T^n given by*

$$S^1 := \left\{ \begin{bmatrix} t^n & 0 & \cdots & 0 \\ 0 & t^{n-1} & & 0 \\ & & & \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & & t \end{bmatrix} \;\middle|\; t \in \mathbb{C},\ \|t\| = 1 \right\} \subseteq \text{diagonal } T^n \subseteq U(n, \mathbb{C})$$

preserves $\mathrm{Hess}(X, h)$.

Thus, a nilpotent Hessenberg variety is naturally an S^1-space. Moreover, there is natural commutative diagram of group actions and inclusions:

$$\begin{array}{ccc} S^1 \times \mathrm{Hess}(X,h) & \hookrightarrow & T^n \times \mathcal{F}\ell ags(\mathbb{C}^n) \\ \downarrow & & \downarrow \\ \mathrm{Hess}(X,h) & \hookrightarrow & \mathcal{F}\ell ags(\mathbb{C}^n). \end{array} \tag{13}$$

Also, from Section 3 we know that the T^n action on $\mathcal{F}\ell ags(\mathbb{C}^n)$ is GKM. On the other hand, since most of the time $\dim_{\mathbb{R}} \mathrm{Hess}(X, h) > 2$, there is virtually no chance for the S^1-action on $\mathrm{Hess}(X, h)$ to be GKM. This brings us to our main strategy:

(14) Exploit the GKM theory on the T^n-action on $\mathcal{F}\ell ags(\mathbb{C}^n)$ to analyze the S^1-action on $\mathrm{Hess}(X, h)$.

The **GKM compatibility** conditions in Definition 4.6 are intended precisely to delineate situations (in more generality than just the case of the Hessenberg varieties in $\mathcal{F}\ell ags(\mathbb{C}^n)$) in which the strategy (14) is actually achievable. We motivate the general definition by analyzing more carefully the case of the S^1-action on nilpotent Hessenberg varieties. For this we need the following.

Theorem 4.5 ([14]). *Let X and $\mathrm{Hess}(X, h)$ be as above, equipped with the S^1-action given above.*

1. *The S^1-fixed points of the Hessenberg variety satisfy $\mathrm{Hess}(X, h)^{S^1} = \mathcal{F}\ell ags(\mathbb{C}^n)^T \cap \mathrm{Hess}(X, h)$. In particular, the Hessenberg fixed points are isolated.*
2. *The S^1-equivariant cohomology $H^*_{S^1}(\mathrm{Hess}(X, h))$ is a free $H^*_{S^1}$-module. In particular, the natural inclusion $i : \mathrm{Hess}(X, h)^{S^1} \to \mathrm{Hess}(X, h)$ induces an injection $i^* : H^*_{S^1}(\mathrm{Hess}(X, h)) \hookrightarrow H^*_{S^1}(\mathrm{Hess}(X, h)^{S^1})$.*

We refer the reader to [14] for details of the proof, commenting only that for item (2), the essential point is that there exists a complex paving-by-affines of X [26] which implies that the ordinary cohomology $H^*(\mathrm{Hess}(X, h))$ of $\mathrm{Hess}(X, h)$ vanishes in even degree. Thus the spectral sequence for

$$\begin{array}{ccc} \mathrm{Hess}(X,h) \hookrightarrow & \mathrm{Hess}(X,h) \times_{S^1} ES^1 \\ & \downarrow \\ & BS^1 \end{array}$$

collapses.

Consider the following commutative diagram, which is the main diagram in this theory:

$$\begin{array}{ccc} H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n)) & \hookrightarrow & H^*_T((\mathcal{F}\ell ags(\mathbb{C}^n))^T) \cong \bigoplus_{w\in S_n} H^*_T(\mathrm{pt}) \\ \downarrow & & \downarrow \\ H^*_{S^1}(\mathrm{Hess}(X,h)) & \hookrightarrow & H^*_{S^1}(\mathrm{Hess}(X,h)^{S^1}) \cong \bigoplus_{w\in \mathrm{Hess}(X,h)^{S^1}} H^*_{S^1}(\mathrm{pt}) \end{array}$$

Some remarks are in order. First, since the horizontal arrows are injections, it suffices for our purposes to understand their images in the right-hand side of the diagram. Second, the left vertical arrow is the composition of the forgetful map and the geometric restriction map, as follows:

$$H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n)) \to H^*_{S^1}(\mathcal{F}\ell ags(\mathbb{C}^n)) \to H^*_{S^1}(\mathrm{Hess}(X,h)).$$

Finally, the right vertical arrows send to 0 the components corresponding to $w \in S_n \setminus \mathrm{Hess}(X,h)^{S^1}$. On the other hand, for a summand $w \in \mathrm{Hess}(X,h)^{S^1}$, this map is the natural projection $H^*_T(\mathrm{pt}) \to H^*_{S^1}(\mathrm{pt})$ induced by $\mathrm{Lie}(S^1) \hookrightarrow \mathfrak{t}^*$.

Let $p_w \in H^*_{S^1}(\mathrm{Hess}(X,h))$ denote the images of σ_w under the left vertical map $H_T(\mathcal{F}\ell ags(\mathbb{C}^n)) \mapsto H_{S^1}(\mathrm{Hess}(X,h))$ in the above diagram. From the remarks above it follows that one can easily and concretely compute, using the Billey formula and the projection $H^*_T \to H^*_{S^1}$, the (images of the) p_w in $H^*_{S^1}(\mathrm{Hess}(X,h)^{S^1})$. Motivated by the first goal stated in (12) we now ask:

(15) Can we pick a subset $\mathcal{B} \subset S_n$ of the Schubert classes whose images $\{p_w\}_{w\in\mathcal{B}}$ form an $H^*_{S^1}$-module basis for $H^*_{S^1}(\mathrm{Hess}(X,h))$?

The naïve guess for a solution to (15) would be to take $\mathcal{B} = \mathrm{Hess}(X,h)^{S^1} \subseteq S_n$. It turns out that this is almost always wrong, for e.g. Betti-count reasons. This is to be expected: in general, the degree of σ_w is unrelated to the degree of the class associated to w in $\mathrm{Hess}(X,h)$. For example, Figure 3 shows the nilpotent Springer variety in $\mathcal{F}\ell ags(\mathbb{C}^4)$ corresponding to the Young diagram $(3,1)$, i.e. to the linear operator $X : \mathbb{C}^4 \to \mathbb{C}^4$ with one Jordan block of size 3 and one of size 1. The current state of the research doesn't yet give a general, proven method for producing a subset $\mathcal{B}$ in all cases. However, in the cases analyzed thus far [15, 14, 2, 7], it has been the case that when the answer to (15) is "Yes", then by construction, the basis $\{p_w\}_{w\in S_n}$ has good vanishing properties with respect to a natural partial order in a sense similar to Corollary 3.16. Finally, note that the answer to (15) is doomed to be "No" if the map $H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n)) \to H^*_{S^1}(\mathrm{Hess}(X,h))$ is not surjective. In practice, in some cases we know the surjectivity in advance, while in other cases, we have access to some other data (e.g., Betti numbers) that help us determine appropriate $\mathcal{B}$ (and thus see as a consequence that the map must be surjective).

We conclude this section with the general definition (cf. also [14, Definition 4.5]).

Definition 4.6. Let T be a compact torus and X a T-space. Suppose the T-action on X is GKM. Suppose given a subspace $Y \subseteq X$ and a subgroup $H \subseteq T$ preserving Y. We say (H,Y) is **GKM-compatible** with (T,X) if

- $Y^H = X^T \cap Y$
- $H^*_H(Y)$ is a free H^*_H-module.

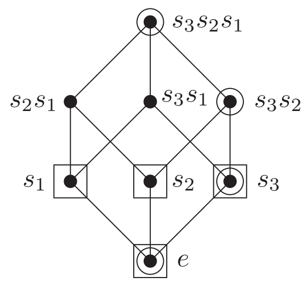


FIGURE 3. The dots in the figure are a subset of S_4, and the edges are the edges in the GKM graph of S_4. The circled dots are the S^1-fixed points of this Springer variety. The boxed dots correspond to a subset $\mathcal{B}$ of S_4 such that $\{p_w\}_{w\in\mathcal{B}}$ is a module basis. The circled dots would *not* correspond to a module basis since the Betti numbers do not match: the boxed dots indicate that this Springer variety has $b_0 = 1$ and $b_2 = 3$, while the circled dots correspond to a set of classes with one element each of degree $0, 2, 4$ and 6.

Remark 4.7. *It is a very interesting problem to find more examples of GKM-compatability other than the nilpotent Hessenberg varieties in flag varieties.*

5. Poset pinball

The goal of this section is to illustrate, using the **Peterson varieties** of Example 4.3, how we can achieve the goals of (12) using GKM-compatibility techniques [15]. In particular this example motivates our introduction in [14] of a general combinatorial strategy, which we dub 'poset pinball', for attacking (12) in more general cases.

Let $Y \subseteq \mathcal{F}\ell ags(\mathbb{C}^n)$ be the Peterson variety defined in Example 4.3. Consider the S^1-action on Y given in Exercise 4.4. The following is an easy computation.

Exercise 5.1. *The S^1-fixed points of Y are precisely the permutation matrices w (really $[w]$) which are block-diagonal matrices, where each block is anti-diagonal. For example, one of the S^1-fixed points for the Peterson variety in $\mathcal{F}\ell ags(\mathbb{C}^8)$ corresponds to the permutation*

$$\left[\begin{array}{ccc|ccc|cc} & & 1 & & & & & \\ & 1 & & & & & & \\ 1 & & & & & & & \\ \hline & & & & & 1 & & \\ & & & & 1 & & & \\ & & & 1 & & & & \\ \hline & & & & & & & 1 \\ & & & & & & 1 & \end{array}\right]$$

Informally, we say that a permutation w of the above form is a "sequence of upward staircases." Next, notice that there is a one-to-one correspondence

$$Y^{S^1} \subseteq \mathcal{F}\ell ags(\mathbb{C}^n) \leftrightarrow \text{subsets of } \{1, 2, \ldots, n-1\} \tag{16}$$

given by taking the indices of the columns in the permutation which are *not* at the top of a staircase. Continuing with the example above, under this correspondence we would identify

$$\left[\begin{array}{ccc|ccc|cc} & & 1 & & & & & \\ & \boxed{1} & & & & & & \\ \boxed{1} & & & & & & & \\ \hline & & & & & 1 & & \\ & & & & \boxed{1} & & & \\ & & & \boxed{1} & & & & \\ \hline & & & & & & & 1 \\ & & & & & & \boxed{1} & \end{array}\right] \leftrightarrow \{1,2,4,5,7\}.$$

(Here the indices "not at the top of a staircase" are boxed for the purpose of illustration.) We set the following notation for the correspondence given in (16):

$$w_{\mathcal{A}} \leftrightarrow \mathcal{A}.$$

Recalling (15), we wish to choose a $v_{\mathcal{A}} \in S_n$ for each $w_{\mathcal{A}}$ such that $\{p_{v_{\mathcal{A}}}\}_{\mathcal{A}\subseteq\{1,2,3,\ldots,n-1\}}$ forms a module basis for $H^*_{S^1}(Y)$. It turns out that in this case we can explicitly describe the appropriate choices of $v_{\mathcal{A}}$. Namely, define

$$v_{\mathcal{A}} := \prod_{i\in\mathcal{A}} s_i.$$

For example, if $\mathcal{A} = \{1,2,4,5,7\}$ then $v_A = s_1s_2s_4s_5s_7$.

We quote the following from [15].

Theorem 5.2. *(1)* $v_{\mathcal{A}} \le w_{\mathcal{A}}$ *for all* $\mathcal{A}$.
(2) $v_{\mathcal{A}} \le w_{\mathcal{B}}$ *if and only if* $w_{\mathcal{A}} \le w_{\mathcal{B}}$.
(3) Degrees of $\{p_{v_{\mathcal{A}}}\}$ *are compatible with (known) Betti numbers of* Y*.*

- $\deg p_{v_{\mathcal{A}}} = 2l(v_A) = 2 \cdot \#\mathcal{A}$.
- $\#\{p_{v_{\mathcal{A}}} \mid \deg p_{v_{\mathcal{A}}} = 2k\} = \binom{n-1}{k}$.
- *From* [22]*:* $b_{2k}(Y) = \binom{n-1}{k}$, *and* $b_{odd} = 0$. *(Note that this matches the preceding point exactly.)*

Proof. We sketch proofs of the first two facts. For (1), the maximal inversion

$$\begin{pmatrix} k & k-1 & k-2 & \cdots & 2 & 1 \end{pmatrix}$$

in S_k has reduced word decomposition

$$s_1(s_2s_1)(s_3s_2s_1)\cdots(s_{k-1}s_{k-2}\cdots s_2s_1),$$

which implies that $p_{v_{\mathcal{A}}}(w_{\mathcal{A}}) \neq 0$. For (2), observe

$$v_{\mathcal{A}} \le w_{\mathcal{B}} \Leftrightarrow \mathcal{A} \subseteq \mathcal{B} \Leftrightarrow w_{\mathcal{A}} \le w_{\mathcal{B}}.$$

□

Theorem 5.2 implies that $\{p_{v_{\mathcal{A}}}\}$ is a module basis of $H^*_{S^1}(Y)$, thus partially achieving (12). In the case of Peterson varieties, we can also do more. The first observation is the following.

Theorem 5.3 ([14])**.** *Let* $p_i := p_{v_{\{i\}}}$. *The elements* $\{p_i\}_{i=1}^{n-1}$ *generate* $H^*_{S^1}(Y)$ *as a ring.*

The above theorem implies that, in order to compute the ring structure of $H^*_{S^1}(Y)$, it suffices to compute products of the form $p_i \cdot p_{v_{\mathcal{A}}}$. This is achieved by the S^1**-equivariant Chevalley-Monk formula for Peterson varieties** [15]. More specifically, we have

$$p_i \cdot p_{\mathcal{A}} = c_{i,\mathcal{A}}^{\mathcal{A}} \cdot p_{\mathcal{A}} + \sum_{\mathcal{A} \subsetneq \mathcal{B} \text{ and } |\mathcal{B}|=|\mathcal{A}|+1} c_{i,\mathcal{A}}^{\mathcal{B}} \cdot p_{\mathcal{B}} \tag{17}$$

for any i and $\mathcal{A}$, where the structure constants are as follows. For any set $\mathcal{C} \subseteq \{1, 2, \ldots, n-1\}$ and any $\ell \in \mathcal{C}$, we denote by $\mathcal{T}_{\mathcal{C}}(\ell)$ and $\mathcal{H}_{\mathcal{C}}(\ell)$ the unique integers such that $\mathcal{T}_{\mathcal{C}}(\ell) \leq \ell \leq \mathcal{H}_{\mathcal{C}}(\ell)$, the consecutive sequence $\{\mathcal{T}_{\mathcal{C}}(\ell), \mathcal{T}_{\mathcal{C}}(\ell)+1, \ldots, \mathcal{H}_{\mathcal{C}}(\ell)-1, \mathcal{H}_{\mathcal{C}}(\ell)\}$ is a subset of $\mathcal{C}$, and such that $\mathcal{T}_{\mathcal{C}}(\ell)-1 \notin \mathcal{C}, \mathcal{H}_{\mathcal{C}}(\ell)+1 \notin \mathcal{C}$. We have first

- $c_{i,\mathcal{A}}^{\mathcal{A}} = 0$ if $i \notin \mathcal{A}$,
- $c_{i,\mathcal{A}}^{\mathcal{A}} = (\mathcal{H}_{\mathcal{A}}(i) - i + 1)(i - \mathcal{T}_{\mathcal{A}}(i) + 1)t$ if $i \in \mathcal{A}$,

where the variable t is the cohomology-degree-2 generator of $H^*_{S^1}(\mathrm{pt}) \cong \mathbb{C}[t]$. Additionally, for a subset $\mathcal{B} \subseteq \{1, 2, \ldots, n-1\}$ which is a disjoint union $\mathcal{B} = \mathcal{A} \cup \{k\}$, we have

- $c_{i,\mathcal{A}}^{\mathcal{B}} = 0$ if $i \notin \{\mathcal{T}_{\mathcal{B}}(k), \mathcal{T}_{\mathcal{B}}(k)+1, \ldots, \mathcal{H}_{\mathcal{B}}(k)-1, \mathcal{H}_{\mathcal{B}}(k)\}$,
- if $k \leq i \leq \mathcal{H}_{\mathcal{B}}(k)$, then
$$c_{i,\mathcal{A}}^{\mathcal{B}} = (\mathcal{H}_{\mathcal{B}}(k) - i + 1) \cdot \binom{\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1}{k - \mathcal{T}_{\mathcal{B}}(k)}$$
- if $\mathcal{T}_{\mathcal{B}}(k) \leq i \leq k-1$, then
$$c_{i,\mathcal{A}}^{\mathcal{B}} = (i - \mathcal{T}_{\mathcal{B}}(k) + 1) \cdot \binom{\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1}{k - \mathcal{T}_{\mathcal{B}}(k) + 1}.$$

From the above description, it is clear that these Monk formulae for the structure constants evidently has many of the properties deemed to be desirable in (12): it is explicit, easily computed, and both **manifestly positive** and **manifestly integral**. Moreover, in follow-up work [1], we also prove a **Giambelli** formula, which gives a concrete and combinatorial formula expressing an arbitrary module generator $p_{v_{\mathcal{A}}}$ in terms of the ring generators p_i.

The results for the case of Peterson varieties raises the following tantalizing question:

Question 5.4. Can we find analogues of $v_{\mathcal{A}}$ for more general Hessenberg varieties? Can we also prove general combinatorial formulae for structure constants using the corresponding module bases?

The game of **poset pinball**, briefly introduced next, is a *combinatorial* approach to answering this question. For more details see [14, Section 3]. Let $(\mathcal{J}, <)$ be a finite partially ordered set (e.g., S_n with the Bruhat order). Identify $\mathcal{J}$ with its Hasse diagram, which is a directed acyclic graph, with an edge $a \mapsto b$ when a covers b in the partial order (i.e., $a > b$ and there does not exist a c with $a > c > b$). We say that $\mathcal{J}$ is the **board**. The vertices are **slots**. At most one pinball can occupy a given slot at any time. The directed edges are called **slides**.

Fix a subset $\mathcal{I} \subset \mathcal{J}$, called the **initial subset**. (The motivating example if $\mathcal{I} = \mathrm{Hess}(X, h)^{S^1}$ and $\mathcal{J} = S_n$). At the start of the game, place a pinball at each slot corresponding to an element of $\mathcal{I}$. During the game, we occasionally place **walls** across some slides, which prevents a ball from rolling down that slide. (When a ball at slot a is released, it may **roll down** a slide from a to a slot b, if $a \mapsto b$ is an edge).

Fix a total order $\prec$ on $\mathcal{I}$ subordinate to the partial order induced from $\mathcal{J}$. We **roll pinballs** in order with respect to $\prec$. By this we mean:

($\bigstar$) Given a pinball at slot a, consider the set: $\{b \in \mathcal{J} : a \mapsto b$ and there is no wall across $a \mapsto b\}$.

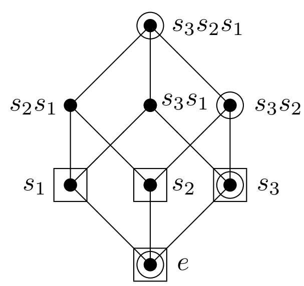
If ($\bigstar$) is non-empty, pick an element b and roll to it. (Note this process is non-deterministic – just like real arcade-game pinball!) Replace a by b, and repeat until the relevant set ($\bigstar$) is empty; i.e., it can roll no further (so all lower slots are already occupied by previously rolled pinballs). Call the result of this procedure (i.e., the name the slot where the pinball ends) **the rolldown of** a and we denote this process by $a \mapsto roll(a)$.

The motivation behind this game is the following.

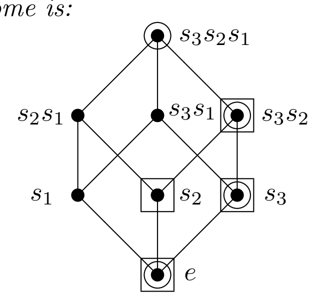
Fact 5.5. *The $v_{\mathcal{A}}$ associated to $w_{\mathcal{A}}$ in the Peterson case is the outcome of a game of poset pinball.*

The real point, however, is that poset pinball can be played for other cases of Hessenberg varieties, not just for Peterson varieties, as the next example shows.

Example 5.6. *We revisit in this context the nilpotent Springer variety in $\mathcal{F}\ell ags(\mathbb{C}^4)$ corresponding to the Young diagram $(3, 1)$, already discussed in Figure 3. One possible outcome of a basic pinball game is as follows:*



where the circled permutations correspond to the S^1-fixed points, and the squared dots are their rolldowns. However, as noted above, basic pinball is not *deterministic. Thus another possible outcome is:*



It is the outcome of the first game in Example 5.6 that actually gives rise to a module basis of the S^1-equivariant cohomology of this Springer variety (as already noted in the discussion of Figure 3). It turns out that one can place additional restrictions on the game of poset pinball in order to 'rule out' the possibility of the second outcome, which is not desirable for several reasons, including that the Betti numbers do not match (cf. discussion of Figure 3). We refer the reader to [14, Section 3] for more details.

Finally, as discussed above, we can also construct geometric representations on S^1-equivariant cohomology rings using our techniques. **Subregular Springer varieties** are those Springer varieties corresponding to the Young diagram $(n-1,1)$ for some $n \in \mathbb{Z}_{>0}$. For fixed n, let $\mathcal{S}_{n-1,1}$ denote the associated subregular Springer variety. Using module bases obtained via poset pinball, we can prove the following result.

Theorem 5.7 ([14]). *The images of* $\{\sigma_l, \sigma_{s_1}, \ldots, \sigma_{s_{n-1}}\}$ *under the natural map* $H^*_T(\mathcal{F}\ell ags(\mathbb{C}^n)) \to H^*_{S^1}(\mathcal{S}_{n-1,1})$ *form a module basis for* $H^*_{S^1}(\mathcal{S}_{n-1,1})$.

Remark 5.8. *By exploiting this, we can build an* S^1*-equivariant lift of the classical Springer action on* $H^*(\mathcal{S}_{n-1,1})$ *to* $H^*_{S^1}(\mathcal{S}_{n-1,1})$. *We refer the reader to* [14, Section 6.3] *for details.*

We close these lectures with a brief description of the results in [2, 7]. The manuscript [2] deals with regular nilpotent Hessenberg varieties with $h = (3, 3, 4, 5, \ldots, n-1, n)$. These should be thought of as regular nilpotent Hessenberg varieties which are very close to the Peterson variety. In the analysis of the Peterson case, it was crucial that we had an explicit formula for the rolldown $v_{\mathcal{A}}$ corresponding to each fixed point $w_{\mathcal{A}}$. In [2] we further develop the poset pinball theory by proposing, for the case of regular nilpotent Hessenberg varieties, an analogous explicit algorithm for producing reduced word decompositions of the rolldowns $roll(w)$ for the fixed points $w \in \mathrm{Hess}(X,h)^{S^1}$. (We dub this the **dimension pair algorithm**, since it is based on the notion of dimension pairs introduced in [19].) While the algorithm is not as easily stated as for the Peterson case, it nevertheless allows us to explicitly analyze the case when the Hessenberg function is of the form $h = (3, 3, 4, 5, ...)$, and in particular to prove using hands-on combinatorics that the result does indeed yield a module basis of S^1-equivariant cohomology.

The manuscript [7], on the other hand, deals with the Springer varieties $\mathcal{S}_{n-2,2}$ corresponding to the Young diagram $(n-2,2)$ for some $n \geq 4$. These should be thought of as Springer varieties which are very close to the subregular Springer varieties $\mathcal{S}_{n-1,1}$ analyzed in Theorem 5.7 above. Using a version of the dimension pair algorithm introduced in [2], Dewitt and I use the Billey formula and some explicit combinatorial computations to prove that, also in the case of $\mathcal{S}_{n-2,2}$, the result of poset pinball obtained by the dimension pair algorithm produces a module basis for S^1-equivariant cohomology.

As is evident from the descriptions above, research in this area is just beginning. We refer the reader to the manuscripts [15, 14, 2, 7] for more detailed lists of open questions and possible future work, but roughly speaking the questions can be summarized as follows:

- Is there a wider class of examples of Hessenberg varieties $\mathrm{Hess}(X,h)$ where one can produce a provable and explicit algorithm for pinball rolldowns which correspond to a module basis of $H^*_{S^1}(\mathrm{Hess}(X,h))$?
- Can we derive explicit, combinatorial formulae for structure constants with respect to well-chosen such module bases?

- Can we use these module bases to construct geometric representations on equivariant cohomology rings?

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